Two generalizations of an over-determined system on a surface

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Abstract We introduce and study two generalizations of an over-determined system on a surface. The first generalization is an over-determined system of polynomial type, which is in the form of \( df = \sum_{i=0}^{n} f^i \theta_i \), where each \( \theta_i \) is a 1-form; the second generalization is represented as \( F_u = (\log \rho^2) + 2(I/\rho)e^F, \) \( F_v = -2Jpe^{-F} \), where \( I, J \) are given by \( \omega = -Jdu + Idv \) for a 1-form \( \omega \) and \( \rho \) is given by \( \Omega = \rho du \wedge dv \) for a nowhere zero 2-form \( \Omega \). Studying these systems, we will obtain generalizations and analogues of results we already have on over-determined systems on surfaces. Results on the second generalization yield corollaries for over-determined systems on surfaces in 3-dimensional non-flat space forms.

1 Introduction

1.1 Over-determined systems on surfaces

Let \( S \) be a surface in a 3-dimensional space form \( N \). Suppose that there exist no umbilical points on \( S \). Let \((u, v)\) be local coordinates on a neighborhood \( U \) of a point \( p \) of \( S \) such that principal distributions \( D_1, D_2 \) on \( S \) are locally given by \( \partial/\partial u, \partial/\partial v \). Then the induced metric \( g \) on \( S \) from \( N \) is locally represented as \( g = A^2 du^2 + B^2 dv^2 \), where \( A, B \) are positive-valued functions. Let \( k_1, k_2 \) be principal curvatures of \( S \) corresponding to \( \partial/\partial u, \partial/\partial v \) on \( U \), respectively, and suppose that \( k_1, k_2 \) are nowhere zero. Then \( F := \log k_1^2 \) is a solution of the following over-determined system

\[
F_u = \alpha + \beta e^F, \quad F_v = \gamma + \delta e^{-F},
\]

where

\[
\alpha := 2(\log |K - L_0|B)u, \quad \beta := -2(\log B)u, \\
\gamma := -2(\log A)v, \quad \delta := 2(K - L_0)(\log A)v,
\]

and \( K \) is the intrinsic curvature of \( S \) and \( L_0 \) denotes the constant sectional curvature of \( N \). The set of solutions of (1.1) with (1.2) does not depend on the choice of such local coordinates as \((u, v)\) and is determined by the induced metric \( g \), principal distributions \( D_1, D_2 \) and the number \( L_0 \). We set

\[
q_1(u, v) := \frac{1}{E(u, v)} \int_{v_0}^{v} (K(u, v) - L_0)E_v(u, v)dv, \\
q_2(u, v) := \frac{1}{G(u, v)} \int_{u_0}^{u} (K(u, v) - L_0)G_u(u, v)du,
\]

(1.3)
where $E := A^2$, $G := B^2$, and a point $(u_0, v_0)$ corresponds to $p$. Functions $q_1, q_2$ on $U$ do not depend on the choice of $(u, v)$ and are determined by $g$, $D_1$, $D_2$, $L_0$ and a point $p$. There exist positive-valued functions \( \phi, \psi \) of one variable satisfying the following ([4]):

\[
    k_1^2 = q_1 + \frac{\phi(u)}{E}, \quad k_2^2 = q_2 + \frac{\psi(v)}{G}.
\]

(1.4)

If there exists another solution $\hat{F}$ of (1.1) with (1.2) than $F = \log k_1^2$, then there exists a surface $\hat{U}$ in $N$ satisfying the following: $\hat{U}$ is isometric to $U$ so that principal directions of $\hat{U}$ correspond to principal directions of $U$; $\hat{k}_1 := e^{\hat{F}/2}$ and $\hat{k}_2 := (K - L_0)/\hat{k}_1$ are principal curvatures of $\hat{U}$. We set

\[
    X := \beta_v + \beta\gamma, \quad Y := \alpha_v - \gamma_u + 2\beta\delta, \quad Z := -\delta_u + \alpha\delta.
\]

Then the compatibility condition of (1.1) is given by $X \equiv 0$, $Y \equiv 0$, $Z \equiv 0$. If (1.1) satisfies the compatibility condition, then the set of solutions is given by a one-parameter family of functions; if (1.1) does not satisfy the compatibility condition, then the set of solutions consists of at most two functions.

We have more concrete information of (1.1) with (1.2), in the case where $L_0 = 0$. We suppose $L_0 = 0$ and $N = E^3$. If (1.1) with (1.2) satisfies the compatibility condition, then the surface in $E^3$ corresponding to each solution is called a molding surface. Any molding surface satisfies $\beta\delta \equiv 0$, which means that the surface has a family of geodesics of curvature, i.e., a family of lines of curvature which consists of geodesics ([7, pp. 152–153], [6, pp. 277–281], [5]). A molding surface is parallel curved, that is, for each molding surface $S$, there exists a plane $P$ in $E^3$ such that at each point of $S$, at least one principal direction is parallel to $P$ ([1]). We can characterize an over-determined system (1.1) with (1.2) which has just two solutions, in terms of the sinh-Gordon equation ([4]). On a surface in $E^3$ with nonzero constant mean curvature which is not part of any surface of revolution, (1.1) with (1.2) has just two solutions. We can characterize an over-determined system (1.1) with (1.2) which has a unique solution so that $Y^2 = 4XZ$ holds, in terms of the equation $f_{uu} + f_{vv} = e^{-2f}$ ([3]). On a minimal surface in $E^3$ which is not part of any surface of revolution, (1.1) with (1.2) has a unique solution so that $Y^2 = 4XZ$ holds.

In the present paper, we will introduce and study two generalizations of an over-determined system on a surface and obtain generalizations and analogues of results we already have on over-determined systems on surfaces.

### 1.2 The first generalization

Let $M$ be an $m$-dimensional manifold with $m \geq 2$. The first generalization is an over-determined system in the form of

\[
    df = \sum_{i=0}^{n} f^i \theta_i, \quad (1.5)
\]
where $n \in \mathbb{N} \cup \{0\}$ and each $\theta_i$ is a 1-form on $M$. In the present paper, such an over-determined system is said to be of polynomial type. If we set $m := 2$, $n := 2$ and $f := e^F$, and if $\theta_0, \theta_1, \theta_2$ are locally represented as

$$
\begin{align*}
\theta_0 &= \delta dv, \\
\theta_1 &= \alpha du + \gamma dv, \\
\theta_2 &= \beta du,
\end{align*}
$$

(1.6)

where $\alpha, \beta, \gamma, \delta$ are functions on a neighborhood $U$ of each point of $M$ and $(u, v)$ are local coordinates on $U$, then (1.5) is locally represented as (1.1). If $n = 0$, then (1.5) has a local solution if and only if $\theta_0$ is closed. In the following, suppose $n \geq 1$. We will see that a system in the form of (1.5) is represented by its pointwise distinct solutions. In particular, we will see that a system (1.5) is determined by $(n + 1)$ pointwise distinct solutions. The representation of (1.5) by its solutions will be used as a basic tool in studying (1.5).

We set $c(1) := 1$, $c(n) := 2n - 2$ for $n \geq 2$. Then for an over-determined system (1.5), there exist 2-forms $\Theta_0, \Theta_1, \ldots, \Theta_{c(n)}$ on $M$ such that for any $t \in \mathbb{R}$,

$$
\sum_{k=0}^{c(n)} t^k \Theta_k = \sum_{i=1}^{n} \sum_{j=0}^{n} it^{i+j-1} \theta_j \wedge \theta_i + \sum_{i=0}^{n} t^i d\theta_i.
$$

For $t \in \mathbb{R}$, we set $P(t) := \sum_{k=0}^{c(n)} t^k \Theta_k$. The compatibility condition of (1.5) is given by $P \equiv 0$, i.e. $\Theta_k \equiv 0$ on $M$ for any $k \in \{0, 1, \ldots, c(n)\}$. If (1.5) satisfies the compatibility condition, then for each $a_0 \in M$ and an arbitrarily given $t_0 \in \mathbb{R}$, there exists a unique solution of (1.5) on a neighborhood of $a_0$ in $M$ such that the value at $a_0$ is equal to $t_0$. The compatibility condition of (1.5) is equivalent to the complete integrability of a Pfaffian equation $\omega = 0$ for a 1-form $\omega := \sum_{i=0}^{n} t^i \theta_i - dt$ on $M \times \mathbb{R}$. If $n = 2$, then we will represent $P(t)$ by two pointwise distinct solutions of (1.5), and from this representation, we will see that the existence of the third solution means that the system satisfies the compatibility condition and we can obtain a condition of the existence of just two solutions of (1.1) with $\gamma \equiv 0$ in the case where (1.1) has at least two solutions, which was originally obtained in [4]. Suppose $n \geq 3$. Then the existence of $(n + 1)$ pointwise distinct solutions of (1.5) does not necessarily mean that the system satisfies the compatibility condition. We will obtain an explicit formula of $P(t)$ by such solutions. In particular, in the case where $n = 3$, we will see that the compatibility condition is represented as a relation among four pointwise distinct solutions and their differentials and that if the four solutions $f_1, f_2, f_3, f_4$ satisfy the relation and if three of them, say, $f_1, f_2, f_3$ satisfy

$$(f_1 - f_2)df_3 + (f_2 - f_3)df_1 + (f_3 - f_1)df_2 \neq 0,$$

then $f_4$ is locally considered as a solution of an initial value problem given by $f_1, f_2, f_3$.

### 1.3 The second generalization

Results on the second generalization will yield corollaries for over-determined systems on surfaces in 3-dimensional non-flat space forms, which correspond to results in the case
where $L_0 = 0$ mentioned in Subsection 1.1.

Let $\alpha, \beta, \gamma, \delta$ be as in (1.2). Then $F$ is a solution of (1.1) if and only if $F' := F + c$ is a solution of $F'_u = \alpha' + \beta' e^{F'}, \ F'_v = \delta' e^{-F'}$, where we set $c := \log A^2$ and

$$\alpha' := \alpha + c_u = (\log (K - L_0)^2 A^2 B^2)_u,$$

$$\beta' := \beta e^{-c} = -\frac{(\log B^2)_u}{(K - L_0) A^2},$$

$$\delta' := \delta e^c = (K - L_0)(A^2)_v.$$

Let $V_K$ be the canonical pre-divergence of the surface $S$: $V_K$ is a vector field on $S$ given by the sum of the geodesic curvature vectors of the two lines of curvature through each point. Let $\theta_K$ be a 1-form on $S$ defined by $\theta_K(w) := g(V_K, w)$ for each tangent vector $w$ at each point of $S$. Let $S$ be oriented and * Hodge’s *-operator. Then for local coordinates $(u, v)$ which satisfy $\partial/\partial u \in \mathcal{D}_1, \partial/\partial v \in \mathcal{D}_2$ and give the positive orientation of $S$, $\omega := -\ast \theta_K$ is represented as $\omega = -J du + I dv$, where $I := B_u/A, J := A_v/B$. Suppose $L_0 = 0$. Then we see that $d\omega = (I_u + J_v) du \wedge dv$ does not vanish and that $\alpha', \beta', \delta'$ are represented as

$$\alpha' = (\log (I_u + J_v)^2)_u, \quad \beta' = \frac{2I}{I_u + J_v}, \quad \delta' = -2(I_u + J_v)J. \quad (1.7)$$

For each solution of an over-determined system $F'_u = \alpha' + \beta' e^{F'}, F'_v = \delta' e^{-F'}$ with (1.7), a triplet $(\omega, \mathcal{D}_1, \mathcal{D}_2)$ gives a family of surfaces in $E^3 ([3])$.

Let $M$ be a two-dimensional manifold. Let $\mathcal{D}_1, \mathcal{D}_2$ be two one-dimensional distributions on $M$ satisfying $\mathcal{D}_1 \neq \mathcal{D}_2$ at any point of $M$. Let $\omega$ be a 1-form on $M$ and $\Omega$ a 2-form on $M$ satisfying $\Omega \neq 0$ at any point of $M$. Let $(u, v)$ be local coordinates on a neighborhood $U$ of each point of $M$. Suppose that $(u, v)$ satisfy $\partial/\partial u \in \mathcal{D}_1, \partial/\partial v \in \mathcal{D}_2$. We represent $\omega$ and $\Omega$ as $\omega = -J du + I dv, \Omega = \rho du \wedge dv$ on $U$, respectively. The second generalization is an over-determined system in the form of

$$F_u = (\log \rho^2)_u + \frac{2I}{\rho} e^{F}, \quad F_v = -2J \rho e^{-F}. \quad (1.8)$$

If $\Omega = d\omega$, then (1.8) is represented as $F_u = \alpha' + \beta' e^{F}, \ F'_v = \delta' e^{-F}$ with (1.7). We can reduce (1.1) with $\alpha, \beta, \gamma, \delta$ which are not necessarily represented as in (1.2) to (1.8).

The set of solutions of (1.8) depends on the choice of local coordinates $(u, v)$ satisfying $\partial/\partial u \in \mathcal{D}_1, \partial/\partial v \in \mathcal{D}_2$, while whether (1.8) has a solution does not depend on the choice of $(u, v)$ and is determined by $\omega, \Omega, \mathcal{D}_1$ and $\mathcal{D}_2$. Referring to [3] and [4], we will obtain a condition of the existence of a solution of (1.8). We set $\alpha'' := (\log \rho^2)_u, \beta'' := 2I/\rho$ and $\delta'' := -2J \rho$. Then (1.8) is represented as $F_u = \alpha'' + \beta'' e^{F}, \ F'_v = \delta'' e^{-F}$. We set

$$X'' := \beta''_v, \quad Y'' := \alpha''_v + 2\beta'' \delta'', \quad Z'' := -\delta''_u + \alpha'' \delta''.$$

Then the compatibility condition of (1.8) is given by $X'' \equiv 0, \ Y'' \equiv 0$ and $Z'' \equiv 0$. Whether (1.8) satisfies the compatibility condition does not depend on the choice of
(u, v) and is determined by ω, Ω, D_1 and D_2. Referring to [5], we will obtain conditions of (ω, Ω, D_1, D_2) satisfying the compatibility condition of (1.8) and in particular, we will see that Liouville’s equation obtained from the compatibility condition of (1.8) does not cause any contradiction. In addition, we will find two kinds of surfaces in N with L_0 ≠ 0 on which over-determined systems satisfy the compatibility condition and in particular, we will see that in the case where L_0 ≠ 0, the compatibility condition of (1.1) with (1.2) does not necessarily imply the existence of a family of geodesics of curvature. In the case where (1.8) does not satisfy the compatibility condition, the number of the solutions of (1.8) and whether (Y'')^2 - 4X''Z'' is positive, zero or negative do not depend on the choice of (u, v) and are determined by ω, Ω, D_1 and D_2. Referring to [4], we will obtain a condition of the existence of just two solutions of (1.8), and based on this condition, we will characterize a surface in N with L_0 ≠ 0 on which an over-determined system has just two solutions. Referring to [3], we will obtain a condition of the existence of a unique solution of (1.8) with (Y'')^2 = 4X''Z'', and based on this condition, we will characterize a surface in N with L_0 ≠ 0 on which an over-determined system has a unique solution so that Y^2 = 4XZ holds. In the case where (1.8) with Ω ≠ dω has a solution, we will see that (ω, Ω, D_1, D_2) gives a family of surfaces in N if and only if an over-determined system for (ω, Ω, D_1, D_2) and L_0 has a solution.

2 The first generalization

2.1 Representations of over-determined systems of polynomial type

Let M be an m-dimensional manifold with m ≥ 2. Suppose n ≥ 1 in (1.5). We will prove

Proposition 2.1 Let p be a positive integer satisfying p ≤ n + 1. Let f_1, f_2, ..., f_p be solutions of (1.5) such that arbitrarily distinct two of them give distinct values at any point of M. If p = 1, then (1.5) is represented as follows:

\[ df = df_1 + (f - f_1) \sum_{k=0}^{n-1} f^k \theta'_k; \]

if 2 ≤ p ≤ n, then (1.5) is represented as follows:

\[ df = \sum_{i=1}^{p} \left( \prod_{j \in I(p;i)} \frac{f - f_j}{f_i - f_j} \right) df_i + \left( \prod_{i=1}^{p} (f - f_i) \right) \sum_{l=0}^{n-p} f^l \theta''_l, \]

where \( I(p;i) := \{ j \in N \ | \ j \leq p, j \neq i \} \); if p = n + 1, then (1.5) is represented as follows:

\[ df = \sum_{i=1}^{n+1} \left( \prod_{j \in I(n+1;i)} \frac{f - f_j}{f_i - f_j} \right) df_i, \]

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where \( \theta'_k, \theta''_k \) are 1-forms on \( M \).

**Proof** Let \( f_1 \) be a solution of (1.5). Then we obtain \( \theta_0 = df_1 - \sum_{i=1}^{n} f_i \theta_i \). Therefore we can represent (1.5) as

\[
df = df_1 + (f - f_1) \sum_{i=1}^{n} \left( \sum_{j=0}^{i-1} f_j f_1^{i-j-1} \right) \theta_i,
\]

and we obtain Proposition 2.1 for \( p := 1 \). Let \( f_2 \) be another pointwise distinct solution of (1.5) than \( f_1 \). Then by Proposition 2.1 for \( p := 1 \), we obtain \( \theta'_0 = d \log |f_2 - f_1| - \sum_{k=1}^{n-1} f_2^k \theta'_k \). Therefore we can represent (1.5) as

\[
df = \frac{f - f_2}{f_1 - f_2} df_1 + \frac{f - f_1}{f_2 - f_1} df_2 + (f - f_1)(f - f_2) \sum_{k=1}^{n-1} \left( \sum_{l=0}^{k-1} f_l f_2^{k-l-1} \right) \theta'_k,
\]

and we obtain Proposition 2.1 for \( p := 2 \). Suppose \( 2 \leq q \leq n - 1 \). Suppose Proposition 2.1 for \( p := q \) and let \( f_1, \ldots, f_q \) be as in Proposition 2.1. Let \( f_{q+1} \) be another pointwise distinct solution of (1.5) than any of \( f_1, \ldots, f_q \). Then we obtain

\[
\theta''_0 = \sum_{i=1}^{q+1} \left( \prod_{j \in I(q+1; i)} \frac{1}{f_i - f_j} \right) df_i - \sum_{k=1}^{q} f_{q+1}^k \theta''_k.
\]

Therefore we can represent (1.5) as

\[
df = \sum_{i=1}^{q} \left( \prod_{j \in I(q; i)} \frac{f - f_j}{f_i - f_j} \right) df_i + \left( \prod_{i=1}^{q} (f - f_i) \right) \sum_{i=1}^{q+1} \left( \prod_{j \in I(q+1; i)} \frac{1}{f_i - f_j} \right) df_i
\]

\[
+ \left( \prod_{i=1}^{q} (f - f_i) \right) \sum_{k=1}^{q} (f^k - f_{q+1}^k) \theta''_k
\]

\[
= \sum_{i=1}^{q} \left( \prod_{j \in I(q; i)} \frac{f - f_j}{f_i - f_j} + \prod_{j \in I(q+1; i)} \frac{f_j}{f_i} \right) df_i
\]

\[
+ \prod_{j=1}^{q} \frac{f - f_j}{f_{q+1} - f_j} df_{q+1} + \left( \prod_{i=1}^{q+1} (f - f_i) \right) \sum_{k=1}^{q} \sum_{l=0}^{n-q} f_l f_{q+1}^{k-l-1} \theta''_k.
\]

Noticing that the first term of the right side of (2.1) is represented as

\[
\sum_{i=1}^{q} \left( \prod_{j \in I(q+1; i)} \frac{f - f_j}{f_i - f_j} \right) df_i,
\]

we obtain Proposition 2.1 for \( p := q + 1 \). Similarly, supposing Proposition 2.1 for \( p := n \), we can obtain Proposition 2.1 for \( p := n + 1 \). \( \square \)
2.2 The compatibility condition

Suppose $n = 1$ and let $f_1$ be a solution of (1.5) with $n = 1$. Then we obtain $P(t) = (t - f_1)d\theta_1$. Therefore the compatibility condition of (1.5) with $n = 1$ is given by $d\theta_1 \equiv 0$. If there exists another pointwise distinct solution $f_2$ of (1.5) than $f_1$, then we obtain $\theta_1 = d\log|f_1 - f_2|$ and therefore we see that (1.5) satisfies the compatibility condition. Referring to the above discussion, we will prove

**Theorem 2.2** Suppose $n = 2$. Let $f_1, f_2$ be solutions of (1.5) which give distinct values at any point of $M$. Then $P$ is represented as

$$P(t) = \frac{(t - f_1)(t - f_2)}{f_1 - f_2} d((f_1 - f_2)\theta_2)$$

for any $t \in \mathbb{R}$, and in particular, the compatibility condition of (1.5) is given by $d((f_1 - f_2)\theta_2) \equiv 0$. If there exists another pointwise distinct solution $f_3$ of (1.5) than $f_1, f_2$, then

$$\theta_2 = \frac{1}{f_1 - f_2} d\log \left| \frac{f_1 - f_3}{f_2 - f_3} \right| = \sum_{i=1}^{3} \frac{d\theta_i}{\prod_{j \in I(3,i)} (f_i - f_j)},$$

and therefore (1.5) satisfies the compatibility condition.

**Remark** We set $m := 2$, $n := 2$ and let $\theta_0, \theta_1, \theta_2$ be as in (1.6). Then

$$\Theta_0 = (\delta_u - \alpha \delta)du \wedge dv,$$
$$\Theta_1 = (-\alpha_v + \gamma_u - 2\beta \delta)du \wedge dv,$$
$$\Theta_2 = (-\beta_v - \beta \gamma)du \wedge dv.$$

Therefore the compatibility condition of (1.5) in this situation is given by

$$-\delta_u + \alpha \delta = 0, \quad \alpha_v - \gamma_u + 2\beta \delta = 0, \quad \beta_v + \beta \gamma = 0,$$

which was originally stated in [5]. Suppose that $\gamma \equiv 0$ and that $f_1, f_2$ are as in Theorem 2.2. Then we obtain $(f_1 - f_2)_v \equiv 0$ and therefore we see that the compatibility condition of (1.5) is given by $\beta_v \equiv 0$, which was originally obtained in [4].

**Proof of Theorem 2.2** By Proposition 2.1 for $n := 2$ and $p := 2$, we can represent (1.5) as

$$df = \frac{f - f_2}{f_1 - f_2} df_1 + \frac{f - f_1}{f_2 - f_1} df_2 + (f - f_1)(f - f_2)\theta_2.$$  \hspace{1cm} (2.4)

Therefore we obtain

$$\theta_0 = -\frac{f_2}{f_1 - f_2} df_1 - \frac{f_1}{f_2 - f_1} df_2 + f_1 f_2 \theta_2;$$
$$\theta_1 = \frac{1}{f_1 - f_2} df_1 + \frac{1}{f_2 - f_1} df_2 - (f_1 + f_2)\theta_2.$$  \hspace{1cm} (2.5)
Therefore we obtain (2.2). If there exists another pointwise distinct solution \( f \) applying (2.5) to (2.6), we obtain
\[
\Theta_0 = f_1 f_2 \left( \frac{1}{f_1 - f_2} (df_1 - df_2) \wedge \theta_2 + d\theta_2 \right),
\]
\[
\Theta_1 = - (f_1 + f_2) \left( \frac{1}{f_1 - f_2} (df_1 - df_2) \wedge \theta_2 + d\theta_2 \right),
\]
\[
\Theta_2 = \frac{1}{f_1 - f_2} (df_1 - df_2) \wedge \theta_2 + d\theta_2.
\]

Therefore we obtain (2.2). If there exists another pointwise distinct solution \( f_3 \) of (1.5) than \( f_1, f_2 \), then applying \( f := f_3 \) to (2.4), we obtain (2.3). \( \square \)

Suppose \( n \geq 3 \). Let \( f_1, f_2, \ldots, f_{n+1} \) be pointwise distinct solutions of (1.5). In this case, since \( c(n) \geq n + 1 \), (1.5) does not necessarily satisfy the compatibility condition. For \( i, j \in \{1, 2, \ldots, n+1\} \) with \( i \neq j \) and \( t \in \mathbb{R} \), we set
\[
A_i(t) := \prod_{k \in I(n+1; i)} (t - f_k), \quad A_{i,j}(t) := \prod_{k \in I(n+1; i, j)} (t - f_k),
\]
where
\[
I(n+1; i, j) := \{ k \in \mathbb{N} \mid k \leq n+1, \ k \neq i, j \}.
\]

We will prove

**Theorem 2.3** Let \( n, f_1, f_2, \ldots, f_{n+1}, A_i, A_{i,j} \) be as above. Then for any \( t \in \mathbb{R} \),
\[
P(t) = - \sum_{1 \leq i < j \leq n+1} \left( \frac{A_{i,j}(t)^2}{A_i(f_i) A_j(f_j)} + \frac{A_j(t)}{A_i(f_i)} + \frac{A_i(t)}{A_j(f_j)} \right) \frac{df_i \wedge df_j}{f_i - f_j}
\]
\[
= - \sum_{1 \leq i < j \leq n+1} \frac{A_{i,j}(t) B_{i,j}(t)}{A_i(f_i) A_{i,j}(f_j) (f_i - f_j)^2}
\]
where
\[
B_{i,j}(t) := (f_i - f_j) A_{i,j}(t) + (t - f_i) A_{i,j}(f_j) - (t - f_j) A_{i,j}(f_i).
\]

**Remark** We see that \( P(f_i) \equiv 0 \) for any \( i \in \{1, 2, \ldots, n+1\} \).

**Remark** In the case where \( n = 3 \), we see from \( c(3) = 3 + 1 \) that if (1.5) has another pointwise distinct solution than \( f_1, f_2, f_3, f_4 \), then (1.5) satisfies the compatibility condition. In the case where \( n \geq 4 \), we see from \( c(n) > n + 1 \) that although (1.5) does not satisfy the compatibility condition, it is possible that (1.5) has another pointwise distinct solution than \( f_1, f_2, \ldots, f_{n+1} \).
Proof of Theorem 2.3 Let $f$ be any function on $M$. Then the following hold:

$$ d \left( \sum_{i=1}^{n+1} \left( \prod_{j \in I(n+1:i)} \frac{f - f_j}{f_i - f_j} \right) df_i \right) $$

$$ = \sum_{i=1}^{n+1} \sum_{j \in I(n+1:i)} \left( \prod_{k \in I(n+1;i,j)} f - f_k \right) df_i \wedge df_i $$

$$ = \sum_{i=1}^{n+1} \sum_{j \in I(n+1;i)} \left( \prod_{k \in I(n+1;i,j)} \frac{f - f_k}{f_i - f_k} \right) \left( \frac{1}{f_i - f_j} df_i \wedge df_i - \frac{f - f_i}{(f_i - f_j)^2} df_i \wedge df_j \right). $$

Exchanging $f$ and $df$ for $t$ and $\sum_{l=1}^{n+1} (A_l(t)/A_l(f_l))df_l$, respectively in the right side of (2.8), we obtain a representation of $P(t)$:

$$ P(t) = \sum_{i=1}^{n+1} \sum_{j \in I(n+1;i)} \frac{A_{i,j}(t)}{A_{i,j}(f_i)} \left( \frac{1}{f_i - f_j} \sum_{l=1}^{n+1} \frac{A_l(t)}{A_l(f_i)} df_i \wedge df_i - \frac{t - f_i}{(f_i - f_j)^2} df_i \wedge df_j \right). $$

We obtain

$$ \sum_{i=1}^{n+1} \sum_{j \in I(n+1;i)} \frac{A_{i,j}(t)}{A_{i,j}(f_i)} \frac{t - f_i}{(f_i - f_j)^2} df_i \wedge df_j $$

$$ = \sum_{1 \leq i < j \leq n+1} \left( \frac{A_j(t)}{A_i(f_i)} + \frac{A_i(t)}{A_i(f_i)} \right) df_i \wedge df_j. $$

If $i$, $j$, $l$ are mutually distinct, then the following holds:

$$ \frac{1}{f_i - f_j} \frac{A_{i,j}(t)}{A_{i,j}(f_i)} \frac{A_i(t)}{A_i(f_i)} df_i \wedge df_i = \frac{A_{i,l}(t)A_j(t)}{A_{i,l}(f_i)A_{i,l}(f_i)(f_i - f_l)^2} df_i \wedge df_i. $$

Therefore we obtain

$$ \sum_{i=1}^{n+1} \sum_{j \in I(n+1;i)} \frac{1}{f_i - f_j} \frac{A_{i,j}(t)}{A_{i,j}(f_i)} \sum_{l=1}^{n+1} \frac{A_l(t)}{A_l(f_i)} df_i \wedge df_i $$

$$ = \sum_{i=1}^{n+1} \sum_{j \in I(n+1;i)} \frac{A_{i,j}(t)^2(t - f_i)}{A_{i,j}(f_i)A_{i,j}(f_j)(f_i - f_j)} df_i \wedge df_j $$

$$ + \sum_{i=1}^{n+1} \sum_{j \in I(n+1;i)} \sum_{l \in I(n+1;i,j)} \frac{A_{i,l}(t)A_j(t)}{A_{i,l}(f_i)A_{i,l}(f_i)(f_i - f_l)^2} df_i \wedge df_i. $$

The second term of the right side of (2.11) is zero. Therefore applying (2.10) and (2.11) to (2.9), we obtain Theorem 2.3. □
2.3 Further studies of Theorem 2.3 for \( n = 3 \)

Suppose \( n = 3 \) and let \( f_1, f_2, f_3, f_4 \) be as in Theorem 2.3. Then (2.7) is represented as

\[
P(t) = \prod_{i=1}^{4} (t - f_i) \sum_{1 \leq i < j \leq 4} (-1)^{i+j} (f_a - f_b) df_i \wedge df_j,
\]

where for \( i, j \in \mathbb{N} \) satisfying \( 1 \leq i < j \leq 4 \), \( a, b \) are integers satisfying \( a < b \) and \( \{i, j, a, b\} = \{1, 2, 3, 4\} \). Therefore we see that (1.5) satisfies the compatibility condition if and only if \( f_1, f_2, f_3, f_4 \) satisfy

\[
\sum_{1 \leq i < j \leq 4} (-1)^{i+j} (f_a - f_b) df_i \wedge df_j = 0. \tag{2.12}
\]

Thus whether (1.5) satisfies the compatibility condition is determined by (2.12) for \( f_1, f_2, f_3, f_4 \). We see from (2.12) that if \( df_1, df_2, df_3, df_4 \) are linearly independent at any point, then (1.5) does not satisfy the compatibility condition. We will prove

**Theorem 2.4** Let \( f_1, f_2, f_3 \) be pointwise distinct functions on a neighborhood \( U \) of a point \( a_0 \) of \( M \) satisfying \( \eta \neq 0 \) at any point, where

\[
\eta := (f_1 - f_2) df_3 + (f_2 - f_3) df_1 + (f_3 - f_1) df_2. \tag{2.13}
\]

Let \( C \) be a curve in \( U \) through \( a_0 \) satisfying \( \eta(w) \neq 0 \) for any nonzero tangent vector \( w \) at any point of \( C \). Let \( \psi \) be a function on \( C \) satisfying \( \psi(a_0) \neq f_i(a_0) \) for \( i = 1, 2, 3 \). Then there exists a unique function \( f_4 \) on a neighborhood \( U_0 \) of \( a_0 \) in \( M \) satisfying \( f_4 \equiv \psi \) on \( C \) and (2.12) on \( U_0 \).

For smooth functions \( f_1, f_2, f_3 \), we set

\[
\Omega := df_2 \wedge df_3 + df_3 \wedge df_1 + df_1 \wedge df_2,
\]

\[
\Omega' := f_1 df_2 \wedge df_3 + f_2 df_3 \wedge df_1 + f_3 df_1 \wedge df_2.
\]

Then \( f_4 \) satisfies (2.12) if and only if \( f := f_4 \) satisfies

\[
\eta \wedge df = f \Omega - \Omega', \tag{2.14}
\]

where \( \eta \) is as in (2.13). Let \( (x^1, x^2, \ldots, x^m) \) be local coordinates on a neighborhood of \( a_0 \) and we represent \( \eta, \Omega, \Omega' \) as

\[
\eta := \sum_{i=1}^{m} \eta_i dx^i, \quad \Omega := \sum_{1 \leq i < j \leq m} \Omega_{ij} dx^i \wedge dx^j, \quad \Omega' := \sum_{1 \leq i < j \leq m} \Omega'_{ij} dx^i \wedge dx^j,
\]

respectively. Then (2.14) is equivalent to

\[
V_{ij}(f) = f \Omega_{ij} - \Omega'_{ij}. \tag{2.15}
\]

10
for $1 \leq i < j \leq m$, where $V_{ij}$ is a vector field given by $V_{ij} := \eta_i \partial/\partial x^j - \eta_j \partial/\partial x^i$. We see that $V_{ij}$ satisfies $\eta(V_{ij}) = 0$.

**Proof of Theorem 2.4** Suppose $m = 2$. Then (2.15) consists of a single equation $V_{12}(f) = f\Omega_{12} - \Omega'_{12}$. Since we suppose $\eta \neq 0$, $V_{12}$ is nowhere zero. Let $C, \psi$ be as in Theorem 2.4. Then on an integral curve of $V_{12}$ through a point $a$ of $C$, there exists a unique solution $f_a$ of $V_{12}(f) = f\Omega_{12} - \Omega'_{12}$ with the initial value $\psi(a)$. Thus on a neighborhood $U_0$ of $a_0$ in $M$, there exists a unique solution $f$ of $V_{12}(f) = f\Omega_{12} - \Omega'_{12}$ satisfying $f \equiv \psi$ on $C$. This means Theorem 2.4 for $m = 2$. Suppose $m \geq 3$. Then (2.15) consists of plural equations. Since $\eta \neq 0$, we can suppose $\eta_1 \neq 0$. Then $V_{1i}$ is nowhere zero for $2 \leq i \leq m$. Noticing $V_{ij} = (\eta_i/\eta_1)V_{1j} - (\eta_j/\eta_1)V_{1i}$ and

$$\eta_1 \Omega_{ij} + \eta_j \Omega_{i1} + \eta_i \Omega_{1j} = 0, \quad \eta_1 \Omega'_{ij} + \eta_j \Omega'_{i1} + \eta_i \Omega'_{1j} = 0 \quad (2.16)$$

for $2 \leq i < j \leq m$, where $\Omega_{ij} := -\Omega_{1j}$, $\Omega'_{ij} := -\Omega'_{1j}$, we see that if $f$ satisfies $V_{ii}(f) = f\Omega_{ii} - \Omega'_{ii}$ for $2 \leq i \leq m$, then $f$ satisfies $V_{ij}(f) = f\Omega_{ij} - \Omega'_{ij}$ for $2 \leq i < j \leq m$. Therefore (2.14) is equivalent to a system of equations in (2.15) satisfying $i = 1$ and $2 \leq j \leq m$. We can choose local coordinates $(x^1, x^2, \ldots, x^m)$ satisfying $\eta_i \equiv 0$ for $2 \leq i \leq m$: if we set

$$x^1 := \log \left| \frac{f_1 - f_3}{f_2 - f_3} \right|, \quad \eta_1 := (f_1 - f_3)(f_2 - f_3),$$

then $\eta = \eta_1 dx^1$. Then an $(m - 1)$-dimensional distribution $\mathcal{D}$ given by $\eta = 0$ is involutive and locally generated by $\partial/\partial x^2, \ldots, \partial/\partial x^m$. In addition, the system of equations in (2.15) satisfying $i = 1$ and $2 \leq j \leq m$ is represented as the following over-determined system on each integral hypersurface of $\mathcal{D}$:

$$\eta_1 \frac{\partial f}{\partial x^i} = f\Omega_{1i} - \Omega'_{1i} \quad (2.17)$$

$(2 \leq i \leq m)$. We can represent (2.17) in the form of (1.5) with $n = 1$. We can suppose $\eta_1 > 0$. The compatibility condition of (2.17) is given by

$$\frac{\partial \log \eta_1}{\partial x^i} \Omega_{1j} - \frac{\partial \log \eta_1}{\partial x^j} \Omega_{1i} + \frac{\partial \Omega_{1i}}{\partial x^j} - \frac{\partial \Omega_{1j}}{\partial x^i} = 0, \quad \frac{\partial \log \eta_1}{\partial x^i} \Omega'_{1j} - \frac{\partial \log \eta_1}{\partial x^j} \Omega'_{1i} + \frac{\partial \Omega'_{1i}}{\partial x^j} - \frac{\partial \Omega'_{1j}}{\partial x^i} = 0. \quad (2.18)$$

Since $\Omega = (1/2)dx^1 \wedge d\eta_1$, we obtain $\Omega_{1i} = (1/2)(\partial \eta_1/\partial x^i)$. Therefore the first equation in (2.18) always holds and the second equation in (2.18) is rewritten into

$$3\frac{\partial \log \eta_1}{\partial x^i} \Omega'_{1j} - 3\frac{\partial \log \eta_1}{\partial x^j} \Omega'_{1i} = 2\frac{\partial \Omega'_{1i}}{\partial x^j} - 2\frac{\partial \Omega'_{1j}}{\partial x^i}. \quad (2.19)$$

We see by the second relation in (2.16) that $\Omega'_{ij} = 0$ for $2 \leq i < j \leq m$. Therefore (2.19) is equivalent to $3d(\log \eta_1) \wedge \Omega' = 2d\Omega'$, and using $\eta_1 = (f_1 - f_3)(f_2 - f_3)$, we see that
we obtain

\[ 3d(\log \eta_1) \wedge \Omega' = 2d\Omega' \] always holds. Therefore the second equation in (2.18) always holds. Hence an over-determined system (2.17) on each integral hypersurface of \( \mathcal{D} \) satisfies the compatibility condition and therefore on an integral hypersurface of \( \mathcal{D} \) through \( a \in C \), there exists a unique solution \( f_a \) of (2.17) with the initial value \( \psi(a) \). Therefore on a neighborhood \( U_0 \) of \( a_0 \) in \( M \), there exists a unique \( f \) of (2.15) satisfying \( f \equiv \psi \) on \( C \). This means Theorem 2.4 for \( m \geq 3 \). \( \square \)

**Example** We set

\[ M := \{(x^1, x^2, x^3) \in \mathbb{R}^3 \mid x^1 > x^2 > x^3 > 0\} \]

and \( f_1 := x^1, f_2 := x^2, f_3 := x^3 \). Then we obtain

\[ \eta_1 = x^2 - x^3, \quad \eta_2 = x^3 - x^1, \quad \eta_3 = x^1 - x^2. \]

Therefore we see \( \eta \neq 0 \) on \( M \). Since

\[
\begin{align*}
V_{12} &= \eta_1 \frac{\partial}{\partial x^2} - \eta_2 \frac{\partial}{\partial x^1} = (x^1 - x^3) \frac{\partial}{\partial x^1} + (x^2 - x^3) \frac{\partial}{\partial x^2}, \\
V_{13} &= \eta_1 \frac{\partial}{\partial x^3} - \eta_3 \frac{\partial}{\partial x^1} = (x^2 - x^1) \frac{\partial}{\partial x^1} + (x^2 - x^3) \frac{\partial}{\partial x^3},
\end{align*}
\]

we obtain

\[
\begin{align*}
\frac{1}{x^2 - x^3} (V_{12} + V_{13}) &= \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}, \\
\frac{1}{x^2 - x^3} (x^2 V_{12} + x^3 V_{13}) &= x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3}. 
\end{align*}
\]

(2.20)

Let \( V, W \) be the first and the second vector fields in (2.20), respectively: \( V := \sum_{i=1}^{3} \partial / \partial x^i, \)

\( W := \sum_{i=1}^{3} x^i \partial / \partial x^i \). Then \( V, W \) form a basis of a two-dimensional subspace of the tangent space of \( M \) at each point defined by \( \eta = 0 \). By \( V_{23} = (\eta_2 / \eta_1)V_{13} - (\eta_3 / \eta_1)V_{12} \) and

\[
\begin{align*}
\Omega_{12} &= 1, \quad \Omega_{13} = -1, \quad \Omega_{23} = 1, \quad \Omega'_{12} = x^3, \quad \Omega'_{13} = -x^2, \quad \Omega'_{23} = x^1,
\end{align*}
\]

we see that a function \( f \) on \( M \) satisfies (2.15) if and only if \( V(f) = 1 \) and \( W(f) = f \) hold. Since the bracket of \( V \) and \( W \) is given by \([V, W] = V, \) a two-dimensional distribution \( \mathcal{D} \) on \( M \) given by \( V, W \) is involutive. Each integral surface of \( \mathcal{D} \) is part of a plane in \( \mathbb{R}^3 \).

Let \( C \) be a curve in \( M \) satisfying \( \eta(w) \neq 0 \) for any nonzero tangent vector \( w \) at any point of \( C \) (for example, we can choose an integral curve of a vector field

\[
(x^2 - x^3) \frac{\partial}{\partial x^1} + (x^3 - x^1) \frac{\partial}{\partial x^2} + (x^1 - x^2) \frac{\partial}{\partial x^3}
\]

as \( C \)). Then on an integral surface of \( \mathcal{D} \) through a point \( a \) of \( C \), there exists a function \( f_a \) satisfying \( V(f_a) = 1 \) and \( W(f_a) = f_a \), and \( f_a \) is uniquely determined by an arbitrarily given initial value at \( a \). Therefore we see that for a function \( \psi \) on \( C \), there exists a unique solution \( \hat{f} \) of the system \( V(f) = 1, W(f) = f \) on a neighborhood of each point of \( C \) in \( M \) satisfying \( f \equiv \psi \) on \( C \).
Remark Suppose \( n = 4 \) and let \( f_1, f_2, f_3, f_4, f_5 \) be as in Theorem 2.3. For \( i, j \in \mathbb{N} \) satisfying \( 1 \leq i < j \leq 5 \), let \( a, b, c \) be integers satisfying \( \{i, j, a, b, c\} = \{1, 2, 3, 4, 5\} \). Then (2.7) is represented as

\[
P(t) = -\left( \prod_{i=1}^{5} (t - f_i) \right) \sum_{1 \leq i < j \leq 5} \frac{t + f_i + f_j - f_i - f_j - f_a - f_b - f_c}{A_{i,j}(f_i)A_{i,j}(f_j)} df_i \wedge df_j.
\]

(2.21)

Thus whether (1.5) satisfies the compatibility condition is determined by \( f_1, f_2, f_3, f_4, f_5 \). Since \( c(4) = 6 \), although (1.5) does not satisfy the compatibility condition, it is possible that (1.5) has the sixth solution \( f_6 \) so that arbitrarily distinct two of \( f_1, \ldots, f_6 \) give distinct values at any point of \( M \). From (2.21), we see that \( f_6 \) satisfies

\[
\sum_{1 \leq i < j \leq 5} \frac{f_6 + f_i + f_j - f_a - f_b - f_c}{A_{i,j}(f_i)A_{i,j}(f_j)} df_i \wedge df_j = 0.
\]

This means that in the case where (1.5) does not satisfy the compatibility condition and has the sixth solution \( f_6 \), the value of \( f_6 \) at each point is determined by \( f_1, f_2, f_3, f_4, f_5 \).

3 The second generalization

3.1 The existence of a solution

Referring to [3] and [4], we will prove

**Theorem 3.1** An over-determined system (1.8) on a neighborhood \( U \) of \( p \in M \) has a solution if and only if there exist local coordinates \( (u, v) \) on \( U \) satisfying \( \partial/\partial u \in \mathcal{D}_1 \), \( \partial/\partial v \in \mathcal{D}_2 \) and

\[
\rho^2 = \left( -2 \int_{u_0}^{u} \rho I du + 1 \right) \left( -2 \int_{v_0}^{v} \rho J dv + 1 \right),
\]

where \( (u_0, v_0) \) corresponds to a point \( p \).

Remark In [4], studying (1.1) with (1.2), the author obtained a prototype of Theorem 3.1. See [3] for Theorem 3.1 in the case where \( \Omega = d\omega \).

**Proof of Theorem 3.1** There exists a solution of (1.1) if and only if we can find functions \( a, b, c, d \) of two variables \( u, v \) satisfying

\[
a_u = \alpha, \quad e^{-b}b_u = \beta e^a, \quad c_v = -\gamma, \quad e^{-d}d_v = -\delta e^c
\]

(3.2)

and \( a + b + c + d = 0 \) ([4]). Suppose that (1.8) has a solution. We set \( a := \log \rho^2 \), \( c := 0 \). Then there exist functions \( b, d \) satisfying \( (e^{-b})_u = -2\rho I \), \( (e^{-d})_v = -2\rho J \) and
Therefore we see that there exist positive-valued functions \( \phi, \psi \) of one variable satisfying

\[
\rho^2 = \left( -2 \int_{u_0}^{u} \rho Id\mu + \psi(v) \right) \left( -2 \int_{v_0}^{v} \rho Jdv + \phi(u) \right). 
\]  

(3.3)

Let \( \tilde{u} \) be a function of one variable \( u \) satisfying \( d\tilde{u}/du = \sqrt{\phi} \) and \( \tilde{v} \) a function of one variable \( v \) satisfying \( d\tilde{v}/dv = \sqrt{\psi} \). Then \((\tilde{u}, \tilde{v})\) are local coordinates satisfying \( \partial/\partial \tilde{u} \in D_1 \) and \( \partial/\partial \tilde{v} \in D_2 \), and we see that (3.3) can be rewritten into (3.1), where we set \( u := \tilde{u}, \ v := \tilde{v} \) and \( \rho := \tilde{\rho}, \ I := \tilde{I}, \ J := \tilde{J} \). Then (3.3) can be rewritten into (3.1), where we set \( u := \tilde{u}, \ v := \tilde{v} \) and \( \rho := \tilde{\rho}, \ I := \tilde{I}, \ J := \tilde{J} \). Therefore (1.8) has a solution. Hence we obtain Theorem 3.1. □

### 3.2 The compatibility condition

Referring to [5], we will prove

**Theorem 3.2** An over-determined system (1.8) on a neighborhood \( U \) of a point of \( M \) satisfies the compatibility condition if and only if there exist local coordinates \( (u, v) \) on \( U \) with \( \partial/\partial u \in D_1, \ \partial/\partial v \in D_2 \) such that \( \omega \) and \( \Omega \) are represented as in the following (a) or (b):

(a) \( \omega = f(u)dv \) or \( g(v)du \), and \( \Omega = du \wedge dv \), where \( f, g \) are functions of one variable;

(b) \( \omega = \rho(-du + dv) \) and \( \Omega = \rho du \wedge dv \), where \( \rho \) is a nowhere zero function such that \( \psi := \log 8\rho^2 \) is a solution of Liouville’s equation \( \psi_{uv} = e^\psi \).

**Remark** In the case where \( \Omega = d\omega \), referring to [5], we can show that the compatibility condition of (1.8) is equivalent to condition (a) in Theorem 3.2.

**Proof of Theorem 3.2** Suppose that (1.8) satisfies the compatibility condition. Then from \( X'' = 0, \ Y'' = 0 \) and \( Z'' = 0 \), we obtain

\[
\left( \frac{I}{\rho} \right)_v = 0, \quad (\log |\rho|)_{uv} = 4IJ, \quad \left( \frac{J}{\rho} \right)_u = 0. 
\]  

(3.4)
If $IJ \equiv 0$, then from the second relation in (3.4), we obtain $\rho = s(u)t(v)$, where $s$ and $t$ are nowhere zero functions of one variable. Therefore in this case, we can suppose that local coordinates $(u, v)$ satisfy $\Omega = du \wedge dv$, and this implies $\rho \equiv 1$. If $I \equiv 0$, then from the third relation in (3.4), we see that $J$ is of one variable $v$ and then $\omega$ is represented as $\omega = g(v)dv$; if $J \equiv 0$, then from the first relation in (3.4), we see that $I$ is of one variable $u$ and then $\omega$ is represented as $\omega = f(u)dv$. In the following, we suppose $IJ \neq 0$. Noticing the first and the third relations in (3.4), we see that there exist nowhere zero functions $p, q$ of one variable satisfying $I = \rho p(u), J = \rho q(v)$. Therefore we can suppose that local coordinates $(u, v)$ satisfy $\partial/\partial u \in D_1, \partial/\partial v \in D_2, A^2 = f_u, B^2 = f_v$, where $f$ is a function of two variables given by
\begin{equation}
 f(u, v) := p(u) + q(v) + \frac{1}{8L_0} \left( \frac{\partial}{\partial u} + \frac{\partial}{\partial v} - 2 \right) \log(s(u) + t(v))^2,
\end{equation}
and $p, s$ are functions of one variable $u$ and $q, t$ are functions of one variable $v$ satisfying $s(u) + t(v) \neq 0$ and
\begin{equation}
 2f^2_{uv} + (\log(s(u) + t(v))^2)_{uv}f_u f_v = 0.
\end{equation}

Remark There exist functions $p, q, s, t$ as in (b) of Corollary 3.3: if we set $p := 0, q := 0, s(u) := e^{\epsilon u}, t(v) := e^{\epsilon v}$ for $\epsilon = +1$ or $-1$, then $p, q, s, t$ satisfy $s(u) + t(v) \neq 0$ and (3.7).
Therefore we see from Corollary 3.3 that if \( L_0 \neq 0 \), then the compatibility condition of (1.1) with (1.2) does not necessarily imply the existence of a family of geodesics of curvature.

Remark The surface \( \hat{S} \) in \( N \) for each solution of (1.1) with (1.2) and (a) of Corollary 3.3 is an analogue of a surface of revolution in \( E^3 \): if any integral curve of \( \mathcal{D}_2 \) is a geodesic and if any integral curve of \( \mathcal{D}_1 \) has constant geodesic curvature, then for any integral curve of \( \mathcal{D}_1 \), the curvature is constant and the torsion vanishes in \( N \) (refer to the proof of Corollary 3.3 below); the torsion of any integral curve of \( \mathcal{D}_2 \) vanishes in \( N \); for each point \( p \) of \( \hat{S} \), there exists a neighborhood \( O \) of \( p \) in \( \hat{S} \) satisfying

(i) arbitrary two integral curves \( C_2, C'_2 \) of \( \mathcal{D}_2 \) in \( O \) are congruent in \( N \) with each other,

(ii) each integral curve \( C_1 \) of \( \mathcal{D}_1 \) in \( O \) has just one point of each integral curve of \( \mathcal{D}_2 \) so that a unique point of \( C_1 \cap C_2 \) corresponds to a unique point of \( C_1 \cap C'_2 \) by the congruence of \( C_2 \) and \( C'_2 \) in \( N \), and

(iii) along \( C_1 \), the angle between the curvature vector of \( C_1 \) and the tangent line of each integral curve of \( \mathcal{D}_2 \) is constant.

Let \( M \) be a two-dimensional manifold. Let \( g \) be a Riemannian metric on \( M \) such that the curvature \( K \) of \( (M,g) \) is nowhere equal to a nonzero number \( L_0 \). Let \( \mathcal{D}_1, \mathcal{D}_2 \) be two one-dimensional distributions on \( M \) such that \( \mathcal{D}_1 \) is orthogonal to \( \mathcal{D}_2 \) with respect to \( g \) at any point of \( M \). Suppose that \( M \) is oriented. Let \( \omega \) be as in the second paragraph of Subsection 1.3 for \( (g, \mathcal{D}_1, \mathcal{D}_2) \) and set \( \Omega := (L_0 - K)d\mu \), where \( d\mu \) is the area element of \( (M,g) \). Let \( (u,v) \) be local coordinates on a neighborhood of a point of \( M \) satisfying \( \partial/\partial u \in \mathcal{D}_1, \partial/\partial v \in \mathcal{D}_2 \). Then \( g \) is locally represented as \( g = A^2 du^2 + B^2 dv^2 \), where \( A, B \) are positive-valued functions. If \( (u,v) \) give the positive orientation of \( M \), then \( d\mu \) is locally represented as \( d\mu = AB du \wedge dv \). We see that (1.1) with (1.2) satisfies the compatibility condition if and only if (a) or (b) in Theorem 3.2 holds for the above \( \omega \) and \( \Omega \).

Proof of Corollary 3.3 Suppose (a) in Theorem 3.2. Then we can suppose \( I \equiv 0 \). Then \((L_0 - K)A \) is a function of one variable \( v \) and \( A \) is represented as \( A = x(u) + y(v) \). From \( K = -A_u/v/A \) and \( L_0 \neq 0 \), we see that \( x \) is constant. Therefore we see that any integral curve of \( \mathcal{D}_2 \) is a geodesic and that any integral curve of \( \mathcal{D}_1 \) has constant geodesic curvature. If we suppose (a) in Corollary 3.3, then we can choose \( (u,v) \) satisfying (a) in Theorem 3.2 for the above \( \omega \) and \( \Omega \). Suppose (b) in Theorem 3.2 for the above \( \omega \) and \( \Omega \). Then we obtain \( \rho = (L_0 - K)AB = I = J \). From \( I = J \), we obtain \( (B^2)_u = (A^2)_v \). Therefore there exists a function \( f \) of two variables \( u, v \) satisfying \( f_u = A^2, f_v = B^2 \). Then we obtain

\[
AB = \sqrt{f_u f_v}, \quad I = J = \frac{f_{uv}}{2 \sqrt{f_u f_v}}.
\]
Let \( \psi \) be as in the proof of Theorem 3.2. Then \( e^\psi = 2f_{uv}^2/f_u f_v \). By this together with \((L_0 - K)AB = I = J\), we obtain
\[
f_{uv} = \frac{1}{8L_0} \left( 2 - \frac{\partial}{\partial u} - \frac{\partial}{\partial v} \right) e^\psi.
\] (3.8)
Since \( \psi \) is a solution of Liouville’s equation, there exist functions \( s, t \) of one variable as in (3.5). This means
\[
e^\psi = -(\log(s(u) + t(v))^2)_{uv}.
\] (3.9)
Therefore we obtain (3.6) with \( s(u) + t(v) \neq 0 \) and (3.7). Suppose that there exist local coordinates \((u, v)\) satisfying \( \partial/\partial u \in \mathcal{D}_1, \partial/\partial v \in \mathcal{D}_2, A^2 = f_u, B^2 = f_v \) for a function \( f \) as in (3.6) with \( s(u) + t(v) \neq 0 \) and (3.7). Let \( \psi \) be a function of two variables given by (3.9). Then by (3.8) together with \( e^\psi = 2f_{uv}^2/f_u f_v \), we obtain \((L_0 - K)AB = I = J\) and we see that \( \rho := I \) satisfies (b) in Theorem 3.2. Hence we have proved Corollary 3.3. □

**Remark** Let \((M, g)\) be a two-dimensional Riemannian manifold such that the curvature \( K \) is nowhere equal to a number \( L_0 \in \mathbb{R} \). Let \( \mathcal{D}_1, \mathcal{D}_2 \) be as above. Suppose that the integral curves of \( \mathcal{D}_2 \) are geodesics. Let \((u, v)\) be local coordinates satisfying \( \partial/\partial u \in \mathcal{D}_1, \partial/\partial v \in \mathcal{D}_2 \). Then we can suppose that \( g \) is locally represented as \( g = A^2 du^2 + dv^2 \).
Suppose that there exists an isometric immersion of a neighborhood of each point of \( M \) into \( N \) such that \( \mathcal{D}_1, \mathcal{D}_2 \) give principal distributions. Then a principal curvature \( k_2 \) corresponding to \( \mathcal{D}_2 \) is of one variable \( v \). From the second relation in (1.1) with (1.2), we see that a principal curvature \( k_1 \) corresponding to \( \mathcal{D}_1 \) satisfies
\[
(k_1)_v = -(\log A)_v \left( k_1 - \frac{K - L_0}{k_1} \right).
\]
By this together with \( K = -A_{vv}/A \), we obtain
\[
(k_1^2 A^2 + L_0 A^2 + A_v^2)_v = 0.
\]
Therefore there exists a function \( \phi \) of one variable \( u \) satisfying
\[
k_1^2 A^2 + L_0 A^2 + A_v^2 = \phi(u).
\]
By this together with \( k_1 k_2 = K - L_0 \), we obtain
\[
(A_{uv} + L_0 A)^2 = k_2(v)^2(\phi(u) - L_0 A^2 - A_v^2).
\] (3.10)
In the case where \( L_0 = 0 \), \( A \) is represented as \( A = A_1(u)A_2(u, v) + 1 \), where \( A_2 \) is a function of two variables \( u, v \) satisfying \((A_2)_v = \sin(\alpha_1(u) + \alpha_2(v))\), and \( A_1, \alpha_1 \) and \( \alpha_2 \) are functions of one variable satisfying \( A_1 > 0 \) and \( \alpha_1(u) + \alpha_2(v) \in (-\pi/2, \pi/2) \); \( k_1, k_2 \) are represented as \( k_1 = A_1(u) \cos(\alpha_1(u) + \alpha_2(v))/A \) and \( k_2 = -\alpha_2'(v) \), respectively (see
[2]). For a function $\phi$ of one variable $u$ and a nowhere zero function $k_2$ of one variable $v$, a function $A$ satisfying (3.10) characterizes such an immersion as above: if $\phi$, $k_2$ and $A$ satisfy (3.10) and if we set $k_1 := -(A_{vv}/A + L_0)/k_2$, then there exists an isometric immersion of a neighborhood of each point of $M$ into $N$, unique up to an isometry of $N$, satisfying

(i) $\mathcal{D}_1, \mathcal{D}_2$ give principal distributions;

(ii) $k_1, k_2$ are principal curvatures corresponding to $\mathcal{D}_1, \mathcal{D}_2$, respectively.

For $\phi, k_2$ and functions $a_0, a_1$ of one variable $u$ satisfying $\phi - L_0 a_0^2 - a_1^2 > 0$, there exists a unique solution $A$ of (3.10) with $A(u,v_0) = a_0(u)$ and $A_v(u,v_0) = a_1(u)$ for some $v_0$. If (1.1) with (1.2) for the above $g$, $\mathcal{D}_1, \mathcal{D}_2$ and $L_0 \neq 0$ satisfies the compatibility condition, then $A$ is of one variable $v$ and therefore $\phi$ is constant. For a positive-valued function $A$ of one variable $v$, each constant $\phi$ satisfying $\phi > L_0 A^2 + A_v^2$ gives a pair of principal curvatures $(k_1, k_2)$ up to a sign, by (3.10) and $k_1 k_2 = K - L_0$.

3.3 The existence of just two solutions

Referring to [4], we will prove

**Theorem 3.4** An over-determined system (1.8) on a neighborhood $U$ of a point of $M$ has just two solutions if and only if there exist local coordinates $(u, v)$ on $U$ with $\partial/\partial u \in \mathcal{D}_1$, $\partial/\partial v \in \mathcal{D}_2$ such that

$$f := -\frac{1}{2} \log \left( \sqrt{\rho^2 + 1} + |\rho| \right)$$

satisfies $(\log |\tanh f|)_{uv} \neq 0$ and $\omega(w) = \Omega(\nabla f, w)$ for any tangent vector $w$ at any point of $U$, where $\nabla f$ is the gradient vector field of $f$ with respect to a metric $|\rho|(du^2 + dv^2)$.

**Remark** The condition $(\log |\tanh f|)_{uv} \neq 0$ corresponds to the condition that (1.8) does not satisfy the compatibility condition.

**Remark** Suppose $\Omega = d\omega$. Then (1.8) as in Theorem 3.4 is just an over-determined system on a surface in $E^3$ with nonzero constant mean curvature which is not part of any surface of revolution (see [4]).

In order to prove Theorem 3.4, we need the following:

**Proposition 3.5 ([4])** An over-determined system $F_u = \alpha'' + \beta' e^F, F_v = \delta'' e^{-F}$ has just two solutions if and only if there exist a function $p$ of two variables and a positive-valued function $q$ of one variable $u$ satisfying

(a) $p(u,v) > q(u)$,
(b) \( \alpha'', \beta'' \) and \( \delta'' \) are represented as

\[
\begin{align*}
\alpha'' &= \left( \log \left( \frac{p^2 - q^2}{q} \right) \right)_u, \\
\beta'' &= -\frac{1}{2p} \left( \log \left( \frac{p^2 - q^2}{q^2} \right) \right)_u, \\
\delta'' &= p_v, 
\end{align*}
\]

(3.12)

(c) \( \beta''_u \neq 0 \).

Proof of Theorem 3.4 Suppose that (1.8) has just two solutions. Then by Proposition 3.5, we see that there exist a function \( p \) of two variables and a positive-valued function \( q \) of one variable \( u \) satisfying \( p(u, v) > q(u) \) and

\[
\begin{align*}
(\log \rho^2)_u &= \left( \log \left( \frac{p^2 - q^2}{q} \right) \right)_u, \\
\frac{2I}{\rho} &= -\frac{1}{2p} \left( \log \left( \frac{p^2 - q^2}{q^2} \right) \right)_u, \\
-2J\rho &= p_v,
\end{align*}
\]

(3.13)

and \( (I/\rho)_v \neq 0 \). Let \( \tilde{u} \) be a function of one variable \( u \) satisfying \( d\tilde{u}/du = \sqrt{q} \). Then \( (\tilde{u}, v) \) are local coordinates satisfying \( \partial/\partial \tilde{u} \in \mathcal{D}_1, \partial/\partial v \in \mathcal{D}_2 \). We set \( R := p/q \). Then (3.13) is represented as

\[
\begin{align*}
\left( \log \left( \frac{R^2 - 1}{\tilde{\rho}^2} \right) \right)_{\tilde{u}} &= 0, \\
\frac{2I}{\tilde{\rho}} &= -\frac{R_{\tilde{u}}}{R^2 - 1}, \\
-2J\tilde{\rho} &= R_v,
\end{align*}
\]

(3.14)

where \( \tilde{J} \) and \( \tilde{\rho} \) are given by \( \omega = -\tilde{J}d\tilde{u} + Idv, \Omega = \tilde{\rho}d\tilde{u} \wedge dv \), respectively. From the first relation in (3.14), we see that there exists a positive-valued function \( g \) of one variable \( v \) satisfying \( R^2 - 1 = g(v)\tilde{\rho}^2 \). Let \( \tilde{v} \) be a function of one variable \( v \) satisfying \( d\tilde{v}/dv = 1/\sqrt{g} \). Then \( (\tilde{u}, \tilde{v}) \) are local coordinates satisfying \( \partial/\partial \tilde{u} \in \mathcal{D}_1, \partial/\partial \tilde{v} \in \mathcal{D}_2 \), and defining \( \tilde{I} \) and \( \tilde{\rho} \) by \( \omega = -\tilde{J}d\tilde{u} + \tilde{I}d\tilde{v} \) and \( \Omega = \tilde{\rho}d\tilde{u} \wedge d\tilde{v} \), respectively, we obtain \( R^2 - 1 = \tilde{\rho}^2 \) and we can represent the second and the third relations in (3.14) as

\[
\begin{align*}
\frac{2\tilde{I}}{\tilde{\rho}} &= -\frac{R_{\tilde{u}}}{R^2 - 1}, \\
\frac{2\tilde{J}}{\tilde{\rho}} &= -\frac{R_v}{R^2 - 1}.
\end{align*}
\]

Therefore if we set

\[
f := -\frac{1}{2} \log \left( \sqrt{\tilde{\rho}^2 + 1} + |\tilde{\rho}| \right),
\]

then we see by \( R^2 - 1 = \tilde{\rho}^2 \) that \( f \) satisfies \( f_{\tilde{u}} = \text{sgn}(\tilde{\rho})\tilde{I} \) and \( f_{\tilde{v}} = \text{sgn}(\tilde{\rho})\tilde{J} \). Therefore \( f \) satisfies \( \omega(w) = \Omega(\nabla f, w) \), where \( \nabla f \) is the gradient vector field of \( f \) with respect to \( |\tilde{\rho}|(d\tilde{u}^2 + d\tilde{v}^2) \). We see by \( f_{\tilde{u}} = \text{sgn}(\tilde{\rho})\tilde{I} \) that the condition \( (I/\rho)_v \neq 0 \) implies \( (\log |\tanh f|)_{\tilde{u}\tilde{v}} \neq 0 \). If there exist local coordinates \((u, v) \) on \( U \) as in Theorem 3.4, then referring to the above discussion, we see that (1.8) has just two solutions. Hence we obtain Theorem 3.4.

Theorem 3.4 has the following corollary.
Corollary 3.6 An over-determined system (1.1) with (1.2) and \( L_0 \neq 0 \) has just two solutions if and only if there exist local coordinates \((u, v)\) with \( \partial/\partial u \in D_1, \partial/\partial v \in D_2 \) such that \( A, B \) satisfy

\[
(\log A)_u = (\log |\Phi(f)|)_u - \frac{f_u}{\Phi(f)} A^2, \quad (\log A)_v = \Phi(f) f_v \frac{1}{A^2} \tag{3.15}
\]

and \( B = \text{sgn}(L_0 - K) \Phi(f)/A \), where

\[
\Phi(f) := -\frac{1}{L_0}(f_{uu} + f_{vv} + \sinh 2f) \tag{3.16}
\]

and \( f \) is a nowhere zero function of two variables \( u, v \) satisfying \( \log |\tanh f|_{uv} \neq 0 \), \( \Phi(f) \neq 0 \) and

\[
\Phi(f)^2 = \left( \int_{u_0}^u 2\Phi(f)f_u du + C_1(v) \right) \left( \int_{v_0}^v 2\Phi(f)f_v dv + C_2(u) \right) \tag{3.17}
\]

for positive-valued functions \( C_1, C_2 \) of one variable.

Remark We will see that for a function \( f \), there exists a positive-valued function \( A \) satisfying (3.15) if and only if \( f \) satisfies (3.17) for positive-valued functions \( C_1, C_2 \) of one variable. There exists a function \( f \) satisfying the latter condition:

(i) for any solution \( f \) of \( \Phi(f) = e^{2\varepsilon f} \) with \( \varepsilon = +1 \) or \(-1\), if we set \( C_1(v) := \varepsilon e^{2\varepsilon f(u_0,v)} \) and \( C_2(u) := \varepsilon e^{2\varepsilon f(u,v_0)} \), then \( f, C_1 \) and \( C_2 \) satisfy (3.17);

(ii) for a function \( \phi \) of one variable satisfying

\[
\varepsilon \phi'' + (\phi')^2 + \frac{1}{2} e^{2\varepsilon \phi} = L_0 C,
\]

where \( \varepsilon = +1 \) or \(-1\), and \( C \) is a positive constant, if we set \( f(u,v) := \phi(u+v) \) and

\[
C_1(v) := -\frac{2}{L_0} \phi'(u_0 + v)^2 - \frac{1}{L_0} \cosh 2\phi(u_0 + v) + 2C, \\
C_2(u) := -\frac{2}{L_0} \phi'(u + v_0)^2 - \frac{1}{L_0} \cosh 2\phi(u + v_0) + 2C,
\]

then we can choose \( C, \phi \) satisfying \( C_i > 0 \), and \( f, C_1 \) and \( C_2 \) satisfy (3.17).

Proof of Corollary 3.6 Let \( M, g, L_0, D_1, D_2, \omega, \Omega \) be as in the paragraph just before the proof of Corollary 3.3. Then by Theorem 3.4, we see that (1.8) has just two solutions if and only if there exist local coordinates \((u, v)\) on \( U \) with \( \partial/\partial u \in D_1, \partial/\partial v \in D_2 \) such that \( f \) as in (3.11) with \( \rho := (L_0 - K)AB \) satisfies \( \log |\tanh f|_{uv} \neq 0 \) and

\[
f_u = \text{sgn}(\rho) B_u \frac{A_u}{A}, \quad f_v = \text{sgn}(\rho) A_v \frac{B}{B}. \tag{3.18}
\]
We see that (3.11) is equivalent to \( \rho = -\text{sgn}(\rho) \sinh 2f \). Since \( \rho = (L_0 - K)AB \), we can rewrite (3.11) into \( \Phi(f) = \text{sgn}(\rho)AB \) by (3.18), where \( \Phi(f) \) is as in (3.16). Then by \( B = \text{sgn}(\rho)\Phi(f)/A \), we can rewrite (3.18) into (3.15). For functions \( \alpha, \beta, \gamma, \delta \) of two variables \( u, v \), there exists a solution of (1.1) if and only if we can find \( a, b, c, d \) satisfying (3.2) and \( a + b + c + d = 0 \). Therefore there exists a solution of (1.1) with

\[
\alpha := (\log \Phi(f))^2_u, \quad \beta := -\frac{2f_u}{\Phi(f)}, \quad \gamma := 0, \quad \delta := 2\Phi(f)f_v,
\]

that is, there exists a positive-valued function \( A \) satisfying (3.15) if and only if \( f \) satisfies (3.17). Hence we have proved Corollary 3.6.

\[ \square \]

**Remark** Let \( (u, v), A, B, f \) be as in Corollary 3.6. Then by (1.3), (3.16) and (3.18), we obtain

\[
q_1 = \frac{1}{A(u, v)^2}(\cosh 2f(u, v) - \cosh 2f(u, v_0)), \quad q_2 = \frac{1}{B(u, v)^2}(\cosh 2f(u, v) - \cosh 2f(u_0, v)). \tag{3.19}
\]

Let \( F, F' \) be the two solutions of (1.1) with (1.2) and \( \iota, \iota' \) immersions into \( N \) given by \( F, F' \), respectively. Let \( k_1, k_2 \) be principal curvatures of \( \iota \) corresponding to \( D_1, D_2 \), respectively, and \( k_1', k_2' \) principal curvatures of \( \iota' \) corresponding to \( D_1, D_2 \), respectively. Then using (1.4) and (3.19), we see that one of pairs \((k_1^2, k_2^2), ((k_1')^2, (k_2')^2)\) is given by \((2/A^2)\cosh^2 f, (2/B^2)\sinh^2 f\) and that the other is given by \((2/A^2)\sinh^2 f, (2/B^2)\cosh^2 f\).

**Remark** Suppose that (1.1) with (1.2) and \( L_0 \neq 0 \) has just two solutions and that we can choose isothermal coordinates as \((u, v)\) in Corollary 3.6. Then we obtain

\[
f_u = \text{sgn}(\rho)(\log A)_u, \quad f_v = \text{sgn}(\rho)(\log A)_v, \quad \Phi(f) = \text{sgn}(\rho)A^2. \tag{3.20}
\]

From the first and the second relations in (3.20), we obtain \( A = ce^\bar{f} \), where \( c \) is a positive number and \( \bar{f} := \text{sgn}(\rho)f \). From (3.20), we obtain \( L_0 - K = -(1/A^2)\sinh 2\bar{f} \). Therefore noticing the previous remark, we see that the surface in \( N \) corresponding to each of the two solutions of the system has nonzero constant mean curvature. On a surface in \( N \) with nonzero constant mean curvature, if (1.1) with (1.2) does not satisfy the compatibility condition, then the system has just two solutions and we can choose isothermal coordinates as \((u, v)\) in Corollary 3.6.

### 3.4 The existence of a unique solution with \((Y'')^2 = 4X''Z''\)

Referring to [3], we will prove

**Theorem 3.7** An over-determined system (1.8) on a neighborhood \( U \) of a point of \( M \) has a unique solution so that \((Y'')^2 - 4X''Z''\) vanishes if and only if there exist local
coordinates \((u, v)\) on \(U\) with \(\partial/\partial u \in \mathcal{D}_1\), \(\partial/\partial v \in \mathcal{D}_2\) such that

\[
f := -\frac{1}{2} \log |\rho|
\]

satisfies \(f_{uv} + 2f_u f_v \neq 0\) and \(\omega(w) = \Omega(\nabla f, w)\), where \(w\) and \(\nabla f\) are as in Theorem 3.4.

**Remark** The condition \(f_{uv} + 2f_u f_v \neq 0\) corresponds to the condition that (1.8) does not satisfy the compatibility condition.

**Remark** Suppose \(\Omega = d\omega\). Then (1.8) as in Theorem 3.7 is just an over-determined system on a minimal surface in \(E^3\) which is not part of any surface of revolution ([3]).

In order to prove Theorem 3.7, we need the following:

**Proposition 3.8 ([3])** An over-determined system \(F_u = \alpha'' + \beta'' e^F\), \(F_v = \delta'' e^{-F}\) has a unique solution so that \((Y'')^2 - 4X''Z''\) vanishes if and only if there exist a function \(h\) of one variable \(u\) and a positive-valued function \(\Delta\) of two variables \(u, v\) satisfying

\[
\alpha'' = -h(u) + 2(\log \Delta)_u, \quad \beta'' = \frac{h(u)}{\Delta} + \left(\frac{1}{\Delta}\right)_u, \quad \delta'' = \Delta_v \tag{3.22}
\]

and \(\beta''_v \neq 0\).

**Proof of Theorem 3.7** Suppose that (1.8) has a unique solution so that \((Y'')^2 - 4X''Z''\) vanishes. Then by Proposition 3.8, we see that there exist a function \(h\) of one variable \(u\) and a positive-valued function \(\Delta\) of two variables \(u, v\) satisfying

\[
(\log \rho^2)_u = -h(u) + 2(\log \Delta)_u, \quad \frac{2I}{\rho} = \frac{h(u)}{\Delta} + \left(\frac{1}{\Delta}\right)_u, \quad -2J \rho = \Delta_v \tag{3.23}
\]

and

\[
(\log \Delta)_{uv} \neq ((\log \Delta)_u - h(u))(\log \Delta)_v. \tag{3.24}
\]

Let \(\phi\) be a positive-valued function of one variable \(u\) satisfying \(d(\log \phi^2)/du = h\). Then from the first relation in (3.23), we see that there exists a positive-valued function \(\psi\) of one variable \(v\) satisfying \(\Delta = |\rho|\phi(u)\psi(v)\). By this together with the second and the third relations in (3.23), we obtain

\[
(\log \tilde{\Delta})_u = -2 \text{sgn}(\rho)I \phi(u)\psi(v), \quad (\log \tilde{\Delta})_v = -\frac{2 \text{sgn}(\rho)J}{\phi(u)\psi(v)}, \tag{3.25}
\]

where \(\tilde{\Delta} := \Delta/\phi^2\). Let \(\tilde{u}, \tilde{v}\) be functions of one variable satisfying \(d\tilde{u}/du = \phi\) and \(d\tilde{v}/dv = 1/\psi\). Then \((\tilde{u}, \tilde{v})\) are local coordinates satisfying \(\partial/\partial \tilde{u} \in \mathcal{D}_1\), \(\partial/\partial \tilde{v} \in \mathcal{D}_2\). If \(\Omega\) is represented as \(\Omega = \tilde{\rho} d\tilde{u} \wedge d\tilde{v}\), then we obtain \(|\tilde{\rho}| = \tilde{\Delta}\). In addition, if \(\omega\) is represented as \(\omega = -\tilde{J} d\tilde{u} + \tilde{I} d\tilde{v}\), then by (3.25), we obtain \(f_{\tilde{u}} = \text{sgn}(\tilde{\rho}) \tilde{I}\) and \(f_{\tilde{v}} = \text{sgn}(\tilde{\rho}) \tilde{J}\), where
Theorem 3.7 has the following corollary.

**Corollary 3.9** An over-determined system (1.1) with (1.2) and $L_0 \neq 0$ has a unique solution so that $Y^2 - 4XZ$ vanishes if and only if there exist local coordinates $(u, v)$ with $\partial/\partial u \in D_1$, $\partial/\partial v \in D_2$ such that $A, B$ satisfy (3.15) and $B = \text{sgn}(L_0 - K)\Phi(f)/A$, where

$$\Phi(f) := -\frac{1}{L_0}(f_{uu} + f_{vv} - e^{-2f})$$

(3.26)

and $f$ is a function of two variables $u, v$ satisfying $f_{uu} + 2f_u f_v \neq 0$, $\Phi(f) \neq 0$ and (3.17) with (3.26) for positive-valued functions $C_1, C_2$ of one variable.

**Remark** For a function $f$, there exists a positive-valued function $A$ satisfying (3.15) with (3.26) if and only if $f$ satisfies (3.17) with (3.26) for positive-valued functions $C_1, C_2$ of one variable. There exists a function $f$ satisfying the latter condition.

**Proof of Corollary 3.9** Let $M, g, L_0, D_1, D_2, \omega, \Omega$ be as in the paragraph just before the proof of Corollary 3.3. Then by Theorem 3.7, we see that (1.8) has a unique solution so that $(Y^m)^2 - 4X^mZ^m$ vanishes if and only if there exist local coordinates $(u, v)$ on $U$ with $\partial/\partial u \in D_1$, $\partial/\partial v \in D_2$ such that $f$ as in (3.21) with $\rho := (L_0 - K)AB$ satisfies $f_{uu} + 2f_u f_v \neq 0$ and (3.18). We see that (3.21) is equivalent to $\rho = \text{sgn}(\rho)e^{-2f}$. Since $\rho = (L_0 - K)AB$, we can rewrite (3.21) into $\Phi(f) = \text{sgn}(\rho)AB$ by (3.18), where $\Phi(f)$ is as in (3.26). Referring to the proof of Corollary 3.6 for the rest discussion, we obtain Corollary 3.9. □

**Remark** Let $(u, v), A, B, f$ be as in Corollary 3.9. Then by (1.3), (3.18) and (3.26), we obtain

$$q_1 = \frac{1}{A(u, v)^2}(e^{-2f(u,v)} - e^{-2f(u,v_0)}), \quad q_2 = \frac{1}{B(u, v)^2}(e^{-2f(u,v)} - e^{-2f(u_0,v)}).$$

Let $F$ be a unique solution of (1.1) with (1.2) and $\iota$ an immersion into $N$ given by $F$. Let $k_1, k_2$ be principal curvatures of $\iota$ corresponding to $D_1, D_2$, respectively. Then we obtain

$$k_1^2 = e^{-2f}/A^2, \quad k_2^2 = e^{-2f}/B^2.$$

**Remark** Suppose that (1.1) with (1.2) and $L_0 \neq 0$ has a unique solution so that $Y^2 - 4XZ$ vanishes and that we can choose isothermal coordinates as $(u, v)$ in Corollary 3.9. Then noticing the previous remark, we see that the surface in $N$ corresponding to the solution
of the system satisfies \( k_1 + k_2 \equiv 0 \) or \( k_1 - k_2 \equiv 0 \), according to \( \text{sgn}(\rho) = +1 \) or \(-1\). Since we do not treat any totally umbilical surface in the present paper, we see that the surface is minimal. On a minimal surface in \( N \), if (1.1) with (1.2) does not satisfy the compatibility condition, then the system has a unique solution so that \( Y^2 - 4XZ \) vanishes and we can choose isothermal coordinates as \((u, v)\) in Corollary 3.9.

### 3.5 A family of surfaces in \( N \) given by (1.8)

Let \( M, D_1, D_2, \omega, \Omega \) be as in the third paragraph of Subsection 1.3. In the case where \( \Omega = d\omega \), we see the following ([3]):

(a) (1.8) with \( \Omega = d\omega \) for a triplet \((\omega, D_1, D_2)\) has a solution on a neighborhood \( U \) of a point of \( M \) if and only if there exist local coordinates \((u, v)\) on \( U \) satisfying \( \partial/\partial u \in D_1, \partial/\partial v \in D_2 \) and (3.1) with \( \rho = I_u + J_v \);

(b) if (1.8) with \( \Omega = d\omega \) for a triplet \((\omega, D_1, D_2)\) has a solution, then \((\omega, D_1, D_2)\) gives a family of surfaces in \( E^3 \) characterized by the following conditions:

(i) \( D_1, D_2 \) give principal distributions on any member of the family,

(ii) on any member, a 1-form \( \theta_K \) for \((g, D_1, D_2)\) as in the second paragraph of Subsection 1.3 is given by \( \theta_K = *\omega \);

(c) for each solution \( F \) of (1.8) with \( \Omega = d\omega \), the family of surfaces in \( E^3 \) given by \( F \) is represented as a disjoint union of one-parameter subfamilies and there exists a one-to-one correspondence between the set of one-parameter subfamilies and the set of solutions of a semilinear hyperbolic differential equation of second order.

In the following, we suppose \( \Omega \neq d\omega \). Theorem 3.1 gives a condition of the existence of a solution of (1.8) for \((\omega, \Omega, D_1, D_2)\). Let \( L_0 \) be a nonzero number. Suppose that \( M \) is oriented so that \((1/L_0)(\Omega - d\omega)\) is a positive 2-form. In each of previous subsections, we treated a Riemannian metric \( g \) on \( M \) such that a 1-form \( \theta_K \) for \((g, D_1, D_2)\) and the area element \( d\mu \) of \( g \) are given by \( \theta_K = *\omega, d\mu = (1/L_0)(\Omega - d\omega) \), respectively. Let \( r \) be a nowhere zero function given by \( r := (1/L_0)(\rho - I_u - J_v) \). Then \((1/L_0)(\Omega - d\omega) = r du \wedge dv \). We will prove

**Proposition 3.10** There exists a Riemannian metric \( g \) on a neighborhood \( U \) of \( p \in M \) satisfying \( \theta_K = *\omega \) and \( d\mu = (1/L_0)(\Omega - d\omega) \) if and only if there exist local coordinates \((u, v)\) on \( U \) which give the orientation of \( M \) and satisfy \( \partial/\partial u \in D_1, \partial/\partial v \in D_2 \) and

\[
r^2 = \left( 2 \int_{u_0}^{u} r I du + 1 \right) \left( 2 \int_{v_0}^{v} r J dv + 1 \right),
\]

(3.27)

where \((u_0, v_0)\) corresponds to \( p \).
Proof We can suppose $r > 0$. There exists a Riemannian metric $g$ as in Proposition 3.10 if and only if there exist positive-valued functions $A, B$ satisfying

$$\frac{B_u}{A} = I, \quad \frac{A_v}{B} = J, \quad AB = r. \tag{3.28}$$

Let $A, B$ satisfy (3.28) and let $h$ be a function given by $B = Ae^h$. Then from (3.28), we see that $h' := -h + \log r$ is a solution of the following over-determined system:

$$h_u' = (\log r^2)_u - \frac{2I}{r} e^{h'}, \quad h_v' = 2r Je^{-h'} \tag{3.29}$$

There exists a solution of (1.1) if and only if we can find $a, b, c, d$ satisfying (3.2) and $a + b + c + d = 0$. Therefore there exists a solution of (1.1) with

$$\alpha := (\log r^2)_u, \quad \beta := -\frac{2I}{r}, \quad \gamma := 0, \quad \delta := 2r J,$$

that is, there exists a function $h'$ satisfying (3.29) if and only if we can choose $(u, v)$ satisfying (3.27). Let $h'$ be a solution of (3.29). Then $A := e^{h'/2}, B := r / e^{h'/2}$ satisfy (3.28). Hence we obtain Proposition 3.10.

Remark Suppose $\Omega \neq d\omega$. We see that if (1.8) for $(\omega, \Omega, D_1, D_2)$ has a solution and if (3.29) for $(\omega, \Omega, D_1, D_2)$ and $L_0$ has a solution, then $(\omega, \Omega, D_1, D_2)$ gives a family of surfaces in $N$ characterized by the following conditions:

(i) $D_1, D_2$ give principal distributions on any member of the family,

(ii) on any member, a 1-form $\theta_K$ for $(g, D_1, D_2)$ and the area element $d\mu$ of $g$ are given by $\theta_K = *\omega, d\mu = (1/L_0)(\Omega - d\omega)$, respectively, where $g$ is the induced metric.

For each solution $F$ of (1.8), there exists a one-to-one correspondence between the family of surfaces in $N$ given by $F$ and the set of solutions of (3.29): for each solution $h'$ of (3.29), the induced metric $g$ of the corresponding surface is represented as $g = e^{h'} du^2 + r^2 e^{-h'} dv^2$ and principal curvatures are represented as $k_1 = e^{F/2-h'/2}, k_2 = -(\rho/r)e^{h'/2-F/2}$.

References


