Surfaces in flat pseudo-Riemannian space forms with zero mean curvature vector

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Abstract Let $F$ be a space-like and conformal immersion of a Riemann surface $M$ into an $(n + 2)$-dimensional flat pseudo-Riemannian space form with zero mean curvature vector. Then $\psi = \partial F = \Psi dw$ is a holomorphic 1-form on $M$ valued in a complex quadric in $C^{n+2}$. We will define a $(-1)$-map $\Psi_{-1}$ of $\Psi$. In addition, for $k = 2, \ldots, [(n + 1)/2]$, we will define a $(-k)$-map $\Psi_{-k}$ of $\Psi$ if $\Psi_{-k+1}$ is already defined, not identically zero and valued in the closure of a complex quadric. We will see that $\Psi_{-1}$ is valued in a suitable complex quadric in $C^n$ if and only if the Hopf quartic differential of $F$ vanishes. We will obtain characterizations of a holomorphic immersion and its analogues in terms of $\Psi_{-1}, \Psi_{-2}, \ldots, \Psi_{-[(n+1)/2]}$.

1 Introduction

A minimal surface in $E^3$ is locally given by a pair of two holomorphic functions. More generally, for a positive integer $n$, a minimal surface in $E^{n+2}$ is locally given by a set of $n + 1$ holomorphic functions. We can define special classes of minimal surfaces in $E^{n+2}$, adopting formulations by smaller sets of holomorphic functions. For example, an isotropic minimal surface in $E^4$ is locally given by a pair of two holomorphic functions. An isotropic minimal surface in $E^4$ is just a minimal surface in $E^4$ with zero Hopf quartic differential. A minimal surface in $E^{n+2}$ with zero Hopf quartic differential is superconformal (we can refer to [4] for superconformal surfaces) and locally given by a set of $n$ holomorphic functions.

Suppose that $n$ is even. Then an immersed complex curve in $C^{n+1} = E^{n+2}$ is a minimal surface. If $n = 2$, then an immersed complex curve in $C = E^4$ is an isotropic minimal surface and any isotropic minimal surface is congruent in $E^4$ with an immersed complex curve in $C^2 = E^4$. On the other hand, if $n > 2$, then the Hopf quartic differential on an immersed complex curve in $C^{n+1} = E^{n+2}$ vanishes and there exists a minimal surface in $E^{n+2}$ with zero Hopf quartic differential which is not congruent in $E^{n+2}$ with any immersed complex curves in $C_{2}^{n+1} = E^{n+2}$.

Let $F$ be a conformal and minimal immersion of a Riemann surface $M$ into $E^{n+2}$ and $w$ a local complex coordinate of $M$. Then $\Psi = \Psi_0 := dF(\partial/\partial w)$ gives a holomorphic map into $C^{n+2}$, and it is valued in the standard complex quadric $Q^{n+1}$ in $C^{n+2}$. Since $F$ is an immersion, we have $\Psi \neq 0$ and $\Psi$ gives the Gauss map of $F$. We see that $\psi := \Psi dw$ does not depend on the choice of $w$ and therefore $\psi$ is a $Q^{n+1}$-valued holomorphic 1-form defined on $M$. In the present paper, we will define a $(-k)$-map $\Psi_{-k}$ of $\Psi$ for $k \in$
A \((-k)\)-map \(\Psi_{-k}\) is a \((\mathbb{CP}^1)^{n+2-2k}\)-valued meromorphic function and we will define a \((-k)\)-map \(\Psi_{-k}\) of \(\Psi\) in the case where \(\Psi_{-k+1}\) is already defined, not identically zero and valued in the closure \(\overline{Q^{n+3-2k}}\) of \(Q^{n+3-2k}\) in \((\mathbb{CP}^1)^{n+4-2k}\). We will see that \(\Psi_{-k} := \Psi_{-k} dw\) does not depend on the choice of \(w\) and that the Hopf quartic differential of \(F : M \rightarrow E^{n+2}\) vanishes if and only if a \((-1)\)-map \(\Psi_{-1}\) of \(\Psi\) is valued in \(\overline{Q^{n-1}}\). Suppose that \(n\) is even. Then a holomorphic immersion \(F : M \rightarrow C_+^{n+1} = E^{n+2}\) satisfies \(\Psi_{-1} \equiv 0\), \(\Psi_{-2} \equiv 0\), ..., \(\Psi_{-\frac{n+1}{2}} \equiv 0\) or a condition that \(\Psi_{-\frac{n}{2}}\) is defined and valued in \(\overline{Q^1}\). We will see that for a minimal immersion \(F : M \rightarrow E^{n+2}\), if \(\Psi_{-l} \equiv 0\) for an integer \(l \in \{1, 2, \ldots, \frac{n}{2} - 1\}\) or if \(\Psi_{-\frac{n}{2}}\) is defined and valued in \(\overline{Q^1}\), then there exists an element \(A\) of \(O(n+2)\) such that \(FA\) is a holomorphic immersion of \(M\) into \(C_+^{n+1} = E^{n+2}\). We will have an analogous discussion on the case where \(n\) is odd: if \(n\) is odd, then for a holomorphic immersion \(F' : M \rightarrow C_+^{n+1}\), \(F := (F', 0)\) is a minimal immersion into \(E^{n+2}\) satisfying \(\Psi_{-1} \equiv 0\), \(\Psi_{-2} \equiv 0\), ..., \(\Psi_{-\frac{n+1}{2}} \equiv 0\) or \(\Psi_{-\frac{n+1}{2}} \equiv 0\); we will see that for a minimal immersion \(F : M \rightarrow E^{n+2}\), if \(\Psi_{-l} \equiv 0\) for an integer \(l \in \{1, 2, \ldots, \frac{n+1}{2}\}\), then there exists an element \(A\) of \(O(n+2)\) such that \(FA\) is represented as \((F', 0)\) by a holomorphic immersion \(F' : M \rightarrow C_+^{n+1}\). In addition, we will have analogous discussions on space-like and conformal immersions of \(M\) into flat pseudo-Riemannian space forms with zero mean curvature vector. For example, if \(n\) is even, then for a holomorphic immersion \(F' : M \rightarrow C_+^{n+1}\), a light-like vector \(c\) of \(E_1^2\) and a harmonic function \(h\) on \(M\), \(F := (F', hc)\) is a space-like and conformal immersion of \(M\) into a flat Lorentz space form \(E_1^{n+2}\) with zero mean curvature vector satisfying \(\Psi_{-1} \equiv 0\), \(\Psi_{-2} \equiv 0\), ..., \(\Psi_{-\frac{n+1}{2}} \equiv 0\) or a condition that \(\Psi_{-\frac{n}{2}}\) is defined and valued in \(\overline{Q^1}\), where \(\overline{Q^1}\) is the closure of \(Q^1 = \{(z^1, z^2) \in C^2 \mid (z^1)^2 - (z^2)^2 = 0\}\) in \((\mathbb{CP}^1)^2\); we will see that for a space-like and conformal immersion \(F : M \rightarrow E_1^{n+2}\) with zero mean curvature vector, if \(\Psi_{-l} \equiv 0\) for an integer \(l \in \{1, 2, \ldots, \frac{n}{2} - 1\}\) or if \(\Psi_{-\frac{n}{2}}\) is defined and valued in \(\overline{Q^1}\), then there exists an element \(A\) of \(O(n+1, 1)\) such that \(FA\) is represented as \((F', hc)\) by a holomorphic immersion \(F' : M \rightarrow C_+^{n+1}\), a light-like vector \(c\) of \(E_1^2\) and a harmonic function \(h\) on \(M\).

Remark The Hopf quartic differential is a complex quartic differential \(Q\) defined on a space-like surface in a pseudo-Riemannian space form. This is considered to be a generalization of the Hopf differential on a surface in a 3-dimensional space form. We see by the equations of Codazzi that if the mean curvature vector of the surface vanishes, then \(Q\) is holomorphic. We can refer to [2] and its references for the Hopf quartic differential.

Remark We can refer to [3], [4] for \(k\)-isotropicity of minimal surfaces in \(E^{n+2}\). We can find a characterization of an immersed complex curve in terms of \(k\)-isotropicity and a representation formula of \(k\)-isotropic minimal surfaces ([3]). We will see that the \(k\)-isotropicity of a minimal immersion \(F\) into \(E^{n+2}\) is characterized in terms of a \((-k)\)-map \(\Psi_{-k}\) of \(\Psi\).
Remark We can find a characterization of an immersed complex curve in \( C^2 = E^4 \) in terms of the induced metric and a holomorphic cubic differential ([1]).

Remark We can refer to [2], [5] and [6] for isotropic minimal surfaces in oriented 4-dimensional Riemannian manifolds.

2 Holomorphic maps valued in complex quadrics and the Hopf quartic differentials

Let \( n \) be a positive integer. Let \( p, q \) be nonnegative integers satisfying \( p + q = n \). Let \( E_n^{n+2} \) be an \((n + 2)\)-dimensional flat pseudo-Riemannian space form with signature \((p + 2, q)\). We suppose that \( E_n^{n+2} \) is simply connected and complete. Then \( E_n^{n+2} \) is diffeomorphic to \( R^{n+2} \) and equipped with a nondegenerate metric \( \langle , \rangle \) defined by

\[
\langle x, y \rangle := \sum_{i=1}^{p+2} x^i y^i - \sum_{j=p+3}^{n+2} x^j y^j \tag{2.1}
\]

for \( x := (x^1, \ldots, x^{n+2}) \), \( y := (y^1, \ldots, y^{n+2}) \in R^{n+2} \). Let \( M \) be a Riemann surface and \( F : M \rightarrow E_n^{n+2} \) a space-like and conformal immersion of \( M \) into \( E_n^{n+2} \) with zero mean curvature vector. If we represent \( F \) as \( F = (F^1, \ldots, F^{n+2}) \), then each \( F^i \) is harmonic. Therefore, for any local complex coordinate \( w \) of \( M \), \( \Psi : = dF(\partial_w) = (\partial_w F^1, \ldots, \partial_w F^{n+2}) \) is considered as a holomorphic map into \( C^{n+2} \), where \( \partial_w := \partial/\partial w \). In addition, since \( F \) is conformal, \( \Psi \) is valued in a complex quadric

\[
Q_{q}^{n+1} := \{ z \in C^{n+2} | \langle z, z \rangle = 0 \},
\]

where

\[
\langle z, z \rangle := \sum_{i=1}^{p+2} (z^i)^2 - \sum_{j=p+3}^{n+2} (z^j)^2, \quad z := (z^1, \ldots, z^{n+2}). \tag{2.2}
\]

We see that \( \Psi dw \) is a \( Q_{q}^{n+1} \)-valued 1-form on \( M \) and does not depend on the choice of a local complex coordinate \( w \). Since \( \langle \Psi, \Psi \rangle = 0 \), we obtain \( \langle \Psi, \Psi_w \rangle = 0 \), where \( \Psi_w := (\partial_w^2 F^1, \ldots, \partial_w^2 F^{n+2}) \).

Let \( \nu_1, \ldots, \nu_n \) form a local frame of the normal bundle on \( M \) with respect to \( F \) satisfying \( \langle \nu_i, \nu_j \rangle = \varepsilon_i \delta_{ij} \) for \( \varepsilon_i = 1 \ (i = 1, \ldots, p) \) and \( \varepsilon_i = -1 \ (i = p+1, \ldots, p+q) \). Considering each \( \nu_i \) to be an \( R^{n+2} \)-valued function, we obtain \( \langle \Psi, \nu_i \rangle = 0 \). Let \( \Phi_1, \ldots, \Phi_n \) be complex quadratic differentials defined by

\[
\Phi_i := \phi_i dw \otimes dw, \quad \phi_i := \langle \Psi_w, \nu_i \rangle \ (i = 1, \ldots, n). \tag{2.3}
\]
Each $\Phi_i$ does not depend on the choice of $w$. We see by (2.3) that

$$Q := \sum_{i=1}^{n} \varepsilon_i \Phi_i \otimes \Phi_i = \left( \sum_{i=1}^{n} \varepsilon_i \phi_i^2 \right) dw \otimes dw \otimes dw \otimes dw$$  \hspace{1cm} (2.4)$$

does not depend on the choice of a local frame $\{\nu_1, \ldots, \nu_n\}$ satisfying $\langle \nu_i, \nu_j \rangle = \varepsilon \delta_{ij}$. Therefore we can define a complex quartic differential $Q$ on $M$ by (2.4). We call $Q$ the **Hopf quartic differential** of $F : M \rightarrow E_{q+2}^n$. By the equations of Codazzi, we see that $\sum_{i=1}^{n} \varepsilon_i \phi_i^2$ is holomorphic. Hence the Hopf quartic differential $Q$ of a space-like and conformal immersion $F : M \rightarrow E_{q+2}^n$ with zero mean curvature vector is holomorphic. If $n = p = 1$, then $\Phi_1$ is the Hopf differential on $M$ with respect to $F$.

We represent $\Psi$ as

$$\Psi = (\Psi, t_1) t_1 + (\Psi, t_2) t_2 + \sum_{i=1}^{n} (\Psi, \nu_i) \nu_i,$$  \hspace{1cm} (2.5)$$

where $t_1, t_2$ are $\mathbb{R}^{n+2}$-valued functions satisfying

$$t_1 - \sqrt{-1} t_2 = \frac{\sqrt{2}}{\langle \Psi, \Psi \rangle} \Psi.$$

We obtain

$$t_1 = \frac{1}{\sqrt{2 \langle \Psi, \Psi \rangle}} (\Psi + \overline{\Psi}), \hspace{1cm} t_2 = \frac{1}{\sqrt{-2 \langle \Psi, \Psi \rangle}} (-\Psi + \overline{\Psi}).$$

Therefore, by $\langle \Psi, \Psi \rangle = 0$, we obtain

$$\langle \Psi, t_1 \rangle t_1 + \langle \Psi, t_2 \rangle t_2 = \frac{\langle \Psi, \overline{\Psi} \rangle}{\langle \Psi, \Psi \rangle} \Psi.$$  \hspace{1cm} (2.6)$$

By (2.3), we obtain

$$\left\langle \sum_{i=1}^{n} (\Psi, \nu_i) \nu_i, \sum_{i=1}^{n} (\Psi, \nu_i) \nu_i \right\rangle = \sum_{i=1}^{n} (\nu_i, \nu_i) \langle \Psi, \nu_i \rangle^2 = \sum_{i=1}^{n} \varepsilon_i \phi_i^2.$$  \hspace{1cm} (2.7)$$

Therefore, from (2.5), (2.6) and (2.7), we obtain

$$\langle \Psi, \Psi \rangle = \sum_{i=1}^{n} \varepsilon_i \phi_i^2.$$  \hspace{1cm} (2.8)$$

In particular, from (2.4) and (2.8), we obtain

**Proposition 2.1** The Hopf quartic differential $Q$ of $F : M \rightarrow E_{q+2}^n$ vanishes if and only if $\langle \Psi, \Psi \rangle = 0$.  

3 A \((-1)\)-map of \(\Psi\)

We represent \(\Psi\) as \(\Psi = (\Psi^1, \ldots, \Psi^{n+2})\). Noticing that \(\text{Re} \, \Psi\) and \(\text{Im} \, \Psi\) are linearly independent, we can suppose that on a neighborhood of a point, \(\Psi^1 + \sqrt{-1} \Psi^2\) is nowhere zero. Let \(f_*\) be a holomorphic function of \(w\) on such a neighborhood satisfying

\[
f_*^2 = -\frac{1}{2} (\Psi^1 + \sqrt{-1} \Psi^2).
\]

Then \(f_*\) satisfies

\[
4f_*^4 + 4\Psi^1 f_*^2 - \sum_{i=1}^{n} \varepsilon_i (\Psi^{i+2})^2 = 0.
\]

We set \(f_i := \Psi^{i+2}/f_*\ (i = 1, 2, \ldots, n)\). Then \(\Psi^1, \ldots, \Psi^{n+2}\) are represented as

\[
\begin{aligned}
\Psi^1 &= -f_*^2 + \sum_{i=1}^{n} \varepsilon_i f_i^2, \\
\Psi^2 &= \sqrt{-1} \left( f_*^2 + \sum_{i=1}^{n} \varepsilon_i f_i^2 \right), \\
\Psi^{i+2} &= 2f_* f_i \quad (i = 1, \ldots, n).
\end{aligned}
\]

We obtain

\[
\langle \Psi_w, \Psi_w \rangle = 4 \sum_{i=1}^{n} \varepsilon_i ((f_i)_w f_* - f_i (f_*)_w)^2.
\]

We set

\[
\Psi_{-1}^i := \left( \frac{f_i}{f_*} \right)_w = - \left( \frac{\Psi^{i+2}}{\Psi^1 + \sqrt{-1} \Psi^2} \right)_w \quad (i = 1, \ldots, n).
\]

Then (3.2) can be rewritten into

\[
\langle \Psi_w, \Psi_w \rangle = 4f_*^4 \sum_{i=1}^{n} \varepsilon_i (\Psi_{-1}^i)^2.
\]

We set \(\Psi_{-1} := (\Psi_{-1}^1, \ldots, \Psi_{-1}^n)\). Then \(\Psi_{-1}\) is a holomorphic map into \(\mathbb{C}^n\). By Proposition 2.1 together with (3.3), we obtain

**Proposition 3.1** The Hopf quartic differential \(Q\) of \(F : M \rightarrow E^{n+2}_q\) vanishes if and only if \(\Psi_{-1}\) is valued in \(Q_{q-1}^{n-1}\).

**Remark** We see that \(\Psi_{-1} dw\) does not depend on the choice of \(w\). Suppose that \(\Psi^1 + \sqrt{-1} \Psi^2\) can become zero but that it is not identically zero on \(M\). Then we can define by \(\Psi_{-1}^i dw\) a meromorphic 1-form on \(M\) for each \(i \in \{1, \ldots, n\}\). We set \(\overline{\mathbb{C}} := \mathbb{C} P^1 = \mathbb{C} \cup \{\infty\}\). We denote by \(Q_{q-1}^{n-1}\) the closure of \(Q_{q-1}^{n-1}\) in \(\overline{\mathbb{C}}^n\). Then we see that the Hopf quartic differential \(Q\) of \(F : M \rightarrow E^{n+2}_q\) vanishes if and only if a \(\overline{\mathbb{C}}^n\)-valued meromorphic 1-form \(\Psi_{-1} dw\) on \(M\) is valued in \(Q_{q-1}^{n-1}\).

We call \(\Psi_{-1}\) a \((-1)\)-map of \(\Psi\).
4 Examples

Example 4.1 Let $F : M \rightarrow E^3$ be a conformal minimal immersion. Then $\Psi$ can be locally represented as

$$\Psi = (-f_s^2 + f_1^2, \sqrt{-1}(f_s^2 + f_1^2), 2f_s f_1),$$

where $f_s, f_1$ are holomorphic functions satisfying $f_s \neq 0$. Then we obtain $\Psi_{-1} = \Psi_{1} = (f_1/f_s)_w$. Therefore $Q$ vanishes if and only if $f_1/f_s$ is constant. If $Q$ vanishes, then $F$ is totally umbilical and therefore the image is part of a plane in $E^3$. If $f_1$ vanishes, then $\Psi = f_s^2 (-1, \sqrt{-1}, 0)$. Suppose $f_1 = cf_s$ for a nonzero complex number $c$. Then

$$\Psi = f_s^2 ((c^2 - 1), \sqrt{-1}(c^2 + 1), 2c).$$

Therefore there exists an element $A$ of $O(3)$ satisfying $\Psi A = \xi (-1, \sqrt{-1}, 0)$, where $\xi := (1 + |c|^2)f_s^2$.

Example 4.2 Let $F : M \rightarrow E^3_1$ be a space-like and conformal immersion with zero mean curvature vector. Then $\Psi$ can be locally represented as

$$\Psi = (-f_s^2 - f_1^2, \sqrt{-1}(f_s^2 - f_1^2), 2f_s f_1),$$

where $f_s, f_1$ are holomorphic functions satisfying $f_s \neq 0$. Then we obtain $\Psi_{-1} = \Psi_{1} = (f_1/f_s)_w$. If $Q$ vanishes, then the image by $F$ is part of a plane in $E^3$. If $f_1 = cf_s$ for a complex number $c$, then noticing that $F$ is space-like, we have $|c| \neq 1$ and we see that there exists an element $A$ of $O(2, 1)$ satisfying $\Psi A = \xi (-1, \sqrt{-1}, 0)$, where $\xi := (1 - |c|^2)f_s^2$.

Example 4.3 Let $F : M \rightarrow E^4$ be a conformal minimal immersion. Then $\Psi$ can be locally represented as

$$\Psi = (-f_s^2 + f_1^2 + f_2^2, \sqrt{-1}(f_s^2 + f_1^2 + f_2^2), 2f_s f_1, 2f_s f_2),$$

where $f_s, f_1, f_2$ are holomorphic functions satisfying $f_s \neq 0$. Then we obtain

$$\Psi_{-1} = (\Psi_{-1}^1, \Psi_{-1}^2) = \left(\frac{f_1}{f_s}_w, \frac{f_2}{f_s}_w\right). \quad (4.1)$$

Therefore $Q$ vanishes if and only if one of $(f_1 \pm \sqrt{-1}f_2)/f_s$ is constant. If $Q$ vanishes, then $F$ is an isotropic minimal immersion. Suppose that one of $(f_1 \pm \sqrt{-1}f_2)/f_s$ is constant. If one of $f_1 \pm \sqrt{-1}f_2$ vanishes, then

$$\Psi = (-f_s^2, \sqrt{-1}f_s^2, 2f_s f_1, \pm 2\sqrt{-1}f_s f_1) \quad (4.2)$$

and therefore $\xi_1 := -f_s^2, \xi_2 := 2f_s f_1$ satisfy $\Psi = (\xi_1, -\sqrt{-1}\xi_1, \xi_2, \pm \sqrt{-1}\xi_2)$. Suppose one of $f_1 \pm \sqrt{-1}f_2 = cf_s$ for a nonzero complex number $c$. Then $\Psi = F_s a_s \mp F_s a_1$, where

$$F_s := f_s, \quad F_1 := 2f_s f_2,$$

$$a_s := (c^2 - 1, \sqrt{-1}(c^2 + 1), 2c, 0), \quad a_1 := (\sqrt{-1}c, -c, \sqrt{-1}, \mp 1).$$
We set
\[ b_1 := a_1, \quad b_* := a_* + c\sqrt{-1}b_1 = (-1, \ \sqrt{-1}, c, \mp c\sqrt{-1}). \]
Then \( \text{Re} b_*, \text{Im} b_*, \text{Re} b_1, \text{Im} b_1 \) form an orthogonal system and the lengths of them are equal to \( \sqrt{|c|^2 + 1} \). We set
\[
\begin{align*}
u_1 &= \frac{1}{\rho} \text{Re} b_* , \quad \nu_2 = -\frac{1}{\rho} \text{Im} b_* , \quad \nu_3 = \frac{1}{\rho} \text{Re} b_1 , \quad \nu_4 = -\frac{1}{\rho} \text{Im} b_1 ,
\end{align*}
\]
where \( \rho := \sqrt{|c|^2 + 1} \). Then \( \nu_1, \nu_2, \nu_3, \nu_4 \) form an orthonormal basis of \( E^4 \) and we obtain
\[
\Psi = \xi_1 \nu_1 - \sqrt{-1}\xi_1 \nu_2 + \xi_2 \nu_3 - \sqrt{-1}\xi_2 \nu_4 ,
\]
where
\[
\xi_1 := \rho F_* , \quad \xi_2 := \rho (-\sqrt{-1}cF_* \mp F_1).
\]
Therefore we see that there exists an element \( A \) of \( O(4) \) such that \( \Psi A \) is represented as in the form of
\[
\Psi A = (\xi_1, -\sqrt{-1}\xi_1, \xi_2, -\sqrt{-1}\xi_2).
\]
This means that \( FA \) is a holomorphic immersion of \( M \) into \( C^2 \).

**Example 4.4** Let \( F : M \rightarrow E^4 \) be a space-like and conformal immersion with zero mean curvature vector. Then \( \Psi \) can be locally represented as
\[
\Psi = (-f_*^2 + f_1^2 - f_2^2, \sqrt{-1}(f_*^2 + f_1^2 - f_2^2), 2f_*f_1, 2f_*f_2),
\]
where \( f_*, f_1, f_2 \) are holomorphic functions satisfying \( f_* \neq 0 \). Then we obtain (4.1). Therefore \( Q \) vanishes if and only if one of \( (f_1 \pm f_2)/f_* \) is constant. If \( Q \) vanishes, then a light-like normal vector field of \( F \) is contained in a constant direction. Suppose that one of \( (f_1 \pm f_2)/f_* \) is constant. If one of \( f_1 \pm f_2 \) vanishes, then
\[
\Psi = (-f_*^2, \sqrt{-1}f_*^2, 2f_*f_1, \mp 2f_*f_1)
\]
and therefore \( \xi_1 := -f_*^2, \xi_2 := 2f_*f_1 \) satisfy \( \Psi = (\xi_1, -\sqrt{-1}\xi_1, \xi_2, \mp \xi_2) \). Suppose one of \( f_1 \pm f_2 = cf_* \) for a nonzero complex number \( c \). Then \( \Psi = F_* a_* \mp \bar{F} \bar{a} \), where
\[
F_* := f_*^2, \quad \bar{F} := 2f_*f_2 , \quad a_* := (-1 + c^2, \sqrt{-1}(1 + c^2), 2c, 0), \quad \bar{a} := (c, \sqrt{-1}c, 1, \mp 1).
\]
We represent \( \bar{a} \) as
\[
\bar{a} = \frac{|c|^2}{c(1 + |c|^2)} a_* + c,
\]
where
\[
c := -\frac{1}{1 + |c|^2}(-2\text{Re} c, 2\text{Im} c, -1 + |c|^2, \pm (1 + |c|^2)).
\]
Therefore, if we set
\[ u_1 := \frac{1}{1 + |c|^2} \Re a_*, \quad u_2 := -\frac{1}{1 + |c|^2} \Im a_*, \]
then we obtain
\[ \Psi = \xi_1 u_1 - \sqrt{-1}\xi_1 u_2 + \xi_2 c, \]
where
\[ \xi_1 := (1 + |c|^2)F_* \mp \bar{\bar{F}}, \quad \xi_2 := \mp \bar{\bar{F}}. \]
We have
\[ \langle u_1, u_1 \rangle = \langle u_2, u_2 \rangle = 1, \quad \langle u_1, u_2 \rangle = \langle u_2, c \rangle = \langle u_1, c \rangle = \langle c, c \rangle = 0. \]
Therefore we see that there exists an element \( A \) of \( O(3,1) \) such that \( \Psi A \) is represented as in the form of
\[ \Psi A = (\xi_1, -\sqrt{-1}\xi_1, \xi_2, \xi_2). \]

**Example 4.5** Let \( F : M \rightarrow E_3^2 \) be a space-like and conformal immersion with zero mean curvature vector. Then \( \Psi \) can be locally represented as
\[ \Psi = (-f_*^2 - f_1^2 - f_2^2, \sqrt{-1}(f_*^2 - f_1^2 - f_2^2), 2f_* f_1, 2f_* f_2), \]
where \( f_*, f_1, f_2 \) are holomorphic functions satisfying \( f_* \neq 0 \). Then we obtain (4.1) and therefore \( Q \) vanishes if and only if one of \(( f_1 \pm \sqrt{-1}f_2) / f_* \) is constant. If \( Q \) vanishes, then \( F \) is an isotropic minimal immersion. Suppose that one of \(( f_1 \pm \sqrt{-1}f_2) / f_* \) is constant. If one of \( f_1 \pm \sqrt{-1}f_2 \) vanishes, then we obtain (4.2) and therefore \( \xi_1 := -f_*^2, \xi_2 := 2f_* f_1 \) satisfy \( \Psi = (\xi_1, -\sqrt{-1}\xi_1, \xi_2, \pm \sqrt{-1}\xi_2). \) Suppose one of \( f_1 \pm \sqrt{-1}f_2 = cf_* \) for a nonzero complex number \( c \). Then \( \Psi = F_* a_* \pm F_1 a_1 \), where
\[ F_* := f_*^2, \quad F_1 := 2f_* f_2, \]
\[ a_* := (-1 - c^2, \sqrt{-1}(1 - c^2), 2c, 0), \]
\[ a_1 := (\sqrt{-1}c, -c, -\sqrt{-1}, \pm 1). \]
Since \( F \) is space-like, we have \( |c| \neq 1 \). We set
\[ b_1 := a_1, \quad b_* := a_* - \sqrt{-1}c b_1 = (-1, \sqrt{-1}, c, \mp \sqrt{-1}c), \]
and
\[ u_1 := \frac{1}{\rho} \Re b_*, \quad u_2 := -\frac{1}{\rho} \Im b_*, \quad u_3 := \frac{1}{\rho} \Re b_1, \quad u_4 := -\frac{1}{\rho} \Im b_1, \]
where \( \rho := |1 - |c|^2|^{1/2} \). Then \( u_1, u_2, u_3, u_4 \) satisfy
\[ \langle u_i, u_j \rangle = \begin{cases} \varepsilon & (i = j = 1, 2), \\ -\varepsilon & (i = j = 3, 4), \\ 0 & (i \neq j), \end{cases} \]
where
\[ \varepsilon := \frac{1 - |c|^2}{|1 - |c|^2|}. \]

We can represent \( \Psi \) as
\[
\Psi = \xi_1 u_1 - \sqrt{-1} \xi_2 u_2 + \xi_2 u_3 - \sqrt{-1} \xi_2 u_4,
\]
where
\[ \xi_1 := \rho F_*, \quad \xi_2 := \rho \left( \sqrt{-1} c F_* \pm F_1 \right). \]

Therefore there exists an element \( A \) of \( \text{O}(2, 2) \) such that \( \Psi A \) is represented as in the form of \( (\xi_1, -\sqrt{-1} \xi_1, \xi_2, -\sqrt{-1} \xi_2) \).

**Example 4.6** Suppose that \( p, q \) are even integers. Then \( n = p + q \) is also even. Let \( F : M \rightarrow E^{n+2}_q \) be a space-like and conformal immersion with zero mean curvature vector satisfying
\[ \Psi^{2i} = -\sqrt{-1} \Psi^{2i-1} \quad (i = 1, 2, \ldots, n+2)^{\frac{1}{2}}. \] (4.3)

Then we have \( \langle \Psi_w, \Psi_w \rangle = 0 \) and therefore \( Q \) vanishes. Since
\[ \Psi^{2i-1}_{-1} = -\frac{1}{2} \left( \frac{\Psi^{2i+1}}{\Psi^1} \right)_w, \quad \Psi^{2i}_{-1} = \frac{\sqrt{-1}}{2} \left( \frac{\Psi^{2i+1}}{\Psi^1} \right)_w, \]
\( \Psi_{-1} \) is valued in \( Q_{n-1}^q \).

**Example 4.7** Suppose that \( p \) is even and that \( q = 2 \). Then \( n = p + 2 \) is even. Let \( F : M \rightarrow E^{n+2}_2 \) be a space-like and conformal immersion with zero mean curvature vector satisfying
\[ \Psi^{2i} = -\sqrt{-1} \Psi^{2i-1} \quad (i = 1, 2, \ldots, \frac{n-2}{2}), \quad \Psi^{-1} = \Psi^{n+1}, \quad \Psi^n = \Psi^{n+2}. \]

Then \( Q \) vanishes and \( \Psi_{-1} \) is valued in \( Q_{n-1}^n \).

**Example 4.8** Suppose that \( p, q \) are odd. Then \( n = p + q \) is even. Let \( F : M \rightarrow E^{n+2}_q \) be a space-like and conformal immersion with zero mean curvature vector satisfying
\[ \Psi^{2i} = -\sqrt{-1} \Psi^{2i-1} \quad (i \in \left\{ 1, 2, \ldots, \frac{p+1}{2} \right\}), \quad \Psi^{2j+p+2} = -\sqrt{-1} \Psi^{2j+p+1} \quad (j \in \left\{ 1, 2, \ldots, \frac{q-1}{2} \right\}), \quad \Psi^{p+2} = \Psi^{n+2}. \] (4.4)

Then \( Q \) vanishes and \( \Psi_{-1} \) is valued in \( Q_{n-1}^q \).
Example 4.9 Suppose that $p$ is even and that $q$ is odd. Then $n = p + q$ is odd. Let $F : M \rightarrow E^{n+2}_q$ be a space-like and conformal immersion with zero mean curvature vector satisfying

$$\Psi^{2i} = -\sqrt{-1}\Psi^{2i-1} \left( i = 1, 2, \ldots, \frac{n+1}{2} \right), \quad \Psi^{n+2} = 0.$$ 

Then $Q$ vanishes and $\Psi_{-1}$ is valued in $Q^{n-1}_q$.

Example 4.10 Suppose that $p$ is even and positive and that $q = 1$. Then $n = p + 1$ is odd. Let $F : M \rightarrow E^{n+2}_1$ be a space-like and conformal immersion with zero mean curvature vector satisfying

$$\Psi^{2i} = -\sqrt{-1}\Psi^{2i-1} \left( i = 1, 2, \ldots, \frac{n-1}{2} \right), \quad \Psi^n = \Psi^{n+2}, \quad \Psi^{n+1} = 0.$$ 

Then $Q$ vanishes and $\Psi_{-1}$ is valued in $Q^{n-1}_1$.

Example 4.11 Suppose that $p$ is odd and that $q$ is even. Then $n = p + q$ is odd. Let $F : M \rightarrow E^{n+2}_q$ be a space-like and conformal immersion with zero mean curvature vector satisfying

$$\Psi^{2i} = -\sqrt{-1}\Psi^{2i-1} \left( i = 1, 2, \ldots, \frac{p+1}{2} \right), \quad \Psi^{p+2} = 0,$$

$$\Psi^{p+2+2j} = -\sqrt{-1}\Psi^{p+2+2j} \left( j = 1, 2, \ldots, \frac{q}{2} \right).$$

Then $Q$ vanishes and $\Psi_{-1}$ is valued in $Q^{n-1}_q$.

Example 4.12 Suppose that $p$ is odd and that $q$ is even and positive. Then $n = p + q$ is odd. Let $F : M \rightarrow E^{n+2}_q$ be a space-like and conformal immersion with zero mean curvature vector satisfying

$$\Psi^{2i} = -\sqrt{-1}\Psi^{2i-1} \left( i = 1, 2, \ldots, \frac{p+1}{2} \right),$$

$$\Psi^{p+2+2j} = -\sqrt{-1}\Psi^{p+2+2j} \left( j = 1, 2, \ldots, \frac{q-2}{2} \right), \quad \Psi^{p+2} = \Psi^{p+1}, \quad \Psi^{n+2} = 0.$$ 

Then $Q$ vanishes and $\Psi_{-1}$ is valued in $Q^{n-1}_q$.

5 A lemma for $\overline{C}^{n+2}$-valued meromorphic functions

Let $\Psi = (\Psi^1, \Psi^2, \ldots, \Psi^{n+2})$ be a $\overline{C}^{n+2}$-valued meromorphic function on $M$: each $\Psi^i$ is a meromorphic function of a local complex coordinate $w$. In the next section, we need the following lemma.
Lemma 5.1 Let $\mu$ be a positive integer. Suppose
\[
\left\langle \frac{\partial^\lambda \Psi}{\partial w^\lambda}, \frac{\partial^\lambda \Psi}{\partial w^\lambda} \right\rangle = 0
\]  
(5.1)
for $\lambda = 0, 1, \ldots, \mu - 1$, where $\langle \ , \ \rangle$ is as in (2.2). In addition, suppose that $\Psi^1 + \sqrt{-1}\Psi^2$ is not identically zero on $M$. Then
\[
\sum_{i=1}^{n} \varepsilon_i \left( \frac{\partial^\mu}{\partial w^\mu} \left( \frac{\Psi^{i+2}}{\Psi^1 + \sqrt{-1}\Psi^2} \right) \right)^2 = \frac{1}{(\Psi^1 + \sqrt{-1}\Psi^2)^2} \left\langle \frac{\partial^\mu \Psi}{\partial w^\mu}, \frac{\partial^\mu \Psi}{\partial w^\mu} \right\rangle ,
\]  
(5.2)
where $\varepsilon_i$ is as in Section 2.

Proof Suppose $\mu = 1$. Then (5.1) means $\langle \Psi, \Psi \rangle = 0$. We set $X := \Psi^1 + \sqrt{-1}\Psi^2$. Then we obtain
\[
\sum_{i=1}^{n} \varepsilon_i \left( \frac{\partial}{\partial w} \left( \frac{\Psi_i^{i+2}}{\Psi^1 + \sqrt{-1}\Psi^2} \right) \right)^2 = \sum_{i=1}^{n} \varepsilon_i \left( \frac{\Psi_i^{i+2} (\Psi^1 + \sqrt{-1}\Psi^2) - \Psi_i^{i+2} (\Psi^1 + \sqrt{-1}\Psi^2)}{(\Psi^1 + \sqrt{-1}\Psi^2)^2} \right)^2
\]  
(5.3)
\[
= \frac{1}{X^2} \sum_{i=1}^{n} \varepsilon_i (\Psi_i^{i+2})^2 - 2 \frac{X_w}{X^3} \sum_{i=1}^{n} \varepsilon_i \Psi_i^{i+2} \Psi^{i+2} + \frac{X_w^2}{X^4} \sum_{i=1}^{n} \varepsilon_i (\Psi_i^{i+2})^2
\]  
\[
= \frac{1}{X^2} \sum_{i=1}^{n} \varepsilon_i (\Psi_i^{i+2})^2 + 2 \frac{X_w}{X^3} (\Psi_i^1 \Psi_i^1 + \Psi_i^2 \Psi_i^2) - \frac{X_w^2}{X^4} (\Psi_i^1)^2 + (\Psi_i^2)^2,
\]
where we use $\langle \Psi, \Psi \rangle = 0$. The sum of the second and the third terms of the right side of (5.3) is given by $((\Psi_i^1)^2 + (\Psi_i^2)^2)/X^2$. Therefore we obtain Lemma 5.1 for $\mu = 1$. Let $\nu$ be a positive integer. Suppose that Lemma 5.1 for $\mu = 1, 2, \ldots, \nu$ is true and that (5.1) holds for $\lambda = 0, 1, \ldots, \nu$. Then we obtain
\[
\sum_{i=1}^{n} \varepsilon_i \left( \frac{\partial^\lambda \xi^{i+2}}{\partial w^\lambda X^2} \right)^2 = 0
\]  
(5.4)
for $\lambda = 0, 1, \ldots, \nu - 1$, where
\[\xi^i := \Psi_i^1 (\Psi^1 + \sqrt{-1}\Psi^2) - \Psi_i^1 (\Psi^1 + \sqrt{-1}\Psi^2).\]

By (5.4), we obtain
\[
\sum_{i=1}^{n} \varepsilon_i \left( \frac{\partial^\nu+1}{\partial w^{\nu+1} X^2} \right)^2 = \sum_{i=1}^{n} \varepsilon_i \left( \frac{\partial^\nu \xi^{i+2}}{\partial w^{\nu} X^2} \right)^2 = \frac{1}{X^4} \sum_{i=1}^{n} \varepsilon_i \left( \frac{\partial^\nu \xi^{i+2}}{\partial w^{\nu}} \right)^2 ,
\]  
(5.5)
Using $\xi^2 = \sqrt{-1}\xi^1$ and (5.1) for $\lambda = 0, 1, \ldots, \nu$, we obtain
\[
\sum_{i=1}^{n} \varepsilon_i \left( \frac{\partial^\nu \xi^{i+2}}{\partial w^{\nu}} \right)^2 = \left\langle \frac{\partial^\nu \xi}{\partial w^{\nu}}, \frac{\partial^\nu \xi}{\partial w^{\nu}} \right\rangle = X^2 \left\langle \frac{\partial^{\nu+1} \Psi}{\partial w^{\nu+1}}, \frac{\partial^{\nu+1} \Psi}{\partial w^{\nu+1}} \right\rangle ,
\]  
(5.6)
where $\xi := (\xi^1, \xi^2, \ldots, \xi^{n+2})$. From (5.5) and (5.6), we obtain (5.2) for $\mu = \nu + 1$ and therefore we obtain Lemma 5.1 for $\mu = \nu + 1$. Thus we obtain Lemma 5.1 by induction.

\[ \square \]

Remark Let $n$ be even and $F : M \longrightarrow E^{n+2}$ a conformal minimal immersion. Then there exists an element $A$ of $O(n + 2)$ such that $\Psi A$ is a holomorphic immersion into $C^{n+1}_T$ if and only if (5.1) holds for any $\lambda \geq 0$ ([7, p. 173]).

6 (−$k$)-maps of $\overline{Q^{n+1}}_q$-valued meromorphic functions

6.1 Case 1: $2k \leq p + 2$

Let $\psi$ be a $\overline{C}^{n+2}$-valued meromorphic 1-form on $M$: $\psi$ is locally represented as $\psi = (\Psi^1, \Psi^2, \ldots, \Psi^{n+2})dw$ by $(n + 2)$ meromorphic functions $\Psi^1, \Psi^2, \ldots, \Psi^{n+2}$ of a local complex coordinate $w$. Suppose that $\psi$ is valued in $\overline{Q^{n-1}}_q$. We set $\Psi := (\Psi^1, \Psi^2, \ldots, \Psi^{n+2})$. Suppose that $\Psi^1 + \sqrt{-1}\Psi^2$ is not identically zero on $M$. Let $\psi_{-1}$ be a $\overline{C}^n$-valued meromorphic 1-form on $M$ defined by

$$\psi_{-1} := (\Psi_{-1}^1, \Psi_{-1}^2, \ldots, \Psi_{-1}^n)dw, \quad \psi_{i} := -\left(\frac{\Psi^{i+2}}{\Psi^1 + \sqrt{-1}\Psi^2}\right)w (i = 1, 2, \ldots, n).$$

Notice that $\Psi^{i+2}/(\Psi^1 + \sqrt{-1}\Psi^2)$ does not depend on the choice of a local complex coordinate $w$. We set $\Psi_{-1} := (\Psi_{-1}^1, \Psi_{-1}^2, \ldots, \Psi_{-1}^n)$ and we call $\Psi_{-1}$ a $(-1)$-map of $\Psi$. Referring to (3.3) or using Lemma 5.1 for $\mu = 1$, we see that $\Psi$ satisfies (5.1) for $\lambda = 1$ if and only if $\psi_{-1}$ is valued in $\overline{Q^{n-1}}_q$.

Remark In the above discussion, it seems that $\Psi^1, \Psi^2$ play special roles. However, referring to (3.3) or using Lemma 5.1 for $\mu = 1$, we see that whether $\psi_{-1}$ is valued in $\overline{Q^{n-1}}_q$ does not depend on the choice of a pair $(\Psi^i, \Psi^j)$ with $i, j \in \{1, 2, \ldots, p + 2\}, i \neq j$ and $\Psi^i + \sqrt{-1}\Psi^j \neq 0$.

In the following, suppose that $\psi_{-1}$ is valued in $\overline{Q^{n-1}}_q$. In addition, suppose $p \geq 2$ and that $\Psi_{-1}^1 + \sqrt{-1}\Psi_{-1}^2$ is not identically zero. Let $\psi_{-2}$ be a $\overline{C}^{n-2}$-valued meromorphic 1-form on $M$ defined by

$$\psi_{-2} := (\Psi_{-2}^1, \Psi_{-2}^2, \ldots, \Psi_{-2}^{n-2})dw, \quad \psi_{i-2} := -\left(\frac{\Psi^{i+2}}{\Psi_{-1}^1 + \sqrt{-1}\Psi_{-1}^2}\right)w (i = 1, 2, \ldots, n - 2).$$

We set $\Psi_{-2} := (\Psi_{-2}^1, \Psi_{-2}^2, \ldots, \Psi_{-2}^{n-2})$ and we call $\Psi_{-2}$ a $(-2)$-map of $\Psi$. This definition implies that $\Psi_{-2}$ is a $(-1)$-map of $\Psi_{-1}$. We see that $\langle (\Psi_{-1})_w, (\Psi_{-1})_w \rangle_{p,q} = 0$ if and only if $\psi_{-2}$ is valued in $\overline{Q^{n-3}}_q$, where $\langle \ , \ \rangle_{p,q}$ denotes the complexification of the metric of $E^n_q$. Therefore by Lemma 5.1 for $\mu = 2$, we see that $\Psi$ satisfies (5.1) for $\lambda = 2$ if and only if $\Psi$ has a $\overline{Q^{n-3}}_q$-valued $(-2)$-map.
Remark By Lemma 5.1 for $\mu = 1, 2$, we see that whether $\psi_{-2}$ is valued in $\Omega^{n-3}_q$ depends on neither the choice of a pair $(\Psi^i, \Psi^j)$ as in the previous remark nor the choice of a pair $(\Psi_{-1}, \Psi_{-1})$ with $i, j \in \{1, 2, \ldots, p\}, \; i \neq j$ and $\Psi_{-1} + \sqrt{-1}\Psi_{-1} \neq 0$.

For a positive integer $k$, suppose the following:

- $2k \leq p + 2$ and $2k < n + 2$;
- for each $l \in \{1, 2, \ldots, k\}$, we have a $\mathcal{C}^{n+2-2l}$-valued meromorphic 1-form $\psi_{-l}$ on $M$;
- for each $l \in \{1, 2, \ldots, k - 1\}$, $\psi_{-l}$ is valued in $\Omega^{n+1-2l}_q$;
- $\psi_{-1}, \psi_{-2}, \ldots, \psi_{-k}$ satisfy $\psi_{-l} = (\psi_{1-l})_{-1}$ for $l = 1, 2, \ldots, k$, where $\psi_0 := \psi$.

We represent $\psi_{-k}$ as $\psi_{-k} = \Psi_{-k} dw$. We call $\Psi_{-k}$ a $(-k)$-map of $\Psi$. We see that

$$\langle (\Psi_{-k+1})^w, (\Psi_{-k+1})^w \rangle_{p+4-2k, q} = 0$$

if and only if $\psi_{-k}$ is valued in $\Omega^{n+1-2k}_q$. Therefore by Lemma 5.1 for $\mu = k$, we see that $\Psi$ satisfies (5.1) for $\lambda = k$ if and only if $\Psi$ has a $\Omega^{n+1-2k}_q$-valued $(-k)$-map.

Remark By Lemma 5.1 for $\mu = 1, 2, \ldots, k$, we see that whether $\psi_{-k}$ is valued in $\Omega^{n+1-2k}_q$ does not depend on the choice of a pair $(\Psi^i_{-l}, \Psi^j_{-l})$ with $i, j \in \{1, 2, \ldots, p + 2 - 2l\}, \; i \neq j$ and $\Psi^i_{-l} + \sqrt{-1}\Psi^j_{-l} \neq 0$ for any $l \in \{0, 1, 2, \ldots, k - 1\}$.

Remark Noticing the representation formula of $k$-isotropic minimal surfaces in $E^{n+2}$ ([3]), we see that $F : M \map E^{n+2}$ is a $k$-isotropic minimal immersion if and only if for each $l \in \{1, \ldots, k\}$, $\Psi_{-l}$ is defined and valued in $\Omega^{n+1-2l}_q$.

### 6.2 Case 2: $2k > p + 2$

Suppose $q \geq 2$ and that we already have a $\Omega^{n+1-2l}_q$-valued meromorphic 1-form $\psi_{-l}$ for each $l \in \{1, 2, \ldots, [p/2] + 1\}$ satisfying $\psi_{-l} = (\psi_{1-l})_{-1}$, where $[p/2]$ denotes the maximum of integers which do not exceed $p/2$. Then we represent $\Psi_{-[p/2]-1}$ as

$$\Psi_{-[p/2]-1} = \begin{cases} 
(\tilde{\Psi}^1_{-[p/2]-1}, \tilde{\Psi}^2_{-[p/2]-1}, \ldots, \tilde{\Psi}^q_{-[p/2]-1}) & \text{if } p \text{ is even}, \\
(\tilde{\Psi}^0_{-[p/2]-1}, \tilde{\Psi}^1_{-[p/2]-1}, \tilde{\Psi}^2_{-[p/2]-1}, \ldots, \tilde{\Psi}^q_{-[p/2]-1}) & \text{if } p \text{ is odd}.
\end{cases}$$

If $p$ is even, then we set

$$\Psi^i_{-[p/2]-2} := -\left(\frac{\tilde{\Psi}^{i+2}_{-[p/2]-1}}{\tilde{\Psi}^1_{-[p/2]-1} + \sqrt{-1}\tilde{\Psi}^2_{-[p/2]-1}}\right)^w.$$
for $i = 1, 2, \ldots, q - 2$; if $p$ is odd, then we set

$$\Psi_{-[p/2]-2}^{-1} := -\left(\frac{\tilde{\Psi}_{-[p/2]-1}^0}{\Psi_{-[p/2]-1}^1 + \sqrt{-1}\Psi_{-[p/2]-1}^2}\right)_w$$

and

$$\Psi_{-[p/2]-2}^{-i} := -\left(\frac{\tilde{\Psi}_{-[p/2]-1}^{i+1}}{\Psi_{-[p/2]-1}^1 + \sqrt{-1}\Psi_{-[p/2]-1}^2}\right)_w$$

for $i = 2, 3, \ldots, q - 1$. Whether $p$ is even or odd, we set

$$\Psi_{-[p/2]-2} := \left(\Psi_{-[p/2]-2}^1, \Psi_{-[p/2]-2}^2, \ldots, \Psi_{-[p/2]-2}^{n-2[p/2]-2}\right).$$

Then we obtain a $\mathcal{O}^{n-2[p/2]-2}$-valued meromorphic 1-form $\psi_{-[p/2]-2}$ on $M$ by $\psi_{-[p/2]-2} := \Psi_{-[p/2]-2} dw$. We denote $\Psi_{-[p/2]-2} dw$ by not only $\psi_{-[p/2]-2}$ but also $(\psi_{-[p/2]-1})^{-1}$. We call $\Psi_{-[p/2]-2}$ a $(-[p/2] - 2)$-map of $\Psi$. We see that

$$\langle(\Psi_{-[p/2]-1})_w, (\Psi_{-[p/2]-1})_w\rangle_{p-2[p/2], q} = 0$$

if and only if $\psi_{-[p/2]-2}$ is valued in $\mathcal{O}^{n-2[p/2]-3}$. Therefore by Lemma 5.1, we see that $\Psi$ satisfies (5.1) for $\lambda = [p/2] + 2$ if and only if $\Psi$ has a $\mathcal{O}^{n-2[p/2]-3}$-valued $(-[p/2] - 2)$-map.

**Remark** By Lemma 5.1, we see that whether $\psi_{-[p/2]-2}$ is valued in $\mathcal{O}^{n-2[p/2]-3}$ depends on neither the choice of a pair $(\Psi_{i-l}^l, \Psi_{j-l}^j)$ with $i, j \in \{1, 2, \ldots, p + 2 - 2l\}$, $i \neq j$ and $\Psi_{i-l}^l + \sqrt{-1}\Psi_{j-l}^j \neq 0$ for any $l \in \{0, 1, 2, \ldots, [p/2]\}$, nor the choice of a pair $\left(\tilde{\Psi}_{-[p/2]-1}^i, \tilde{\Psi}_{-[p/2]-1}^j\right)$ with $i, j \in \{1, 2, \ldots, q\}$, $i \neq j$ and $\tilde{\Psi}_{-[p/2]-1}^i + \sqrt{-1}\tilde{\Psi}_{-[p/2]-1}^j \neq 0$.

For a positive integer $k$, suppose the following:

- $p + 2 < 2k < n + 2$;
- for each $l \in \{1, 2, \ldots, k\}$, we have a $\mathcal{O}^{n+2-2l}$-valued meromorphic 1-form $\psi_l$ on $M$;
- for each $l \in \{1, 2, \ldots, [p/2] + 1\}$, $\psi_l$ is valued in $\mathcal{O}^{n+1-2l}$;
- for each $l \in \{[p/2] + 2, \ldots, k - 1\}$, $\psi_l$ is valued in $\mathcal{O}^{n+1-2l}$;
- $\psi_{-1}, \psi_{-2}, \ldots, \psi_{-k}$ satisfy $\psi_{-l} = (\psi_{1-l})^{-1}$ for $l = 1, 2, \ldots, k$.

We represent $\psi_{-k}$ as $\psi_{-k} = \Psi_{-k} dw$. We call $\Psi_{-k}$ a $(-k)$-map of $\Psi$. We see that

$$\langle(\Psi_{-k+1})_w, (\Psi_{-k+1})_w\rangle_{p-2[p/2], q+2[p/2]+4-2k} = 0$$

if and only if $\psi_{-k}$ is valued in $\mathcal{O}^{n+1-2k}$. Therefore by Lemma 5.1, we see that $\Psi$ satisfies (5.1) for $\lambda = k$ if and only if $\Psi$ has a $\mathcal{O}^{n+1-2k}$-valued $(-k)$-map.
Remark By Lemma 5.1, we see that whether \( \psi_{-k} \) is valued in \( \frac{Q^{n+1-2k}}{q+2[p/2]+2-2k} \) depends on neither the choice of a pair \((\Psi^i_{-l}, \Psi^j_{-l})\) with \( i, j \in \{1, 2, \ldots, p + 2 - 2l\}, i \neq j \) and \( \Psi^i_{-l} + \sqrt{-1}\Psi^j_{-l} \neq 0 \) for any \( l \in \{0, 1, 2, \ldots, [p/2]\} \), nor the choice of a pair \((\tilde{\Psi}^i_{-l}, \tilde{\Psi}^j_{-l})\) with \( i, j \in \{1, 2, \ldots, q + 2[p/2] + 2 - 2l\}, i \neq j \) and \( \tilde{\Psi}^i_{-l} + \sqrt{-1}\tilde{\Psi}^j_{-l} \neq 0 \) for any \( l \in \{[p/2] + 1, \ldots, k - 1\}. \)

7 The main theorems

Let \( p, q \) be nonnegative integers with \( n = p + q \geq 1 \). We set

\[
K := \max\{k \in \mathbb{N} \mid 2k < n + 2\} = \left\lfloor \frac{n+1}{2} \right\rfloor, \quad a(n) := n + 2 - 2K.
\]

We define an integer \( b(p, q) \) as follows:

- if \( q = 0 \), then we set \( b(p, 0) := 0 \);
- if \( q \geq 1 \), then we set \( b(p, q) := q + 2[p/2] + 2 - 2K \).

Then we see the following:

- if \( p \) is even and if \( q = 0 \), then \((a(n), b(p, 0)) = (2, 0)\);
- if \( p, q \) are even and if \( q \) is positive, then \((a(n), b(p, q)) = (2, 2)\);
- if \( p, q \) are odd, then \((a(n), b(p, q)) = (2, 1)\);
- if \( p \) is even and if \( q \) is odd, then \((a(n), b(p, q)) = (1, 1)\);
- if \( p \) is odd and if \( q \) is even, then \((a(n), b(p, q)) = (1, 0)\).

We will prove

**Theorem 7.1** Let \( \psi \) be a \( \mathcal{C}^{n+2} \)-valued meromorphic 1-form on \( M \) which is locally represented as \( \psi = \Psi dw \). Suppose that \( p, q \) are even. Then \( \Psi_{-1} \equiv 0, \Psi_{-2} \equiv 0, \ldots, \Psi_{-K+1} \equiv 0 \) or \( \Psi \) has a \( Q^{a(n)-1}_{b(p, q)} \)-valued \((-K)\)-map if and only if there exists an element \( A \) of \( \text{O}(p+2, q) \) such that \( \hat{\Psi} := \Psi A \) satisfies

\[
\hat{\Psi}^{2i} = -\sqrt{-1}\hat{\Psi}^{2i-1} \quad (i = 1, \ldots, \frac{p + 2 - 2r}{2}, \frac{p + 4}{2}, \ldots, \frac{n + 2 - 2r}{2}),
\]

\[
\hat{\Psi}^{p+2-2r+j} = \hat{\Psi}^{n+2-2r+j} \quad (j = 1, \ldots, 2r),
\]

where \( r \) is a nonnegative integer satisfying

\[
r \leq \min\left\{ \frac{p + 2}{2}, \frac{q}{2} \right\}.
\]
Remark Theorem 7.1 says that for a space-like and conformal immersion $F : M \rightarrow E^{n+2}_q$ with zero mean curvature vector, $\psi = \partial F = \Psi dw$ satisfies $\Psi_{-1} \equiv 0$, $\Psi_{-2} \equiv 0$, $\ldots$, $\Psi_{-K+1} \equiv 0$ or a condition that $\Psi$ has a $Q(a(n)-1)_{b(p,q)}$-valued $(-K)$-map if and only if

$$F : M \rightarrow E^{n+2}_q \cong E^{p+2-2r}_p \oplus E^{4r}_2 \oplus E^{q-2r}_2$$

is represented as $F = (F_+, F_0, F_-)$, where

$$F_+ : M \rightarrow C^{p+2-2r}_p = E^{p+2-2r}_p, \quad F_- : M \rightarrow C^{2-2r}_2 = E^{q-2r}_2$$

are holomorphic maps and

$$F_0 : M \rightarrow E^{4r}_2 \cong \bigoplus_{i=1}^{2r} E^2_i$$

is represented as $F_0 = (h_1 c, \ldots, h_{2r} c)$ for harmonic functions $h_i (i = 1, \ldots, 2r)$ and a light-like vector $c \in E^2_1$.

Suppose that there exists an element $A$ of $O(p+2, q)$ as in Theorem 7.1. Then we obtain (5.1) for any nonnegative integer $\lambda$. Therefore by Lemma 5.1, we see that $\Psi$ has a $Q(a(n)-1)_{b(p,q)}$-valued $(-K)$-map, unless $\Psi_{-l} \equiv 0$ for $l = 1, 2, \ldots, K-1$. In the following, we will show that the converse is true. In particular, we suppose that $p, q$ are even and that $\Psi$ has a $Q(a(n)-1)_{b(p,q)}$-valued $(-K)$-map, and we will show the existence of such an element $A$ of $O(p+2, q)$ as in Theorem 7.1.

**Proposition 7.2** Suppose that $p$ is even and that $q = 0$. Let $\Psi = (\Psi^1, \Psi^2, \ldots, \Psi^{n+2})$ be a $C^{n+2}$-valued meromorphic function on $M$ satisfying

$$\Psi^2_i = -\sqrt{-1}\Psi^2_{-1}^{-1} \quad (i = 1, 2, \ldots, n/2). \quad (7.3)$$

Then there exists an element $A$ of $O(n+2)$ such that $\Psi A$ is represented as in the form of (4.3).

**Proof** Referring to Example 4.3, we can obtain Proposition 7.2 with $p = 2$. In the following, suppose $p \geq 4$. We can suppose that on a neighborhood of a point of $M$, $\Psi$ is represented as in (3.1) with $\varepsilon_i = 1$. Then (7.3) yields

$$\left(\frac{f_{2j-1} - \sqrt{-1}f_{2j}}{f_*}\right)_w = 0 \quad (j = 1, \ldots, n/2). \quad (7.4)$$

Therefore for each $j \in \{1, \ldots, n/2\}$, there exists a complex number $c_j$ satisfying $f_{2j-1} = c_j f_* + \sqrt{-1} f_{2j}$. Then we can represent $\Psi$ as

$$\Psi = F_* a_* + \sum_{j=1}^{n/2} F_j a_j, \quad (7.5)$$
\[ F_* := f_*^2, \quad F_j := 2f_*f_{2j} \quad (j = 1, \ldots, \frac{p}{2}), \]

\[
a_* := \left( -1 + \sum_{j=1}^{\frac{p}{2}} c_j^2, \sqrt{-1} \left( 1 + \sum_{j=1}^{\frac{p}{2}} c_j^2 \right), 2c_1, 0, 2c_2, 0, \ldots, 2c_{p/2}, 0 \right),
\]

\[
a_j := c_j \left( \sqrt{-1}, -1, 0, \ldots, 0 \right) + \sqrt{-1}e_{2j+1} + e_{2j+2},
\]

and \(e_\alpha\) is an element of \(R^{p+2}\) such that the \(\beta\)-th component is given by \(\delta_{\alpha,\beta}\). We have

\[
\langle a_i, a_j \rangle = 0, \quad \langle a_*, a_i \rangle = 0, \quad \langle a_*, a_* \rangle = 0. \tag{7.6}
\]

In particular, the length \(|\text{Re} a_j|\) of \(\text{Re} a_j\) is equal to the length \(|\text{Im} a_j|\) of \(\text{Im} a_j\) and \(\text{Re} a_j\) is orthogonal to \(\text{Im} a_j\) in \(E^{p+2}\) for each \(j \in \{1, \ldots, p/2, *\}\). We see that \(\{a_1, \ldots, a_{p/2}, a_*\}\) are linearly independent in \(C^{p+2}\). Referring to the Gram-Schmidt orthogonalization, we can obtain vectors \(b_j\) \((j = 1, \ldots, p/2)\), \(b_*\) of \(C^{p+2}\) as follows:

- we set \(b_1 := a_1\) and \(\rho_1 := |\text{Re} b_1| = |\text{Im} b_1|\);

- we set

\[
b_2 := a_2 - \frac{1}{2\rho_1^2} \langle a_2, \overline{b}_1 \rangle b_1,
\]

and noticing that (7.6) implies \(\langle b_2, b_2 \rangle = 0\), we set \(\rho_2 := |\text{Re} b_2| = |\text{Im} b_2|\);

- for \(j \in \{1, \ldots, p/2 - 1\}\), if we already have \(b_1, \ldots b_j\), then we set \(\rho_i := |\text{Re} b_i| = |\text{Im} b_i|\) for \(i = 1, \ldots, j\) and

\[
b_{j+1} := a_{j+1} - \frac{1}{2} \sum_{i=1}^{j} \frac{1}{\rho_i^2} \langle a_{j+1}, \overline{b}_i \rangle b_i,
\]

and noticing that (7.6) implies \(\langle b_{j+1}, b_{j+1} \rangle = 0\), we set \(\rho_{j+1} := |\text{Re} b_{j+1}| = |\text{Im} b_{j+1}|\);

- by the above procedure, we obtain vectors \(b_j\) \((j = 1, \ldots, p/2)\), and we set

\[
b_* := a_* - \frac{1}{2} \sum_{j=1}^{p/2} \frac{1}{\rho_j^2} \langle a_*, \overline{b}_j \rangle b_j
\]

and noticing that (7.6) implies \(\langle b_*, b_* \rangle = 0\), we set \(\rho_* := |\text{Re} b_*| = |\text{Im} b_*|\).

Noticing that \(\text{Re} b_j, \text{Im} b_j\) \((j = 1, \ldots, p/2, *)\) form an orthogonal system, we set

\[
u_1 := \frac{1}{\rho_*} \text{Re} b_*, \quad u_2 := -\frac{1}{\rho_*} \text{Im} b_*,
\]

\[
u_{2j+1} := \frac{1}{\rho_j} \text{Re} b_j, \quad u_{2j+2} := -\frac{1}{\rho_j} \text{Im} b_j \quad (j = 1, \ldots, \frac{p}{2}).
\]

Then \( \{ u_1, \ldots, u_{p+2} \} \) is an orthonormal basis of \( E^{p+2} \) and we see from (7.5) that there exist meromorphic functions \( \xi_1, \ldots, \xi_{p/2+1} \) satisfying

\[
\Psi = \sum_{j=1}^{q+1} \left( \xi_j u_{2j-1} - \sqrt{-1} \xi_j u_{2j} \right).
\]

Therefore there exists an element \( A \) of \( O(p + 2) \) such that \( \Psi A \) is represented as in (4.3). Hence we have proved Proposition 7.2. \( \square \)

**Proposition 7.3** Suppose that \( p = 0 \) and that \( q \) is an even and positive integer. Let \( \Psi = (\Psi^1, \Psi^2, \ldots, \Psi^{n+2}) \) be a \( C^{n+2} \)-valued meromorphic function on \( M \) satisfying (7.3). Then there exists an element \( A \) of \( O(2, q) \) such that \( \hat{\Psi} = \Psi A \) is represented as in the form of either (4.3) or

\[
\hat{\Psi}^1 = \hat{\Psi}^3, \quad \hat{\Psi}^2 = \hat{\Psi}^4, \quad \hat{\Psi}^{2j+4} = -\sqrt{1} \hat{\Psi}^{2j+3} \quad (j = 1, 2, \ldots, \frac{q-2}{2}).
\]  

(7.7)

**Proof** If \( q = 2 \), then referring to Example 4.5, we can find \( A \in O(2, 2) \) satisfying either

\[
\hat{\Psi}^i_{2i-1} = -\sqrt{-1} \hat{\Psi}^{2i-1}_{2i-1} \quad (i = 1, 2) \quad \text{or} \quad \hat{\Psi}^i = \hat{\Psi}^{i+2} \quad (i = 1, 2).
\]

In the following, suppose \( q \geq 4 \). We can suppose that \( \Psi \) is locally represented as in (3.1) with \( \varepsilon_i = -1 \). Then (7.3) yields (7.4) and for each \( j \in \{ 1, \ldots, n/2 \} \), there exists a complex number \( c_j \) satisfying

\[
f_{2j-1} = c_j f_s + \sqrt{-1} f_{2j}.
\]

Then we can represent \( \Psi \) as in (7.5), where

\[
F_s := f_s^2, \quad F_j := 2 f_s f_{2j} \quad (j = 1, \ldots, \frac{q}{2}),
\]

\[
a_* := \left( -1 - \sum_{j=1}^{q/2} c_j^2, \sqrt{-1} \left( 1 - \sum_{j=1}^{q/2} c_j^2 \right), 2c_1, 0, 2c_2, 0, \ldots, 2c_{q/2}, 0 \right),
\]

\[
a_j := c_j \left( -\sqrt{-1}, 1, 0, \ldots, 0 \right) + \sqrt{-1} e_{2j+1} + e_{2j+2}.
\]

We have (7.6) and we see that \( \{ a_1, \ldots, a_{q/2}, a_* \} \) are linearly independent in \( C^{q+2} \). If \( c_1 = \cdots = c_{q/2} = 0 \), then we immediately obtain Proposition 7.3. In the following, suppose that at least one of \( c_1, \ldots, c_{q/2} \) is not zero. Then we suppose \( c_{q/2} \neq 0 \). We set

\[
\tilde{a}_j := a_j - \frac{c_j}{c_{q/2}} a_{q/2} \quad (j = 1, \ldots, \frac{q}{2} - 1).
\]  

(7.8)

Then noticing that the first and the second components of \( \tilde{a}_j \) are zero for each \( j = 1, \ldots, q/2 - 1 \), we see that the real and the imaginary parts of a linear combination of \( \tilde{a}_1, \ldots, \tilde{a}_{q/2-1} \) are time-like. We set \( b_1 := \tilde{a}_1 \) and

\[
\rho_1 := \sqrt{-\langle \text{Re} b_1, \text{Re} b_1 \rangle} = \sqrt{-\langle \text{Im} b_1, \text{Im} b_1 \rangle}.
\]

By \( b_1 \) and \( \rho_1 \), we set

\[
b_2 := \tilde{a}_2 + \frac{1}{2 \rho_1^2} \langle \tilde{a}_2, \bar{b}_1 \rangle b_1.
\]

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Noticing that \( \text{Re} \, b_2 \) and \( \text{Im} \, b_2 \) are time-like, we set

\[
\rho_2 := \sqrt{-\langle \text{Re} \, b_2, \text{Re} \, b_2 \rangle} = \sqrt{-\langle \text{Im} \, b_2, \text{Im} \, b_2 \rangle}.
\]

For \( j \in \{1, \ldots, q/2 - 2\} \), if we already obtain \( b_1, \ldots, b_j \) as linear combinations of \( a_1, \ldots, a_j \), then noticing that the real and the imaginary parts are time-like, we set

\[
\rho_i := \sqrt{-\langle \text{Re} \, b_i, \text{Re} \, b_i \rangle} = \sqrt{-\langle \text{Im} \, b_i, \text{Im} \, b_i \rangle} \quad (i = 1, \ldots, j)
\]

and

\[
b_{j+1} := a_{j+1} + \frac{1}{2} \sum_{i=1}^{j} \frac{1}{\rho_i} \langle \tilde{a}_{j+1}, \tilde{b}_i \rangle b_i,
\]

\[
\rho_{j+1} := \sqrt{-\langle \text{Re} \, b_{j+1}, \text{Re} \, b_{j+1} \rangle} = \sqrt{-\langle \text{Im} \, b_{j+1}, \text{Im} \, b_{j+1} \rangle}.
\]

Then \( \text{Re} \, b_j, \text{Im} \, b_j \) \( (j = 1, \ldots, q/2 - 1) \) are time-like and form an orthogonal system. We set

\[
b_{q/2} := \frac{1}{2} \sum_{j=1}^{q-1} \rho_j^{-1} \langle a_{q/2}, \tilde{b}_j \rangle b_j.
\]

Then \( \text{Re} \, b_{q/2} \) and \( \text{Im} \, b_{q/2} \) are orthogonal to each of \( \text{Re} \, b_j \) and \( \text{Im} \, b_j \) for \( j = 1, \ldots, q/2 - 1 \). We set

\[
b_* := a_* + \sqrt{-1} \sum_{j=1}^{q} c_j a_j
\]

\[
= (-1, \sqrt{-1} c_1, \sqrt{-1} c_1, \ldots, c_{q/2}, \sqrt{-1} c_{q/2})
\]

Since \( b_* \) and \( a_j \) satisfy \( \langle b_*, a_j \rangle = \langle b_*, \bar{a}_j \rangle = 0 \), \( \text{Re} \, b_* \) and \( \text{Im} \, b_* \) are orthogonal to each of \( \text{Re} \, b_j \) and \( \text{Im} \, b_j \) for \( j = 1, \ldots, q/2 \). In addition, we obtain

\[
b_{q/2} = \frac{c_{q/2}}{c} (\sqrt{-1}c_1, c, \sqrt{-1}c_1, \ldots, \sqrt{-1}c_{q/2}),
\]

\[
c := -\sqrt{-1}(|c_1|^2 + \cdots + |c_{q/2}|^2).
\]

We see the following:

• if \( \text{Re} \, b_{q/2} \) and \( \text{Im} \, b_{q/2} \) are space-like, then \( \text{Re} \, b_* \) and \( \text{Im} \, b_* \) are time-like;

• if \( \text{Re} \, b_{q/2} \) and \( \text{Im} \, b_{q/2} \) are light-like, then \( \text{Re} \, b_* \) and \( \text{Im} \, b_* \) are light-like;

• if \( \text{Re} \, b_{q/2} \) and \( \text{Im} \, b_{q/2} \) are time-like, then \( \text{Re} \, b_* \) and \( \text{Im} \, b_* \) are space-like.

We set \( \varepsilon := 1 \) (respectively, \( -1 \)) if \( \text{Re} \, b_{q/2}, \text{Im} \, b_{q/2} \) are space-like (respectively, time-like).

Suppose that \( \text{Re} \, b_{q/2} \) and \( \text{Im} \, b_{q/2} \) are space-like or time-like. Then for

\[
u_{2j-1} := \frac{1}{\rho_j} \text{Re} \, b_j, \quad u_{2j} := -\frac{1}{\rho_j} \text{Im} \, b_j \quad (j = 1, \ldots, q/2),
\]

\[
u_{q+1} := \frac{1}{\rho_*} \text{Re} \, b_*, \quad u_{q+2} := -\frac{1}{\rho_*} \text{Im} \, b_*
\]
with
\[
\rho_{q/2} := \sqrt{\varepsilon(\operatorname{Re} b_{q/2}, \operatorname{Re} b_{q/2})} = \sqrt{\varepsilon(\operatorname{Im} b_{q/2}, \operatorname{Im} b_{q/2})},
\]
\[
\rho_* := \sqrt{-\varepsilon(\operatorname{Re} b_*, \operatorname{Re} b_*)} = \sqrt{-\varepsilon(\operatorname{Im} b_*, \operatorname{Im} b_*)},
\]
there exist meromorphic functions \(\xi_1, \ldots, \xi_{q/2+1}\) satisfying
\[
\Psi = \sum_{j=1}^{q+1} (\xi_j u_{2j-1} - \sqrt{-1} \xi_j u_{2j}).
\]

Therefore there exists an element \(A\) of \(O(2, q)\) such that \(\hat{\Psi} = \Psi A\) is represented as in (4.3). Suppose that \(\operatorname{Re} b_{q/2}\) and \(\operatorname{Im} b_{q/2}\) are light-like. Then we obtain \(c = -\sqrt{-1}\) and
\[
b_{q/2} = \sqrt{-1} c_{q/2} (-1, -\sqrt{-1}, \overline{c}_1, -\sqrt{-1} \overline{c}_1, \ldots, \overline{c}_{q/2}, -\sqrt{-1} \overline{c}_{q/2}) = \sqrt{-1} c_{q/2} \overline{b}_x.
\]

Then for
\[
u_{2j-1} := \frac{1}{\rho_j} \operatorname{Re} b_j, \quad \nu_{2j} := -\frac{1}{\rho_j} \operatorname{Im} b_j \quad (j = 1, \ldots, \frac{q}{2} - 1),
\]
\[
c := \operatorname{Re} b_*, \quad c' := \operatorname{Im} b_*,
\]
there exist meromorphic functions \(\xi_1, \ldots, \xi_{q/2-1}, \xi, \xi'\) satisfying
\[
\Psi = \sum_{j=1}^{q-1} (\xi_j u_{2j-1} - \sqrt{-1} \xi_j u_{2j}) + \xi c + \xi' c'.
\]

Therefore there exists an element \(A\) of \(O(2, q)\) such that \(\hat{\Psi} = \Psi A\) is represented as in (7.7). Hence we have proved Proposition 7.3. \(\square\)

Referring to the above proof of Proposition 7.3, we can obtain

**Proposition 7.4** Suppose that \(p, q\) are even and positive. Let \(\Psi = (\Psi^1, \Psi^2, \ldots, \Psi^{n+2})\) be a \(\mathbb{C}^{n+2}\)-valued meromorphic function on \(M\) satisfying (7.3). Then there exists an element \(A\) of \(O(p+2, q)\) such that \(\hat{\Psi} = \Psi A\) is represented as in the form of either (4.3) or (7.1) with \(r = 1\).

We will prove

**Proposition 7.5** Suppose that \(p, q\) are even and positive. Let \(\Psi = (\Psi^1, \Psi^2, \ldots, \Psi^{n+2})\) be a \(\mathbb{C}^{n+2}\)-valued meromorphic function on \(M\) satisfying
\[
\Psi_{-1}^{2i} = -\sqrt{-1} \Psi_{-1}^{2i-1} \quad (i = 1, \ldots, \frac{p-2g}{2}, \frac{p+2}{2}, \ldots, \frac{n-2g}{2}),
\]
\[
\Psi_{-1}^{p-2g+j} = \Psi_{-1}^{n-2g+j} \quad (j = 1, \ldots, 2g),
\]
where \(g\) is a positive integer satisfying \(g \leq \min\{p/2, q/2\}\). Then there exists an element \(A\) of \(O(p+2, q)\) such that \(\hat{\Psi} = \Psi A\) is represented as in the form of (7.1) with \(r = g - 1, g\) or \(g + 1\).
Proof. We represent $\Psi$ as in (3.1). Then by (7.10), we have
\[
\left(\frac{f_{2i}}{f_n}\right)_{w} = -\sqrt{-1}\left(\frac{f_{2i-1}}{f_n}\right)_{w} \quad (i = 1, \ldots, \frac{p}{2} - g, \frac{p}{2} + 1, \ldots, \frac{n}{2} - g),
\]
\[
\left(\frac{f_{p-2g+j}}{f_n}\right)_{w} = \left(\frac{f_{n-2g+j}}{f_n}\right)_{w} \quad (j = 1, \ldots, 2g).
\]

Therefore there exist complex numbers $c_i, d_j$ satisfying
\[
f_{2i-1} = c_if_* + \sqrt{-1}f_{2i}, \quad f_{p-2g+j} = d_jf_* + f_{n-2g+j}.
\]

Then we can represent $\Psi$ as
\[
\Psi = F_*a_* + \sum_{i=1}^{\frac{n}{2}-g} F_i a_i + \sum_{j=1}^{2g} F'_j a'_j + \sum_{i=\frac{p}{2}+1}^{\frac{n}{2}-g} F_i a_i,
\]
where
\[
F_* := f_*^2, \quad F_i := 2f_*f_{2i} \quad (i = 1, \ldots, \frac{p}{2} - g, \frac{p}{2} + 1, \ldots, \frac{n}{2} - g),
\]
\[
F'_j := 2f_*f_{n-2g+j} \quad (j = 1, \ldots, 2g)
\]
and
\[
a_* := \left(-1 + \sum_{i=1}^{\frac{n}{2}-g} c_i^2 - \sum_{i=\frac{p}{2}+1}^{\frac{n}{2}-g} c_i^2 + \sum_{j=1}^{2g} d_j^2\right) e_1
\]
\[
+ \sqrt{-1} \left(1 + \sum_{i=1}^{\frac{n}{2}-g} c_i^2 - \sum_{i=\frac{p}{2}+1}^{\frac{n}{2}-g} c_i^2 + \sum_{j=1}^{2g} d_j^2\right) e_2
\]
\[
+ 2 \sum_{i=1}^{\frac{n}{2}-g} c_i e_{2i+1} + 2 \sum_{j=1}^{2g} d_j e_{2+p-2g+j} + 2 \sum_{i=\frac{p}{2}+1}^{\frac{n}{2}-g} c_i e_{2i+1},
\]
\[
a_i := c_i \left(\sqrt{-1}, -1, 0, \ldots, 0\right) + \sqrt{-1} e_{2i+1} + e_{2i+2},
\]
\[
a'_j := d_j \left(1, \sqrt{-1}, 0, \ldots, 0\right) + e_{2+p-2g+j} + e_{2+n-2g+j}.
\]

We see the following:

- $\text{Re} \ a_i, \text{Im} \ a_i \ (i = 1, \ldots, p/2 - g)$ are space-like;

- if $c_i = 0$ for $i = p/2 + 1, \ldots, n/2 - g$, then $\text{Re} \ a_i, \text{Im} \ a_i \ (i = p/2 + 1, \ldots, n/2 - g)$ are time-like, and if at least one of $c_i \ (i = p/2 + 1, \ldots, n/2 - g)$ is not zero and if $c_{n/2-g} \neq 0$, then the real and the imaginary parts of
\[
\bar{a}_i := a_i - \frac{c_i}{c_{n/2-g}} a_{n/2-g} \quad (i = \frac{p}{2} + 1, \ldots, \frac{n}{2} - g - 1)
\]
are time-like.
\begin{itemize}
    \item if \(d_j = 0\) (respectively, \(d_j \neq 0\)) for \(j \in \{1, \ldots, 2g\}\), then \(\text{Re} \, a'_j\), \(\text{Im} \, a'_j\) are light-like (respectively, space-like).
\end{itemize}

We can suppose that there exists a nonnegative integer \(k\) satisfying \(d_j = 0\) for \(j = 1, \ldots, k\) and \(d_j \neq 0\) for \(j = k + 1, \ldots, 2g\). Then from \(a_i = i = 1, \ldots, p/2 - g\) and \(a'_j = (j = k + 1, \ldots, 2g)\), we obtain vectors \(b_i = i = 1, \ldots, p/2 + g - k\) such that \(\text{Re} \, b_i\), \(\text{Im} \, b_i\) are space-like and form an orthogonal system. Suppose \(c_i = 0\) for \(i = p/2 + 1, \ldots, n/2 - g\). Then \(\text{Re} \, a_s\), \(\text{Im} \, a_s\) are space-like and from \(a_s\), we obtain \(b_s\) such that \(\text{Re} \, b_s\), \(\text{Im} \, b_s\) are space-like and form an orthogonal system. Noticing Proposition 7.2, we suppose

\[
\hat{\Psi} = \Psi_{k+1} \Psi_k^{-1} \quad (k = 1, \ldots, p/2 - g - 1) \quad \text{and} \quad \hat{\Psi} = \Psi_k^{-1} \quad (k = p/2 + 1, \ldots, n/2 - g).
\]

This implies \(k \geq g\). We see that for vectors \(\hat{b}_i = a_i = i = p/2 + 1, \ldots, n/2 - g\), \(\text{Re} \, \hat{b}_i\), \(\text{Im} \, \hat{b}_i\) are time-like and form an orthogonal system. Let \(W\) be a subspace of \(E^{n+2}_q\) generated by \(\text{Re} \, \hat{b}_i\), \(\text{Im} \, \hat{b}_i\) and \(\hat{W}\) a subspace of \(E^{n+2}_q\) generated by \(\text{Re} \, \hat{b}_i\), \(\text{Im} \, \hat{b}_i\), \(i = p/2 + 1, \ldots, n/2 - g\). Then \(W\) is orthogonal to \(\hat{W}\).

Noticing \(a'_j = e_{2+p-2g+j} + e_{2+n-2g+j}\) and that \(e_{2+p-2g+j} + e_{2+n-2g+j}\) are orthogonal to both \(W\) and \(\hat{W}\), we obtain \(k = 2g\). Then we obtain (7.1) with \(r = g\). In the following, suppose \(c_{n/2-g} \neq 0\). From \(\hat{a}_i = i = p/2 + 1, \ldots, n/2 - g\), we obtain vectors \(\hat{b}_i = i = p/2 + 1, \ldots, n/2 - g\) such that \(\text{Re} \, \hat{b}_i\), \(\text{Im} \, \hat{b}_i\) are time-like and form an orthogonal system. Let \(W\) be a subspace of \(E^{n+2}_q\) generated by \(\text{Re} \, \hat{b}_i\), \(\text{Im} \, \hat{b}_i\) and \(\hat{W}\) a subspace of \(E^{n+2}_q\) generated by \(\text{Re} \, \hat{b}_i\), \(\text{Im} \, \hat{b}_i\), \(i = p/2 + 1, \ldots, n/2 - g\). Then \(W\) is orthogonal to \(\hat{W}\).

Applying Proposition 7.3 to \(\hat{\Psi} = \Psi_{-[p/2]-1} A_{-[p/2]-1}\) such that \(\hat{\Psi} = \Psi_{-[p/2]-1} A_{-[p/2]-1}\) satisfies

\[
\hat{\Psi}^2_{-[p/2]-1} = \sqrt{-1} \hat{\Psi}^{2i-1}_{-[p/2]-1} \quad (i = 1, \ldots, q/2).
\]

Applying Proposition 7.3 to \(\hat{\Psi} = \Psi_{-[p/2]} A_{-[p/2]}\) such that \(\hat{\Psi} = \Psi_{-[p/2]} A_{-[p/2]}\) satisfies

\[
\hat{\Psi}^2_{-[p/2]} = \sqrt{-1} \hat{\Psi}^{2i-1}_{-[p/2]} \quad (i = 1, \ldots, q/2 + 1)
\]

or

\[
\hat{\Psi}^3 = \hat{\Psi}^3, \quad \hat{\Psi}^2 = \hat{\Psi}^4, \quad \hat{\Psi}^{2j+4} = \sqrt{-1} \hat{\Psi}^{2j+3} \quad (j = 1, 2, \ldots, q/2 - 2).
\]

Applying Proposition 7.5 to \(\hat{\Psi} = \Psi_{-[p/2]-1} A_{-[p/2]-1}\) such that \(\hat{\Psi} = \Psi_{-[p/2]-1} A_{-[p/2]-1}\) satisfies (7.1) for a nonnegative integer \(r\) with (7.2). If \(\hat{\Psi} = 0\) for an integer \(l \in \{1, 2, \ldots, K - 1\}\), then we similarly obtain the result. Hence we have proved Theorem 7.1. \(\square\)
We will prove

**Theorem 7.6** Let $\psi$ be a $C^{n+2}$-valued meromorphic 1-form on $M$ which is locally represented as $\psi = \Psi dw$. Suppose that $p, q$ are odd. Then $\Psi_{-1} \equiv 0$, $\Psi_{-2} \equiv 0, \ldots, \Psi_{-K+1} \equiv 0$ or $\Psi$ has a $\overline{Q}_1$-valued $(-K)$-map if and only if there exists an element $A$ of $O(p+2, q)$ such that $\hat{\Psi} := \Psi A$ satisfies

$$\hat{\Psi}^{2i} = -\sqrt{-1} \hat{\Psi}^{2i-1} \quad (i = 1, \ldots, \frac{p+1-2r}{2}),$$

$$\hat{\Psi}^{2i+1} = -\sqrt{-1} \hat{\Psi}^{2i} \quad (i = \frac{p+3}{2}, \ldots, \frac{n-2r}{2}),$$

$$\hat{\Psi}^{p+1-2r+j} = \hat{\Psi}^{p+1-2r+j} \quad (j = 1, \ldots, 2r + 1),$$

where $r$ is a nonnegative integer satisfying

$$r \leq \min \left\{ \frac{p+1}{2}, \frac{q-1}{2} \right\}. \quad (7.12)$$

**Remark** Theorem 7.6 says that for a space-like and conformal immersion $F : M \rightarrow E_q^{n+2}$ with zero mean curvature vector, $\psi = \partial F = \Psi dw$ satisfies $\Psi_{-1} \equiv 0$, $\Psi_{-2} \equiv 0, \ldots, \Psi_{-K+1} \equiv 0$ or a condition that $\Psi$ has a $\overline{Q}_1$-valued $(-K)$-map if and only if

$$F : M \rightarrow E_q^{n+2} \cong E_{p+1-2r}^p \oplus E_{2r+1}^{q-1-2r}$$

is represented as $F = (F_+, F_0, F_-)$, where

$$F_+ : M \rightarrow C^{p+1-2r} = E_{p+1-2r}^p, \quad F_- : M \rightarrow C^{q-1-2r} = E_{q-1-2r}^{q-1-2r}$$

are holomorphic maps and

$$F_0 : M \rightarrow E_{2r+1}^{4r+2} \cong \bigoplus_{i=1}^{2r+1} E_1^2$$

is represented as $F_0 = (h_1 e, \ldots, h_{2r+1} e)$ for harmonic functions $h_i (i = 1, \ldots, 2r + 1)$ and a light-like vector $c \in E_1^2$.

If there exists an element $A$ of $O(p+2, q)$ as in Theorem 7.6, then we see that $\Psi$ has a $\overline{Q}_1$-valued $(-K)$-map, unless $\Psi_{-l} \equiv 0$ for $l = 1, 2, \ldots, K-1$. The main part of the proof of Theorem 7.6 consists of discussions on the existence of $A \in O(p+2, q)$ as in Theorem 7.6 for odd $p, q$ and $\Psi$ with a $\overline{Q}_1$-valued $(-K)$-map.

**Proposition 7.7** Suppose that $p$ is odd and that $q = 1$. Let $\Psi = (\Psi^1, \Psi^2, \ldots, \Psi^{n+2})$ be a $C^{n+2}$-valued meromorphic function on $M$ satisfying

$$\Psi_{-1}^{2i} = -\sqrt{-1} \Psi_{-1}^{2i-1} \quad (i = 1, 2, \ldots, \frac{p-1}{2}), \quad \Psi_{-1}^p = \Psi_{-1}^{p+1}. \quad (7.13)$$

Then there exists an element $A$ of $O(n+1, 1)$ such that $\Psi A$ is represented as in the form of (4.4) with $q = 1.$
We set
\[ c_j := \left( -1 + \sum_{j=1}^{\frac{p-1}{2}} c_j^2 + \tilde{c}^2 \right) e_1 + \sqrt{1} \left( 1 + \sum_{j=1}^{\frac{p-1}{2}} c_j^2 + \tilde{c}^2 \right) e_2 + 2 \sum_{i=1}^{\frac{p-1}{2}} c_i e_{2i+1} + 2 \tilde{c} e_{n+1}, \]
\[ a_j := c_j \left( \sqrt{-1}, -1, 0, \ldots, 0 \right) + \sqrt{1} e_{2j+1} + e_{2j+2}, \]
\[ \tilde{a} := \tilde{c} \left( 1, \sqrt{-1}, 0, \ldots, 0 \right) + e_{n+1} + e_{n+2}. \]

We have
\[ \langle a_i, a_j \rangle = \langle a_*, a_i \rangle = \langle a_*, a_* \rangle = \langle a_i, \tilde{a} \rangle = \langle a_*, \tilde{a} \rangle = \langle \tilde{a}, \tilde{a} \rangle = 0. \] (7.15)

For \( a_1, \ldots, a_{\frac{p-1}{2}}, a_* \), let \( b_1, \ldots, b_{\frac{p-1}{2}}, b_* \) be vectors obtained as in the proof of Proposition 7.2. Then there exist complex numbers \( \alpha_j \) \( (j = 1, \ldots, (p-1)/2, *) \) satisfying
\[ \text{Im} \tilde{a} = \text{Im} \left( \alpha_* b_* + \sum_{j=1}^{\frac{p-1}{2}} \alpha_j b_j \right). \]

We set
\[ c := \tilde{a} - \alpha_* b_* - \sum_{j=1}^{\frac{p-1}{2}} \alpha_j b_j. \]

Then \( c \) is a real vector, and we see by (7.15) that \( c \) is zero or light-like and that \( c \) satisfies \( \langle c, a_j \rangle = \langle c, a_* \rangle = 0 \). Therefore we see by (7.14) that \( \Psi \) can be represented as in the form of
\[ \Psi = \sum_{j=1}^{\frac{p-1}{2}} \left( \xi_j u_{2j-1} - \sqrt{-1} \xi_j u_{2j} \right) + \xi c, \]

Proof Referring to Example 4.4, we can obtain Proposition 7.7 with \( p = 1 \). In the following, suppose \( p \geq 3 \). We can suppose that on a neighborhood of a point of \( M \), \( \Psi \) is represented as in (3.1) with \( \varepsilon_i = 1 \) \( (i = 1, \ldots, n-1) \), \( -1 \) \( (i = n) \). Then (7.13) yields
\[ \left( \frac{f_{2j-1} - \sqrt{-1} f_{2j}}{f_*} \right)_w = 0 \quad (j = 1, \ldots, \frac{p-1}{2}), \quad \left( \frac{f_p - f_{p+1}}{f_*} \right)_w = 0. \]

Therefore we see that for each \( j \in \{1, \ldots, (p-1)/2\} \), there exists a complex number \( c_j \) satisfying \( f_{2j-1} = c_j f_* + \sqrt{-1} f_{2j} \) and that there exists a complex number \( \tilde{c} \) satisfying \( f_p = \tilde{c} f_* + f_{p+1} \). Then we can represent \( \Psi \) as
\[ \Psi = F_* a_* + \sum_{j=1}^{\frac{p-1}{2}} F_j a_j + \tilde{F} \tilde{a}, \quad (7.14) \]
where
\[ F_* := f_*^2, \quad F_j := 2 f_* f_{2j} \quad (j = 1, \ldots, \frac{p-1}{2}), \quad \tilde{F} := 2 f_* f_{p+1}, \]
\[ a_* := \left( -1 + \sum_{j=1}^{\frac{p-1}{2}} c_j^2 + \tilde{c}^2 \right) e_1 + \sqrt{1} \left( 1 + \sum_{j=1}^{\frac{p-1}{2}} c_j^2 + \tilde{c}^2 \right) e_2 + 2 \sum_{i=1}^{\frac{p-1}{2}} c_i e_{2i+1} + 2 \tilde{c} e_{n+1}, \]
\[ a_j := c_j \left( \sqrt{-1}, -1, 0, \ldots, 0 \right) + \sqrt{1} e_{2j+1} + e_{2j+2}, \]
\[ \tilde{a} := \tilde{c} \left( 1, \sqrt{-1}, 0, \ldots, 0 \right) + e_{n+1} + e_{n+2}. \]
where $\xi_j (j = 1, \ldots, (p + 1)/2)$ and $\xi$ are meromorphic functions. Then there exists an element $A$ of $O(n + 1, 1)$ such that $\Psi A$ is represented as in (4.4) with $q = 1$. Hence we have proved Proposition 7.7.

**Proof of Theorem 7.6** Noticing Proposition 7.7, we suppose $q \geq 3$. We have $K = n/2$ and $(a(n), b(p, q)) = (2, 1)$. Suppose that $\Psi$ has a $\overline{O(1)}$-valued $(-K)$-map. Then we can suppose $\Psi^2_K = \Psi^1_K$. Then applying analogues of Proposition 7.7 to $\Psi_{-l}$ $(l = K - 1, \ldots, [p/2] + 1)$, we see that there exists an element $A_{- [p/2] - 1}$ of $O(1, q)$ such that

\[
\hat{\Psi}_{- [p/2] - 1} := \Psi_{- [p/2] - 1} A_{- [p/2] - 1}
\]

satisfies

\[
\hat{\Psi}^1_{- [p/2] - 1} = \hat{\Psi}^2_{- [p/2] - 1}, \quad \hat{\Psi}^{2i+2}_{- [p/2] - 1} = -\sqrt{-1} \hat{\Psi}^{2i+1}_{- [p/2] - 1} \quad (i = 1, \ldots, q - 1/2).
\]

Applying an analogue of Proposition 7.3 to $\Psi_{- [p/2]}$, we see that there exists an element $A_{- [p/2]}$ of $O(3, q)$ such that $\hat{\Psi}_{- [p/2]} := \Psi_{- [p/2]} A_{- [p/2]}$ satisfies

\[
\hat{\Psi}^2_{- [p/2]} = -\sqrt{-1} \Psi^1_{- [p/2]}, \quad \hat{\Psi}^3_{- [p/2]} = \hat{\Psi}^4_{- [p/2]},
\]

\[
\hat{\Psi}^{2i}_{- [p/2]} = -\sqrt{-1} \hat{\Psi}^{2i-1}_{- [p/2]} \quad (i = 3, \ldots, q + 3/2)
\]

or

\[
\hat{\Psi}^i = \hat{\Psi}^{i+3} (i = 1, 2, 3), \quad \hat{\Psi}^{2j+6} = -\sqrt{-1} \hat{\Psi}^{2j+5} (j = 1, 2, \ldots, q - 3/2).
\]

Applying analogues of Proposition 7.5 to $\Psi_{- l}$ $(l = [p/2] - 1, \ldots, 1, 0)$, we see that there exists an element $A$ of $O(p + 2, q)$ such that $\hat{\Psi} := \Psi A$ satisfies (7.11) for a nonnegative integer $r$ with (7.12). If $\Psi_{- l} \equiv 0$ for an integer $l \in \{1, 2, \ldots, K - 1\}$, then we similarly obtain the result. Hence we have proved Theorem 7.6.

Referring to the proof of Theorem 7.1, we can obtain

**Theorem 7.8** Let $\psi$ be a $\overline{C^{n+2}}$-valued meromorphic 1-form on $M$ which is locally represented as $\psi = \Psi dw$. Suppose that $p$ is even and that $q$ is odd. Then $\Psi_{- 1} \equiv 0$, $\Psi_{- 2} \equiv 0, \ldots, \Psi_{- K+1} \equiv 0$ or $\Psi_{- K} \equiv 0$ if and only if there exists an element $A$ of $O(p + 2, q)$ such that $\hat{\Psi} := \Psi A$ satisfies

\[
\hat{\Psi}^{2i} = -\sqrt{-1} \hat{\Psi}^{2i-1} \quad (i = 1, \ldots, p + 2 - 2r), \quad \hat{\Psi}^{p+2-2r+j} = \hat{\Psi}^{n+1-2r+j} \quad (j = 1, \ldots, 2r), \quad \hat{\Psi}^{n+2} = 0,
\]

where $r$ is a nonnegative integer satisfying $r \leq \min\{(p + 2)/2, (q - 1)/2\}$, or

\[
\hat{\Psi}^{2i} = -\sqrt{-1} \hat{\Psi}^{2i-1} \quad (i = 1, \ldots, p - 2r), \quad \hat{\Psi}^{p-2r+j} = \hat{\Psi}^{n+1-2r+j} \quad (j = 1, 2, \ldots, 2r + 1), \quad \hat{\Psi}^{p+2} = 0,
\]

where $r$ satisfies $r \leq \min\{p/2, (q - 1)/2\}$.  

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Remark} Theorem 7.8 says that for a space-like and conformal immersion $F : M \to E^{n+2}$ with zero mean curvature vector, $\psi = \partial F = \Psi dw$ satisfies $\Psi_{-1} \equiv 0$, $\Psi_{-2} \equiv 0, \ldots, \Psi_{-K+1} \equiv 0$ or $\Psi_{-K} \equiv 0$ if and only if one of the following holds:

- the immersion

$$F : M \to E^{n+2} \cong E^{p+2-2r} \oplus E^{4r}_2 \oplus E^{q-1-2r}_q \oplus E^1_1$$

is represented as $F = (F_+, F_0, F_-, 0)$, where

$$F_+ : M \to C^{p+2-2r} = E^{p+2-2r}, \quad F_- : M \to C^{q-1-2r} = E^{q-1-2r}_q$$

are holomorphic maps and

$$F_0 : M \to E^{4r}_2 \cong \bigoplus_{i=1}^{2r} E^i_1$$

is represented as $F_0 = (h_1 c, \ldots, h_{2r} c)$ for harmonic functions $h_i (i = 1, \ldots, 2r)$ and a light-like vector $c \in E^2$;

- the immersion

$$F : M \to E^{n+2} \cong E^{p-2r} \oplus E^{4r+2}_{2r+1} \oplus E^{q-1-2r}_q \oplus E^1_1$$

is represented as $F = (F_+, F_0, F_-, 0)$, where

$$F_+ : M \to C^{p-2r} = E^{p-2r}, \quad F_- : M \to C^{q-1-2r} = E^{q-1-2r}_q$$

are holomorphic maps and

$$F_0 : M \to E^{4r+2}_{2r+1} \cong \bigoplus_{i=1}^{2r+1} E^i_1$$

is represented as $F_0 = (h_1 c, \ldots, h_{2r+1} c)$ for harmonic functions $h_i (i = 1, \ldots, 2r+1)$ and a light-like vector $c \in E^2$.

Referring to the proof of Theorem 7.6, we can obtain

**Theorem 7.9** Let $\psi$ be a $C^{n+2}$-valued meromorphic 1-form on $M$ which is locally represented as $\psi = \Psi dw$. Suppose that $p$ is odd and that $q$ is even. Then $\Psi_{-1} \equiv 0$, $\Psi_{-2} \equiv 0, \ldots, \Psi_{-K+1} \equiv 0$ or $\Psi_{-K} \equiv 0$ if and only if there exists an element $A$ of $O(p+2, q)$ such that $\hat{\Psi} := \Psi A$ satisfies

$$\hat{\Psi}^{2i} = -\sqrt{1} \hat{\Psi}^{2i-1} \left( i = 1, \ldots, \frac{p+1-2r}{2} \right),$$

$$\hat{\Psi}^{2i+1} = -\sqrt{1} \hat{\Psi}^{2i} \left( i = \frac{p+3}{2}, \ldots, \frac{n+1-2r}{2} \right),$$

$$\hat{\Psi}^{p+1-2r+j} = \hat{\Psi}^{n+2-2r+j} \quad (j = 1, \ldots, 2r), \quad \hat{\Psi}^{p+2} = 0,$$
where $r$ is a nonnegative integer satisfying $r \leq \min \{ (p+1)/2, q/2 \}$, or

$$\hat{\Psi}^{2i} = -\sqrt{-1} \hat{\Psi}^{2i-1} \quad \left( i = 1, \ldots, \frac{p+1-2r}{2} \right),$$

$$\hat{\Psi}^{2i+1} = -\sqrt{-1} \hat{\Psi}^{2i} \quad \left( i = \frac{p+3}{2}, \ldots, \frac{n-1-2r}{2} \right),$$

$$\hat{\Psi}^{n+1-2r+j} = \hat{\Psi}^{n-2r+j} \quad (j = 1, 2, \ldots, 2r + 1), \quad \hat{\Psi}^{n+2} = 0,$$

where $r$ satisfies $r \leq \min \{ (p+1)/2, (q-2)/2 \}$.

**Remark** Theorem 7.9 says that for a space-like and conformal immersion $F : M \rightarrow E^{n+2}_q$ with zero mean curvature vector, $\psi = \partial F = \Psi dw$ satisfies $\Psi^{-1} \equiv 0$, $\Psi^{-2} \equiv 0$, ..., $\Psi^{-K+1} \equiv 0$ or $\Psi^{-K} \equiv 0$ if and only if one of the following holds:

- the immersion
  $$F : M \rightarrow E^{n+2}_q \cong E^{p+1-2r}_2 \oplus E^{q-2r}_2 \oplus E^{4r}_2$$
  is represented as $F = (F_+, F_0, F_-)$, where
    $$F_+ : M \rightarrow \mathbb{C}^{p+1-2r}_2 = E^{p+1-2r}_2, \quad F_- : M \rightarrow \mathbb{C}^{q-2r}_2 = E^{q-2r}_2$$
  are holomorphic maps and
    $$F_0 : M \rightarrow E^{4r}_2 \cong \bigoplus_{i=1}^{2r} E^2_1$$
  is represented as $F_0 = (h_1 c, \ldots, h_{2r} c)$ for harmonic functions $h_i$ ($i = 1, \ldots, 2r$) and a light-like vector $c \in E^2_1$;

- the immersion
  $$F : M \rightarrow E^{n+2}_q \cong E^{p+1-2r}_2 \oplus E^{4r+2r+1}_2 \oplus E^{q-2r-1}_2 \oplus E^{1}_1$$
  is represented as $F = (F_+, F_0, F_-)$, where
    $$F_+ : M \rightarrow \mathbb{C}^{p+1-2r}_2 = E^{p+1-2r}_2, \quad F_- : M \rightarrow \mathbb{C}^{q-2r}_2 = E^{q-2r}_2$$
  are holomorphic maps and
    $$F_0 : M \rightarrow E^{4r+2r+1}_2 \cong \bigoplus_{i=1}^{2r+1} E^2_1$$
  is represented as $F_0 = (h_1 c, \ldots, h_{2r+1} c)$ for harmonic functions $h_i$ ($i = 1, \ldots, 2r + 1$) and a light-like vector $c \in E^2_1$.

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References


