Surfaces with zero mean curvature vector in neutral 4-manifolds

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Abstract Space-like surfaces and time-like surfaces with zero mean curvature vector in oriented neutral 4-manifolds are isotropic and compatible with the orientations of the spaces if and only if their lifts to the space-like and the time-like twistor spaces respectively are horizontal. In neutral Kähler surfaces and paraKähler surfaces, complex curves and para-complex curves respectively are such surfaces and characterized by one additional condition. In neutral 4-dimensional space forms, the holomorphic quartic differentials defined on such surfaces vanish. There exist time-like surfaces with zero mean curvature vector and zero holomorphic quartic differential which are not compatible with the orientations of the spaces and the conformal Gauss maps of time-like surfaces of Willmore type and their analogues give such surfaces.

1 Introduction

Space-like surfaces with zero mean curvature vector are interesting objects of study in various 4-dimensional spaces. In 4-dimensional Riemannian space forms, isotropicity of such surfaces is just a condition that holomorphic quartic differentials defined on the surfaces vanish and characterized by horizontality of their twistor lifts. Bryant showed that minimal spheres in $S^4$ are just given by horizontal lifts from $S^2$ into the twistor space $\mathbb{C}P^3$ associated with $S^4$ ([5]). Friedrich showed that in an oriented 4-dimensional Riemannian manifold $N$, isotropic minimal surfaces compatible with the orientation of $N$ are just given by horizontal lifts into the twistor space $\hat{N}$ associated with $N$ ([9]). See [8] for construction of the twistor space and studies of isotropic minimal surfaces in various 4-dimensional spaces in terms of twistor lifts. If $N$ is a Kähler surface, then a complex curve in $N$ is just an isotropic minimal surface which is with at least one complex point and compatible with the orientation of $N$ ([2]). In particular, if $N$ is hyperKähler, then an isotropic minimal surface in $N$ compatible with the orientation of $N$ is just a complex curve with respect to a complex structure given by the hyperKähler structure of $N$ ([9], [2]). In 4-dimensional Lorentz space forms, a holomorphic quartic differential defined on a space-like surface with zero mean curvature vector vanishes if and only if a light-like normal vector field is contained in a constant direction. Almost all space-like surfaces with zero mean curvature vector in the de Sitter 4-space $S^4_1$ are given by the conformal Gauss maps of Willmore surfaces in 3-dimensional Riemannian space forms.
holomorphic quartic differentials defined on Willmore surfaces coincide with holomorphic quartic differentials given by the conformal Gauss maps up to nonzero constants ([3]). A Willmore surface with zero holomorphic quartic differential has a property that a light-like normal vector field of the conformal Gauss map is contained in a constant direction and this is important: based on it, Bryant showed that Willmore spheres in $S^3$ are given by complete minimal surfaces in $E^3$ with finite total curvature and embedded flat ends ([6]). Almost all space-like surfaces with zero mean curvature vector in the anti-de Sitter 4-space $H^4_1$ are given by the conformal Gauss maps of Willmore surfaces in 3-dimensional Lorentz space forms and holomorphic quartic differentials defined on Willmore surfaces coincide with holomorphic quartic differentials given by the conformal Gauss maps up to nonzero constants ([3]). We can have analogous discussions and results for space-like surfaces with zero mean curvature vector in the Minkowski 4-space $E^4_1$ ([3]).

In the present paper, we will study space-like surfaces with zero mean curvature vector in neutral 4-dimensional manifolds (pseudo-Riemannian manifolds with signature (2,2)). The space-like twistor space $\hat{N}_+$ associated with an oriented neutral 4-dimensional manifold $N$ will be defined and we will characterize space-like surfaces in $N$ which are with zero mean curvature vector, isotropic and compatible with the orientation of $N$, by horizontality of their lifts into $\hat{N}_+$. This characterization is an analogue of Friedrich’s result mentioned in the previous paragraph. In a neutral Kähler surface, we will obtain a characterization of complex curves in terms of isotropicity and this characterization is an analogue of the above characterization of complex curves in a Kähler surface. In particular, in a neutral hyperKähler 4-manifold, we can obtain a characterization of space-like surfaces which are with zero mean curvature vector, isotropic and compatible with the orientation of the space in terms of the neutral hyperKähler structure. In the flat neutral space form $E^2_3$, using the neutral hyperKähler structure of $E^2_3$, we will characterize complex curves in terms of the induced metrics and holomorphic cubic differentials, and this is an analogue of a result in [1]. In neutral space forms $S^4_2, E^4_2, H^4_2$, holomorphic quartic differentials are defined on space-like surfaces with zero mean curvature vector ([3]) and a condition that the differentials vanish is equivalent to horizontality of the lifts of the surfaces into the space-like twistor spaces associated with $S^4_2, E^4_2, H^4_2$.

Further more parts of the present paper are devoted to studies of time-like surfaces with zero mean curvature vector in neutral 4-dimensional manifolds. The time-like twistor space $\hat{N}_-$ associated with an oriented neutral 4-dimensional manifold $N$ can be defined (see [10], [11] for the reflector space associated with an even-dimensional manifold with neutral signature) and we can characterize time-like surfaces in $N$ which are with zero mean curvature vector, isotropic and compatible with the orientation of $N$, by horizontality of their lifts into $\hat{N}_-$. In a paraKähler surface, we will obtain a characterization of paracomplex curves in terms of isotropicity. In particular, in a neutral hyperKähler 4-manifold, we can obtain a characterization of isotropicity of time-like surfaces which are with zero mean curvature vector in neutral 4-dimensional manifolds.
curvature vector, isotropic and compatible with the orientation of the space in terms of the neutral hyperKähler structure. Paracomplex curves in $E^4_2$ will be characterized in terms of the induced metrics and holomorphic cubic differentials. In $S^4_2$, $E^4_2$, $H^4_2$, holomorphic quartic differentials will be defined on time-like surfaces with zero mean curvature vector, in terms of the paracomplex structures of the surfaces. We will see that a condition that the differentials vanish is not equivalent to horizontality of the lifts of the surfaces into the time-like twistor spaces: there exist time-like surfaces in $S^4_2$, $E^4_2$, $H^4_2$ with zero mean curvature vector and zero holomorphic quartic differential such that the covariant derivatives of the lifts are light-like. The conformal Gauss maps of time-like surfaces in 3-dimensional Lorentz space forms $S^3_1$, $E^3_1$, $H^3_1$ are maps from the surfaces into $S^4_2$. We will obtain equations for surfaces such that the mean curvature vectors of the conformal Gauss maps vanish and we will say that such surfaces are of Willmore type. Holomorphic quartic differentials will be defined on the time-like surfaces of Willmore type and we will see that they coincide with the holomorphic quartic differentials given by the conformal Gauss maps up to nonzero constants, if the differentials are not null. In addition, we will see that the holomorphic quartic differential on a time-like surface of Willmore type vanishes if and only if a light-like normal vector field of the conformal Gauss map of the surface is contained in a constant direction and this condition means that for almost all such time-like surfaces of Willmore type, the covariant derivatives of the lifts of their conformal Gauss maps are light-like. We will define the conformal Gauss maps of time-like surfaces in suitable 3-dimensional spaces into $H^4_2$ or $E^4_2$ in similar ways and have analogous discussions. In particular, we will have the definitions, discussions and results on holomorphic quartic differentials.

2 $SO(2, 2)$-invariant subspaces of the space of 2-vectors

Let $X$ be an oriented 4-dimensional vector space and suppose that $X$ has a symmetric and indefinite bilinear form $h_X$ with signature (2,2). Then there exists an ordered basis $(e_1, e_2, e_3, e_4)$ of $X$ which gives the orientation of $X$ and satisfies $h_X(e_i, e_j) = 0$ ($i \neq j$), $h_X(e_i, e_i) = 1$ ($i = 1, 2$), $h_X(e_j, e_j) = -1$ ($j = 3, 4$). Let $B_X$ be the set of ordered bases of $X$ as $(e_1, e_2, e_3, e_4)$.

The bilinear form $h_X$ defines a bilinear form $\hat{h}_X$ of the 2-fold exterior power $\Lambda^2 X$ so that

$$\hat{h}_X(e_i \wedge e_j, e_k \wedge e_l) = h_X(e_i, e_k)h_X(e_j, e_l) - h_X(e_i, e_l)h_X(e_j, e_k).$$

We set $\omega_{ij} := e_i \wedge e_j$ and

$$E_{\pm, 1} := \frac{1}{\sqrt{2}}(\omega_{12} \pm \omega_{34}), \quad E_{\pm, 2} := \frac{1}{\sqrt{2}}(\omega_{13} \pm \omega_{42}), \quad E_{\pm, 3} := \frac{1}{\sqrt{2}}(\omega_{14} \pm \omega_{23}).$$
Then arbitrarily distinct two of \( E_{+,1}, E_{-,1}, E_{+,2}, E_{-,2}, E_{+,3}, E_{-,3} \) are orthogonal to each other with respect to \( \hat{h}_X \) and we obtain
\[
\hat{h}_X(E_{\pm,1}, E_{\pm,1}) = 1, \quad \hat{h}_X(E_{\pm,2}, E_{\pm,2}) = \hat{h}_X(E_{\pm,3}, E_{\pm,3}) = -1.
\]
Hence we see that \( \hat{h}_X \) is a symmetric and indefinite bilinear form of \( \wedge^2 X \) with signature \((2, 4)\).

Let \( T \) be a linear transformation of \( X \) which preserves the orientation of \( X \) and the bilinear form \( h_X \). Then for each \( (e_1, e_2, e_3, e_4) \in \mathcal{B}_X \), \((T(e_1), T(e_2), T(e_3), T(e_4))\) is an element of \( \mathcal{B}_X \). Therefore there exists an element \( A \) of \( SO(2, 2) \) satisfying
\[
(T(e_1) \quad T(e_2) \quad T(e_3) \quad T(e_4)) = (e_1 \quad e_2 \quad e_3 \quad e_4)A.
\]
Let \( \Phi_T \) be a linear transformation of \( \wedge^2 X \) defined by \( \Phi_T(\omega_{ij}) := T(e_i) \wedge T(e_j) \). We see that \( \Phi_T \) is determined by \( T \) and does not depend on the choice of \( (e_1, e_2, e_3, e_4) \). For a linear transformation \( T' \) of \( X \) which preserves the orientation of \( X \) and the bilinear form \( h_X \), we have \( \Phi_{T \circ T'} = \Phi_T \circ \Phi_{T'} \). We naturally consider \( SU(1, 1) \) to be a subgroup of \( SO(2, 2) \):
\[
G_+ := \left\{ B = \begin{pmatrix} b_1 & -b_2 & b_3 & b_4 \\ b_2 & b_1 & -b_4 & b_3 \\ b_3 & -b_4 & b_1 & b_2 \\ b_4 & b_3 & -b_2 & b_1 \end{pmatrix} \bigg| \begin{array}{c} b_1, b_2, b_3, b_4 \in \mathbb{R}, \\ b_1^2 + b_2^2 - b_3^2 - b_4^2 = 1 \end{array} \right\}
\]
is isomorphic to \( SU(1, 1) \) and a subgroup of \( SO(2, 2) \). We also consider \( SO(1, 2) \) to be a subgroup of \( SO(2, 2) \):
\[
H := \left\{ C = (c_{ij}) \in SO(1, 2) \bigg| \begin{array}{c} 1 \quad 0 \quad 0 \quad 0 \\ 0 \quad c_{11} \quad c_{12} \quad c_{13} \\ 0 \quad c_{21} \quad c_{22} \quad c_{23} \\ 0 \quad c_{31} \quad c_{32} \quad c_{33} \end{array} \right\}
\]
is a subgroup of \( SO(2, 2) \). We denote by \( C \) not only an element of \( SO(1, 2) \) but also an element of \( H \). We see the following:

- if \( A \) is an element \( B \) of \( G_+ \), then
  \[
  (\Phi_T(E_{-,1}) \quad \Phi_T(E_{+,2}) \quad \Phi_T(E_{+,3})) = (E_{-,1} \quad E_{+,2} \quad E_{+,3}),
  \]
  \[
  (\Phi_T(E_{+,1}) \quad \Phi_T(E_{-,2}) \quad \Phi_T(E_{-,3})) = (E_{+,1} \quad E_{-,2} \quad E_{-,3})\tilde{B}
  \]
  for an element \( \tilde{B} \) of \( SO_0(1, 2) \);

- if \( A \) is an element \( C \) of \( H \), then
  \[
  (\Phi_T(E_{+,1}) \quad \Phi_T(E_{+,2}) \quad \Phi_T(E_{+,3})) = (E_{+,1} \quad E_{+,2} \quad E_{+,3})C.
  \]
We see that there exists a subgroup $G_-$ of $SO(2,2)$ which is conjugate to $G_+$ and satisfies
\[
(\Phi_T(E_{+1}) \Phi_T(E_{-1}) \Phi_T(E_{-3})) = (E_{+1} E_{-2} E_{-3})
\]
for any $A = B \in G_-$. Let $\wedge_{+}^2 X$, $\wedge_{-}^2 X$ be subspaces of $\wedge^2 X$ generated by $E_{-1}, E_{+2}, E_{+3}$, and $E_{+1}, E_{-2}, E_{-3}$, respectively:
\[
\wedge_{+}^2 X := \langle E_{-1}, E_{+2}, E_{+3} \rangle, \quad \wedge_{-}^2 X := \langle E_{+1}, E_{-2}, E_{-3} \rangle.
\]
Then noticing $SO(2,2) = G_+ H = G_- H$, we see that $\wedge_{+}^2 X$ do not depend on the choice of $(e_1, e_2, e_3, e_4) \in \mathcal{B}_X$. We see
\begin{itemize}
  \item $\wedge_{-}^2 X = \wedge_{+}^2 X \oplus \wedge_{-}^2 X$, and $\wedge_{+}^2 X$ is orthogonal to $\wedge_{-}^2 X$ with respect to $\hat{h}_X$;
  \item the restriction of $\hat{h}_X$ on $\wedge_{\pm}^2 X$ for $\varepsilon \in \{+, -\}$ has signature $(1, 2)$.
\end{itemize}

We set
\[
U_+ (\wedge_{+}^2 X) := \{ \omega \in \wedge_{+}^2 X \mid \hat{h}_X(\omega, \omega) = 1 \}, \quad U_- (\wedge_{-}^2 X) := \{ \omega \in \wedge_{-}^2 X \mid \hat{h}_X(\omega, \omega) = -1 \}.
\]

Let $H_+, H_-$ be subgroups of $H$ defined by
\[
H_+ := \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \theta & -\sin \theta \\ 0 & 0 & \sin \theta & \cos \theta \end{pmatrix} \middle| \theta \in [0, 2\pi) \right\}, \quad H_- := \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh t & 0 & \sinh t \\ 0 & 0 & 1 & 0 \\ 0 & \sinh t & 0 & \cosh t \end{pmatrix} \middle| t \in \mathbb{R} \right\}.
\]

Then for $\varepsilon \in \{+, -\}$, $G'_\varepsilon := G_\varepsilon H_\varepsilon$ is a subgroup of $SO(2,2)$. For $e = (e_1, e_2, e_3, e_4)$, $e' = (e'_1, e'_2, e'_3, e'_4) \in \mathcal{B}_X$, we write $e \sim \varepsilon e'$ if there exists $A \in G'_\varepsilon$ satisfying $e' = eA$. We see that $\sim_\varepsilon$ is an equivalence relation in $\mathcal{B}_X$. For $e \in \mathcal{B}_X$, we denote by $[e]_\varepsilon$ the equivalence class of $e$ with respect to $\sim_\varepsilon$. Then maps $\Xi_\varepsilon : \mathcal{B}_X / \sim_\varepsilon \rightarrow U_\varepsilon (\wedge_{\varepsilon}^2 X)$ ($\varepsilon = +, -$) defined by
\[
\Xi_+ ([e]_+) := \frac{1}{\sqrt{2}} (e_1 \wedge e_2 - e_3 \wedge e_4), \quad \Xi_- ([e]_-) := \frac{1}{\sqrt{2}} (e_1 \wedge e_3 - e_4 \wedge e_2)
\]
are bijective. In other words, $SO(2,2)/G'_+$ (respectively, $SO(2,2)/G'_-$) is considered to be a hyperboloid of two sheets (respectively, one sheet).

Let $I$ be a complex structure of $X$. We say that $I$ is $h_X$-preserving if the pull-back of $h_X$ by $I$ coincides with $h_X$. Let $I$ be an $h_X$-preserving complex structure of $X$. We say
that \( I \) is compatible with the orientation of \( X \) if \((e, I(e), e^\perp, I(e^\perp))\) \( \in \mathcal{B}_X \) for any pair of vectors \( e, e^\perp \) satisfying

\[
h_X(e, e) = 1, \quad h_X(e^\perp, e^\perp) = -1, \quad h_X(e, e^\perp) = 0, \quad h_X(I(e), e^\perp) = 0. \quad (2.1)
\]

Let \( J \) be a paracomplex structure of \( X \), i.e., a linear transformation of \( X \) satisfying \( J \neq \pm \text{id}_X \) and \( J^2 = \text{id}_X \). We say that \( J \) is \( h_X \)-reversing if the pull-back of \( h_X \) by \( J \) coincides with \(-h_X\). Let \( J \) be an \( h_X \)-reversing paracomplex structure of \( X \). We say that \( J \) is compatible with the orientation of \( X \) if \((e, J(e), J(e), e^\perp)\) \( \in \mathcal{B}_X \) for any pair of vectors \( e, e^\perp \) satisfying \( (2.1) \) with \( I := J \).

Let \( I_+ \) be an \( h_X \)-preserving complex structure of \( X \) compatible with the orientation of \( X \). Then for a pair \((e, e^\perp)\) with \( (2.1) \) and \( I := I_+ \), \( \omega_{I_+} := (1/\sqrt{2}) (e \wedge I_+(e) - e^\perp \wedge I_+(e^\perp)) \) is an element of \( U_+ (\bigwedge^2_+ X) \) and does not depend on the choice of \((e, e^\perp)\) with \( (2.1) \) and \( I = I_+ \). In addition, each element of \( U_+ (\bigwedge^2_+ X) \) gives a unique \( h_X \)-preserving complex structure of \( X \) compatible with the orientation of \( X \). Hence we find a one-to-one correspondence between \( U_+ (\bigwedge^2_+ X) \) and the set of \( h_X \)-preserving complex structures of \( X \) compatible with the orientation of \( X \). Let \( J_+ \) be an \( h_X \)-reversing paracomplex structure of \( X \) and suppose that \( J_+ \) is not compatible with the orientation of \( X \) and that \( J_+ \) satisfies \( I_+ J_+ = -J_+ I_+ \). We set \( K_+ := I_+ J_+ \). Then \( K_+ \) is an \( h_X \)-reversing paracomplex structure of \( X \) which is not compatible with the orientation of \( X \) and for each \((a, b, c) \in \mathbb{R}^3 \) with \( a^2 - b^2 - c^2 = 1 \), \( aI_+ + bJ_+ + cK_+ \) is an \( h_X \)-preserving complex structure of \( X \) compatible with the orientation of \( X \).

Let \( J_- \) be an \( h_X \)-reversing paracomplex structure of \( X \) compatible with the orientation of \( X \). Then for a pair \((e, e^\perp)\) with \( (2.1) \) and \( I := J_- \), \( \omega_{J_-} := (1/\sqrt{2}) (e \wedge J_-(e) - e^\perp \wedge J_-(e^\perp)) \) is an element of \( U_- (\bigwedge^2_- X) \) and does not depend on the choice of \((e, e^\perp)\) with \( (2.1) \) and \( I = J_- \). In addition, each element of \( U_- (\bigwedge^2_- X) \) gives a unique \( h_X \)-reversing paracomplex structure of \( X \) compatible with the orientation of \( X \). Hence we find a one-to-one correspondence between \( U_- (\bigwedge^2_- X) \) and the set of \( h_X \)-reversing paracomplex structures of \( X \) compatible with the orientation of \( X \). Let \( I_- \) be an \( h_X \)-preserving complex structure of \( X \) and suppose that \( I_- \) is not compatible with the orientation of \( X \) and that \( I_- \) satisfies \( I_- J_- = -J_- I_- \). We set \( K_- := I_- J_- \). Then \( K_- \) is an \( h_X \)-reversing paracomplex structure of \( X \) compatible with the orientation of \( X \) and for each \((a, b, c) \in \mathbb{R}^3 \) with \( a^2 - b^2 - c^2 = -1 \), \( aI_- + bJ_- + cK_- \) is an \( h_X \)-reversing paracomplex structure of \( X \) compatible with the orientation of \( X \).

### 3 The space-like twistor space

Let \( M \) be a connected smooth manifold and \( E \) an oriented vector bundle over \( M \) of rank 4. Let \( h \) be a non-degenerate metric of \( E \) with signature \((2, 2)\). Let \( \hat{E}_+ \) be a fiber bundle
Let \( \hat{\nabla} \) with respect to \( \phi \) with respect to \( \nabla \) with respect to \( \hat{\nabla} \). Then \( \Omega \) means that the right side of (3.1) is zero; for \( \hat{\nabla} \) of \( \hat{\nabla} \) with respect to \( \hat{\nabla} \) if \( \hat{\nabla} = 0 \). Let \( \hat{\nabla} \) be the induced connection of \( E \). We represent \( \Omega \) as

\[
\Omega = \frac{1}{\sqrt{2}}(\xi_1 \wedge \xi_2 - \xi_3 \wedge \xi_4),
\]

where \( \xi_k \) \((k = 1, 2, 3, 4)\) form a local frame of \( E \) on an open set \( U \) of \( M \) so that \((\xi_1, \xi_2, \xi_3, \xi_4)\) gives an element of \( B_{E_a} \) for any \( a \in U \). Let \( \omega^k_l \) \((k, l = 1, 2, 3, 4)\) be 1-forms on \( U \) given by

\[
\nabla \xi_l = \sum_{k=1}^{4} \omega^k_l \xi_k.
\]

Suppose that \( \Omega \) is horizontal with respect to \( \hat{\nabla} \). Then we have

\[
0 = \sqrt{2} \hat{\nabla} \Omega = \nabla \xi_1 \wedge \xi_2 + \xi_1 \wedge \nabla \xi_2 - \nabla \xi_3 \wedge \xi_4 - \xi_3 \wedge \nabla \xi_4 = (\omega^3_1 \xi_3 + \omega^4_1 \xi_4) \wedge \xi_2 + \xi_1 \wedge (\omega^3_2 \xi_3 + \omega^4_2 \xi_4)
- (\omega^3_1 \xi_1 + \omega^3_2 \xi_2) \wedge \xi_4 - \xi_3 \wedge (\omega^3_1 \xi_1 + \omega^3_2 \xi_2).
\]

Therefore we obtain \( \omega^3_1 = \omega^2_3, \omega^4_1 = -\omega^3_2 = 0 \). We set \( \Omega^*e, e' := h(I_{\Omega} e, e') \) for vectors \( e, e' \). Then \( \Omega^* \) is a section of \( \Lambda^2 E^* \) and represented as \( \Omega^* = \xi^*_1 \wedge \xi^*_2 - \xi^*_3 \wedge \xi^*_4 \), where \((\xi^*_1, \xi^*_2, \xi^*_3, \xi^*_4)\) is the dual frame of \((\xi_1, \xi_2, \xi_3, \xi_4)\). We have

\[
(\nabla \Omega^*)(\xi_k, \xi_l) = -\Omega^*(\nabla \xi_k, \xi_l) - \Omega^*(\xi_k, \nabla \xi_l). \tag{3.1}
\]

For \((k, l) = (1, 2), (3, 4)\), the right side of (3.1) is zero; for \((k, l) = (1, 3), (2, 4)\), \( \omega^3_1 + \omega^4_2 = 0 \) means that the right side of (3.1) is zero; for \((k, l) = (1, 4), (2, 3)\), \( \omega^3_1 = \omega^3_2 \) means that the right side of (3.1) is zero. Therefore if \( \Omega \) is horizontal with respect to \( \hat{\nabla} \), then \( \Omega^* \) is parallel with respect to \( \nabla \), which implies that \( I_{\Omega} \) is parallel with respect to \( \nabla \). Referring to the above discussions, we see that if \( I_{\Omega} \) is parallel with respect to \( \nabla \), then \( \Omega \) is horizontal with respect to \( \hat{\nabla} \). 

\[\square\]
Let $M = N$ be a connected, oriented, 4-dimensional pseudo-Riemannian manifold of signature $(2,2)$. Let $h$ be the metric of $N$ and $\nabla$ the Levi-Civita connection of $h$. Let $E = TN$ be the tangent bundle of $N$ and set $\hat{N}_+ := \hat{T}\hat{N}_+$. Then from Proposition 3.1, we obtain

**Corollary 3.2** Let $\Omega$ be a section of $\hat{N}_+$. Then $\Omega$ is horizontal with respect to $\hat{\nabla}$ if and only if the corresponding almost complex structure $\mathcal{I}_\Omega$ of $N$ is parallel with respect to $\nabla$.

Let $\mathcal{I}$ be an almost complex structure of $N$. We say that the triplet $(N, h, \mathcal{I})$ is Kähler if $h$-preserving, compatible with the orientation of $N$ and parallel with respect to $\nabla$. We also say that $N$ equipped with $h$ and $\mathcal{I}$ as above is a neutral Kähler surface.

**Example** We can find a natural almost complex structure $\mathcal{I}$ of $E_2^4$ given by $\mathcal{I}(e_1) = e_2$ and $\mathcal{I}(e_3) = e_4$ for $e_1 := (1, 0, 0, 0)$, $e_2 := (0, 1, 0, 0)$, $e_3 := (0, 0, 1, 0)$, $e_4 := (0, 0, 0, 1)$. We see that $E_2^4$ equipped with $\mathcal{I}$ is a neutral Kähler surface.

**Example** For $x = (x^1, x^2, x^3, x^4, x^5, x^6) \in \mathbb{R}^6$, we set $z^k := x^{2k-1} + \sqrt{-1}x^{2k}$ for $k = 1, 2, 3$. Then we can identify $S^5_2 \subset E_2^6$ with $\{z = (z^1, z^2, z^3) \in \mathbb{C}^3 \mid |z^1|^2 + |z^2|^2 - |z^3|^2 = 1\}$. There exists a natural action of $S^1 = \{e^{\sqrt{-1}\theta} \mid \theta \in [0, 2\pi)\}$ on $S^5_2$ given by $e^{\sqrt{-1}\theta} \cdot (z) \longmapsto e^{\sqrt{-1}\theta}z$ and then the orbit space $N = S^5_2/S^1$ is a 4-dimensional neutral manifold. In addition, a natural complex structure of $E_2^6$ induces an almost complex structure $\mathcal{I}$ of $N$ and $\mathcal{I}$ is parallel with respect to the Levi-Civita connection of $N$. Hence we see that $N = S^5_2/S^1$ equipped with $\mathcal{I}$ is a neutral Kähler surface. We can consider $N = S^5_2/S^1$ to be an analogue of the complex projective plane $\mathbb{C}P^2$ with the Fubini-Study metric.

**Example** Let $z^k$ be as in the previous example. We can identify $H^5_3 \subset E_4^6$ with $\{z = (z^1, z^2, z^3) \in \mathbb{C}^3 \mid |z^1|^2 - |z^2|^2 - |z^3|^2 = -1\}$. There exists a natural action of $S^1$ on $H^5_3$ given by $e^{\sqrt{-1}\theta} \cdot (z) \longmapsto e^{\sqrt{-1}\theta}z$ and then the orbit space $N = H^5_3/S^1$ is a 4-dimensional neutral manifold. In addition, a natural complex structure of $E_4^6$ induces an almost complex structure $\mathcal{I}$ of $N$ and $\mathcal{I}$ is parallel with respect to the Levi-Civita connection of $N$. Hence we see that $N = H^5_3/S^1$ equipped with $\mathcal{I}$ is a neutral Kähler surface. We can consider $N = H^5_3/S^1$ to be an analogue of the complex hyperbolic plane $\mathbb{C}H^2$ with the Bergman metric.

Let $M$ be a Riemann surface. Let $N$ be a connected, oriented, 4-dimensional neutral manifold and $F : M \longrightarrow N$ a space-like and conformal immersion. Let $E$ be the pull-back bundle over $M$ by $F$: $E = F^*TN$. Then the metric $h$ of $N$ and its Levi-Civita connection $\nabla$ induce a metric and a connection of $E$, which are also denoted by $h$, $\nabla$, respectively. For each $a \in M$, let $(e_1, e_2, e_3, e_4)$ be an element of $\mathcal{B}_{T_{F(a)}N}$ with $e_1, e_2 \in dF(T_aM)$ such that $(e_1, e_2)$ gives the orientation of $M$. Then

$$
\hat{F}(a) := \frac{1}{\sqrt{2}}(e_1 \wedge e_2 - e_3 \wedge e_4) \in U_+(\Lambda^2 T_{F(a)}N)
$$
does not depend on the choice of \((e_1, e_2, e_3, e_4)\). Therefore \(F\) defines a unique section \(\hat{F}\) of \(\hat{E}_+\). We call \(\hat{F}\) the lift of \(F\). We consider \(\hat{F}\) to be not only a map to \(\hat{E}_+\) with \(E = F^*TN\) but also a map to \(\hat{N}_+ = \hat{T}\hat{N}_+\). Let \(I_F\) be the almost complex structure of \(E\) corresponding to \(\hat{F}\). Then from Proposition 3.1, we see that \(\hat{F}\) is horizontal with respect to \(\hat{\nabla}\) if and only if \(I_F\) is parallel with respect to \(\nabla\).

Let \(w = u + \sqrt{-1}v\) be a local complex coordinate of \(M\) and set \(T_1 := dF(\partial/\partial u), T_2 := dF(\partial/\partial v)\). Suppose that \(F\) has zero mean curvature vector. Then \(\nabla_{T_1}T_1 + \nabla_{T_2}T_2 = 0\).

We say that \(F\) is isotropic if \(F\) satisfies

\[
\begin{align*}
h \left( (\nabla_{T_1}T_1)^\perp, (\nabla_{T_1}T_1)^\perp \right) &= h \left( (\nabla_{T_1}T_2)^\perp, (\nabla_{T_1}T_2)^\perp \right), \\
h \left( (\nabla_{T_1}T_1)^\perp, (\nabla_{T_1}T_2)^\perp \right) &= 0,
\end{align*}
\]

where \((\nabla_{T_k}T_l)^\perp\) is the normal component of \(\nabla_{T_k}T_l\) with respect to \(F\); we say that \(F\) is strictly isotropic if \(F\) is isotropic and compatible with the orientation of \(N\), that is, if \(I (\nabla_{T_1}T_1)^\perp = (\nabla_{T_1}T_2)^\perp\) at each \(a \in M\), where \(I\) is a complex structure of \(T_{F(a)}N\) given by \(I_F\). For a space-like and conformal immersion \(F : M \rightarrow N\), \(I_F\) is parallel with respect to \(\nabla\) if and only if \(F\) is with zero mean curvature vector and strictly isotropic. Hence Proposition 3.1 yields the following analogue of a result in [9]:

**Corollary 3.3** For a space-like and conformal immersion \(F : M \rightarrow N\), \(F\) is with zero mean curvature vector and strictly isotropic if and only if the lift \(\hat{F} : M \rightarrow \hat{N}_+\) of \(F\) is horizontal with respect to \(\hat{\nabla}\).

Suppose \(N = S^4_2, E^4_2\) or \(H^4_2\). Let \(F : M \rightarrow N\) be a space-like and conformal immersion with zero mean curvature vector. Then \(F\) defines a holomorphic quartic differential \(Q\) on \(M\) (see [3]). Then \(Q \equiv 0\) just means that \(F\) is isotropic. Therefore space-like surfaces in \(S^4_2, E^4_2, H^4_2\) with zero mean curvature vector and zero holomorphic quartic differential are characterized by horizontality of their lifts into the space-like twistor spaces associated with the above space forms.

### 4 The time-like twistor space

Let \(M, E, h\) be as in the beginning of Section 3. Let \(\hat{E}_-\) be a fiber bundle over \(M\) given by

\[
\hat{E}_- := \bigcup_{a \in M} U_-(\bigwedge^2_+ E_a) \cong \bigcup_{a \in M} \mathcal{B}_{E_a}/\sim_-.
\]

We call \(\hat{E}_-\) the time-like twistor space associated with \(E\). We see that \(\hat{E}_-\) is connected. There exists a one-to-one correspondence between the set of sections of \(\hat{E}_-\) and the set of almost paracomplex structures of \(E\) (sections of \(\text{End}(E)\) giving paracomplex structures of each fiber of \(E\)) which are \(h\)-reversing and compatible with the orientation of \(E\). Let
\( \nabla, \hat{\nabla} \) be as in the beginning of Section 3. A section \( \Omega \) of \( \hat{E}_- \) is said to be *horizontal* with respect to \( \hat{\nabla} \) if \( \hat{\nabla} \Omega = 0 \).

Referring to the proof of Proposition 3.1, we can prove

**Proposition 4.1** Let \( \Omega \) be a section of \( \hat{E}_- \). Then \( \Omega \) is horizontal with respect to \( \hat{\nabla} \) if and only if the corresponding almost paracomplex structure \( J_{\Omega} \) of \( E \) is parallel with respect to \( \nabla \).

**Proof** We represent \( \Omega \) as
\[
\Omega = \frac{1}{\sqrt{2}} (\xi_1 \wedge \xi_3 - \xi_4 \wedge \xi_2),
\]
where \( \xi_k \) \((k = 1, 2, 3, 4)\) are as in the proof of Proposition 3.1. Let \( \omega_{kl} \) \((k, l = 1, 2, 3, 4)\) be as above. If \( \Omega \) is horizontal with respect to \( \hat{\nabla} \), then we can show that \( \Omega' := \xi_1^* \wedge \xi_3^* - \xi_4^* \wedge \xi_2^* \) is parallel with respect to \( \nabla \), which implies that \( J_{\Omega} \) is parallel with respect to \( \nabla \). In addition, we can show that if \( J_{\Omega} \) is parallel with respect to \( \nabla \), then \( \Omega \) is horizontal with respect to \( \hat{\nabla} \). \( \square \)

Let \( M = N \) be a connected, oriented, 4-dimensional neutral manifold. Let \( h \) be the metric of \( N \) and \( \nabla \) the Levi-Civita connection of \( h \). Let \( E = TN \) be the tangent bundle of \( N \) and set \( \hat{N}_- := \hat{T}N_- \). Then from Proposition 4.1, we obtain the following:

**Corollary 4.2** Let \( \Omega \) be a section of \( \hat{N}_- \). Then \( \Omega \) is horizontal with respect to \( \hat{\nabla} \) if and only if the corresponding almost paracomplex structure \( J_{\Omega} \) of \( N \) is parallel with respect to \( \nabla \).

Let \( J \) be an almost paracomplex structure of \( N \). We say that the triplet \((N, h, J)\) is *paraKähler* if \( J \) is \( h \)-reversing, compatible with the orientation of \( N \) and parallel with respect to \( \nabla \). We also say that \( N \) equipped with \( h \) and \( J \) as above is a *paraKähler surface*.

**Example** We can find a natural almost paracomplex structure \( J \) of \( E_2^4 \) given by \( J(e_1) = e_3 \) and \( J(e_4) = e_2 \). We see that \( E_2^4 \) equipped with \( J \) is a paraKähler surface.

**Example** For \( x = (x^1, x^2, x^3, x^4, x^5, x^6) \in R^6 \), we set \( z^k := x^k + jx^{k+3} \) for \( k = 1, 2, 3 \) (see the appendix for paracomplex numbers). Then we can identify \( S_3^5 \subset E_3^6 \) with \( \{z = (z^1, z^2, z^3) \in C^3 \mid |z|^2 + |z_2|^2 + |z_3|^2 = 1\} \). There exists a natural action of \( G = \{e^{jt} := \cosh t + jsinh t \mid t \in R\} \) on \( S_3^5 \) given by \( (e^{jt}, z) \longrightarrow e^{jt}z \) and then the orbit space \( N = S_3^5/G \) is a 4-dimensional neutral manifold. In addition, a natural paracomplex structure \( J_0 \) of \( E_3^6 \) given by \( J_0(e_k) = e_{k+3} \) for \( k = 1, 2, 3 \) induces an almost paracomplex structure \( J \) of \( N \) and \( J \) is parallel with respect to the Levi-Civita connection of \( N \). Hence we see that \( N = S_3^5/G \) equipped with \( J \) is a paraKähler surface. We can consider \( N = S_3^5/G \) to be an analogue of the above neutral Kähler surface \( S_2^5/S^1 \).
Example Let $z^k$ be as in the previous example. We can identify $H^5_2 \subset E^6_3$ with \{\(z = (z^1, z^2, z^3) \in \mathbb{C}^3 \mid |z|^2 + |z|^2 + |z|^2 = -1\)\}. There exists a natural action of $G$ on $H^5_2$ given by $(\hat{e}^t, z) \mapsto \hat{e}^t z$ and then the orbit space $N = H^5_2/G$ is a 4-dimensional neutral manifold. In addition, the paracomplex structure $\mathcal{J}_0$ of $E^6_3$ induces an almost paracomplex structure $\mathcal{J}$ of $N$ and $\mathcal{J}$ is parallel with respect to the Levi-Civita connection of $N$. Hence we see that $N = H^5_2/G$ equipped with $\mathcal{J}$ is a paraKähler surface. We can consider $N = H^5_2/G$ to be an analogue of the above neutral Kähler surface $H^3_3/S^1$.

Let $M$ be a Lorentz surface (see the appendix for Lorentz surfaces). Let $N$ be a connected, oriented, 4-dimensional neutral manifold and $F : M \rightarrow N$ a time-like and conformal immersion. Let $E$ be the pull-back bundle over $M$ by $F$: $E = F^*TN$. Then the metric $h$ of $N$ and its Levi-Civita connection $\nabla$ induce a metric and a connection of $E$, which are also denoted by $h$, $\nabla$, respectively. For each $a \in M$, let $(e_1, e_2, e_3, e_4)$ be an element of $\mathcal{B}_{T_F(a)N}$ with $e_1, e_3 \in dF(T_aM)$ such that $(e_1, e_3)$ gives the orientation of $M$. Then

$$\hat{F}(a) := \frac{1}{\sqrt{2}}(e_1 \wedge e_3 - e_4 \wedge e_2) \in U_-(T^\wedge_2T_{F(a)N})$$

do not depend on the choice of $(e_1, e_2, e_3, e_4)$. Therefore $F$ defines a unique section $\hat{F}$ of $\hat{E}_-$. We call $\hat{F}$ the lift of $F$. We consider $\hat{F}$ to be not only a map to $\hat{E}_-$ with $E = F^*TN$ but also a map to $\hat{N}_- = \hat{T}N_-$. Let $\mathcal{J}_F$ be the almost paracomplex structure of $E$ corresponding to $\hat{F}$. Then from Proposition 4.1, we see that $\hat{F}$ is horizontal with respect to $\nabla$ if and only if $\mathcal{J}_F$ is parallel with respect to $\nabla$.

For each $a \in M$, let $\xi_1, \xi_2, \xi_3, \xi_4$ be local sections of $E$ on a neighborhood $U$ of $a$ such that for each $x \in U$,

- $(\xi_1(x), \xi_2(x), \xi_3(x), \xi_4(x)) \in \mathcal{B}_{T_F(x)N}$,
- $\xi_1(x), \xi_3(x) \in dF(T_xM)$,
- $(\xi_1(x), \xi_3(x))$ gives the orientation of $M$.

We set

$$\mathcal{E}_{+,1} := \frac{1}{\sqrt{2}}(\xi_1 \wedge \xi_2 + \xi_3 \wedge \xi_4),$$

$$\mathcal{E}_{-,2} := \frac{1}{\sqrt{2}}(\xi_1 \wedge \xi_3 - \xi_4 \wedge \xi_2), \quad \mathcal{E}_{-,3} := \frac{1}{\sqrt{2}}(\xi_1 \wedge \xi_4 - \xi_2 \wedge \xi_3).$$

Then on $U$, $\hat{F}$ is represented as $\hat{F} = \mathcal{E}_{-,2}$.

Let $w = u + j v$ be a local paracomplex coordinate of $M$ and set $T_1 := dF(\partial/\partial u)$, $T_2 := dF(\partial/\partial v)$. Suppose that $F$ has zero mean curvature vector. Then we have $\nabla_{T_1}T_1 = \nabla_{T_2}T_2$. We say that $F$ is isotropic if $F$ satisfies

$$h \left( (\nabla_{T_1}T_1)^\perp, (\nabla_{T_1}T_1)^\perp \right) = -h \left( (\nabla_{T_1}T_2)^\perp, (\nabla_{T_1}T_2)^\perp \right),$$

$$h \left( (\nabla_{T_1}T_1)^\perp, (\nabla_{T_1}T_2)^\perp \right) = 0.$$
For each point \( a \in M \), let \( J \) be a paracomplex structure of \( T_{F(a)}N \) given by \( J_F \). We say that a normal vector of \( F \) at \( F(a) \) is \textit{properly light-like} if it is nonzero and \( J \)-invariant. We say that \( F \) is \textit{strictly isotropic} if \( F \) is isotropic and compatible with the orientation of \( N \), that is, if \( J (\nabla_{T_1} T_2) = (\nabla_{T_1} T_2) \) at each \( a \in M \).

We say that \( \nabla_{T_k} \hat{F} \) is \textit{properly light-like} if \( \nabla_{T_k} \hat{F} \) is represented as \( \nabla_{T_k} \hat{F} = \rho(\mathcal{E}_{+1} + \mathcal{E}_{-3}) \), where \( \rho \) is a nowhere zero function. We will prove

\begin{theorem}
Let \( F : M \rightarrow N \) be a time-like and conformal immersion. Then the following hold:

(a) \( F \) has zero mean curvature vector so that \( (\nabla_T T)^\perp \) is zero or properly light-like for \( T := T_1 + T_2 \) if and only if \( \nabla_{T_1} \hat{F} \) and \( \nabla_{T_2} \hat{F} \) are zero or properly light-like so that \( \nabla_T \hat{F} = 0 \);

(b) \( \nabla_{T_1} \hat{F} \) and \( \nabla_{T_2} \hat{F} \) are zero if and only if \( F \) is strictly isotropic;

(c) on condition that \( \nabla_{T_1} \hat{F} \) and \( \nabla_{T_2} \hat{F} \) are properly light-like so that \( \nabla_T \hat{F} = 0 \), \( F \) is isotropic if and only if \( F \) has properly light-like second fundamental form.
\end{theorem}

\textit{Proof} \quad We represent the induced metric \( g \) by \( g = e^{2\alpha} dw d\overline{w} = e^{2\alpha}(du^2 - dv^2) \). Let \( N_1, N_2 \) be normal vector fields of \( F \) on a neighborhood \( U \) of each point of \( M \) satisfying \( h(N_k, N_l) = (-1)^{k+l+2} e^{2\alpha} \delta_{kl} \). Suppose that \( (T_1, N_1, T_2, N_2) \) gives the orientation of \( N \). We set

\[ \xi_1 := \frac{1}{e^{\alpha}} T_1, \quad \xi_2 := \frac{1}{e^{\alpha}} N_1, \quad \xi_3 := \frac{1}{e^{\alpha}} T_2, \quad \xi_4 := \frac{1}{e^{\alpha}} N_2 \]

and

\[ \nabla_{\xi_k} \xi_l = \sum_{i=1}^{4} c_{hk}^i \xi_l \]

for \( h \in \{1, 3\}, \ k \in \{1, 2, 3, 4\} \). Then we have \( c_{hk}^l = \varepsilon(k, l)c_{hl}^k \), where

\[ \varepsilon(k, l) = \begin{cases} -1 & (k, l \in \{1, 2\} \text{ or } k, l \in \{3, 4\}), \\ 1 & (\text{otherwise}). \end{cases} \]

Therefore we obtain

\[ \nabla_{\xi_1} \hat{F} = (c_{13}^2 - c_{11}^4)\mathcal{E}_{+1} + (c_{13}^4 - c_{11}^2)\mathcal{E}_{-3}, \]

\[ \nabla_{\xi_3} \hat{F} = (c_{33}^2 - c_{31}^4)\mathcal{E}_{+1} + (c_{33}^4 - c_{31}^2)\mathcal{E}_{-3}. \]  \hspace{1cm} (4.1)

Therefore \( \nabla_{\xi_1} \hat{F} \) and \( \nabla_{\xi_3} \hat{F} \) are zero or properly light-like so that \( \nabla_{\xi_1} \hat{F} = -\nabla_{\xi_3} \hat{F} \) if and only if

\[ c_{13}^2 - c_{11}^4 = c_{13}^4 - c_{11}^2 = -(c_{33}^2 - c_{31}^4) = -(c_{33}^4 - c_{31}^2). \]  \hspace{1cm} (4.2)
We see that (4.2) holds if and only if \( F \) has zero mean curvature vector so that \( c_1^2 + c_3^2 = c_1^4 + c_3^4 \). We can exchange \( c_1^2 + c_3^2 = c_1^4 + c_3^4 \) for the condition that \((\nabla T)^+\) is zero or properly light-like. Therefore we obtain (a) in Theorem 4.3. From (4.1), we obtain (b) in Theorem 4.3. Suppose that \( \hat{\nabla}_T \hat{F} \) and \( \hat{\nabla}_{T_2} \hat{F} \) are properly light-like so that \( \hat{\nabla}_T \hat{F} = 0 \). Then we have \( c_1^2 + c_3^2 = c_1^4 + c_3^4 \). Then the isotropicity means either \((c_1^3, c_1^3) = (c_1^4, c_1^4)\) or \( c_{1k} = c_{1k}^4 \) for \( k = 1, 3 \). We see that \((c_1^3, c_1^3) = (c_1^4, c_1^4)\) if and only if \( F \) is strictly isotropic. Since \( \hat{\nabla}_T \hat{F} \) and \( \hat{\nabla}_{T_2} \hat{F} \) are not zero, the isotropicity is equivalent to \( c_{1k} = c_{1k}^4 \) for \( k = 1, 3 \) and then we obtain (c) in Theorem 4.3.

\[ \square \]

**Remark** We can refer to [10], [11] for (b) of Theorem 4.3.

**Remark** See Subsection 7.3 for further discussions on (c) of Theorem 4.3 with \( N = S_2^4 \), \( E_2^4 \) or \( H_2^4 \).

## 5 Complex curves in neutral Kähler surfaces and paracomplex curves in paraKähler surfaces

Let \((N, h, I)\) be a neutral Kähler surface. Let \( M \) be a connected Riemann surface and \( I^M \) its complex structure. Let \( F : M \rightarrow N \) be a space-like and conformal immersion. We say that a point \( a \) of \( M \) is complex with respect to \((F, I)\) if \( dF \circ I^M = I \circ dF \) on \( T_a M \). We will prove

**Theorem 5.1** Let \((N, h, I)\) be a neutral Kähler surface. Let \( M \) be a connected Riemann surface and \( F : M \rightarrow N \) a space-like and conformal immersion. Then the following (a), (b) are equivalent to each other:

(a) \( F \) is a holomorphic immersion with respect to \( I \);

(b) \( F \) has at least one complex point with respect to \((F, I)\) and zero mean curvature vector so that \( F \) is strictly isotropic.

**The first proof** Referring to [2], we can prove Theorem 5.1. Let \( F : M \rightarrow N \) be a holomorphic immersion with respect to \( I \). We have \( dF \circ I^M = I \circ dF \). Let \( w = u + \sqrt{-1}v \) be a local complex coordinate of \( M \) and set \( T_1 := dF(\partial/\partial u) \), \( T_2 := dF(\partial/\partial v) = IT_1 \). Then we have \( \nabla_{T_1} T_1 + \nabla_{T_2} T_2 = 0 \) and this means that \( F \) has zero mean curvature vector. By \( \nabla_{T_1} T_1 = \nabla_{T_1} T_2 \), we see that \( F \) is strictly isotropic. Let \( F \) be as in (b). Let \( g \) be the induced metric by \( F \). Then \( g \) is locally represented as \( g = e^{2\alpha} (du^2 + dv^2) \). Let \( N_1, N_2 \) be normal vector fields of \( F \) and suppose \( h(N_k, N_l) = -e^{2\alpha} \delta_{kl} \) and that \((T_1, T_2, N_1, N_2)\) gives
the orientation of \( N \). Then we have

\[
(\nabla_{T_1} T_1 \nabla_{T_1} T_2 \nabla_{T_1} N_1 \nabla_{T_1} N_2) = (T_1 T_2 N_1 N_2) \begin{pmatrix}
\alpha_u & \alpha_v & \mu_1 & \mu_2 \\
-\alpha_v & \alpha_u & -\mu_2 & \mu_1 \\
\mu_1 & -\mu_2 & \alpha_u & -\beta \\
\mu_2 & \mu_1 & \beta & \alpha_u
\end{pmatrix}, \tag{5.1}
\]

where \( \mu_1, \mu_2 \) and \( \beta \) are functions. Therefore, since \( \nabla I = 0 \), we obtain

\[
(-\nabla_{T_1} T_2 \nabla_{T_1}, T_1 - \nabla_{T_1} N_2 \nabla_{T_1} N_1) = (-TT_2 TT_1 - TN_2 TN_1) \begin{pmatrix}
\alpha_u & \alpha_v & \mu_1 & \mu_2 \\
-\alpha_v & \alpha_u & -\mu_2 & \mu_1 \\
\mu_1 & -\mu_2 & \alpha_u & -\beta \\
\mu_2 & \mu_1 & \beta & \alpha_u
\end{pmatrix}. \tag{5.2}
\]

Let \( a_0 \) be a complex point of \( M \) with respect to \((F, I)\). Then \( TT_1 = T_2 \) and \( TN_1 = N_2 \) at \( a_0 \). Comparing (5.1) with (5.2), and noticing uniqueness of the solution of a system of ordinary differential equations for a given initial value, we see that \((T_1, T_2, N_1, N_2)\) and \((-TT_2, TT_1, -TN_2, TN_1)\) coincide with each other on the integral curve of \( \partial/\partial u \) through \( a_0 \). Therefore, by analogous discussions on \( \nabla_{T_1} \) instead of \( \nabla_{T_1} \), we see that \((T_1, T_2, N_1, N_2)\) and \((-TT_2, TT_1, -TN_2, TN_1)\) coincide with each other on integral curves of \( \partial/\partial v \). In particular, we obtain \( TT_1 = T_2 \) on a neighborhood of \( a_0 \). Since \( M \) is connected, \( F \) satisfies \( I \circ dF = dF \circ I^M \) on \( T_a M \) for any \( a \in M \). This means that \( F \) is a holomorphic immersion with respect to \( I \). \( \square \)

The second proof Let \( E \) be the pull-back bundle over \( M \) by \( F \). Suppose that \( F \) is holomorphic. Then the lift \( \hat{F} \) of \( F \) coincides with the section \( \Omega_\tau \) of \( \hat{E}_+ \) given by the complex structure \( I \). Since \( I \) is parallel with respect to \( \nabla \), we see from Proposition 3.1 that \( \Omega_\tau = \hat{F} \) is horizontal with respect to \( \hat{\nabla} \). Therefore we see from Corollary 3.3 that \( F \) is with zero mean curvature vector and strictly isotropic. Since any point of \( M \) is complex with respect to \((F, I)\), we obtain (b) from (a). Suppose (b). We see from Corollary 3.3 that the lift \( \hat{F} \) is horizontal with respect to \( \hat{\nabla} \). Therefore \( \hat{F} \) is uniquely determined by an initial value at a point. Noticing that there exists a complex point \( a_0 \) of \( M \) with respect to \((F, I)\) and that the section \( \Omega_\tau \) of \( \hat{E}_+ \) given by \( I \) is horizontal with respect to \( \hat{\nabla} \) and coincides with \( \hat{F} \) at \( a_0 \), we see that \( \Omega_\tau \) coincides with \( \hat{F} \) on \( M \). This means that \( F \) is holomorphic with respect to \( I \). Hence we obtain (a) from (b). \( \square \)

Let \((N, h, J)\) be a paraKähler surface. Let \( M \) be a connected Lorentz surface and \( J^M \) its paracomplex structure. Let \( F : M \rightarrow N \) be a time-like and conformal immersion. We say that a point \( a \) of \( M \) is paracomplex with respect to \((F, J)\) if \( dF \circ J^M = J \circ dF \) on \( T_a M \). We say that \( F \) is holomorphic with respect to \( J \) if any point of \( M \) is paracomplex with respect to \((F, J)\). Referring to the proofs of Theorem 5.1, we can prove
**Theorem 5.2** Let \((N, h, \mathcal{J})\) be a paraKähler surface. Let \(M\) be a connected Lorentz surface and \(F : M \to N\) a time-like and conformal immersion. Then the following (a), (b) are equivalent to each other:

(a) \(F\) is a holomorphic immersion with respect to \(\mathcal{J}\);

(b) \(F\) has at least one paracomplex point with respect to \((F, \mathcal{J})\) and zero mean curvature vector so that \(F\) is strictly isotropic.

### 6 Complex curves and paracomplex curves in neutral hyperKähler 4-manifolds

Let \(N\) be an oriented 4-dimensional neutral manifold and \(h\) the metric of \(N\). Let \(\mathcal{I}_+\) be an \(h\)-preserving almost complex structure of \(N\) and \(\mathcal{J}_+\) an \(h\)-reversing almost paracomplex structure of \(N\). Suppose \(\mathcal{I}_+\mathcal{J}_+ = -\mathcal{J}_+\mathcal{I}_+\) and set \(\mathcal{K}_+ := \mathcal{I}_+\mathcal{J}_+\). Then \(\mathcal{K}_+\) is an \(h\)-reversing almost paracomplex structure of \(N\). Suppose that \(\mathcal{I}_+\) is compatible with the orientation of \(N\). Then \(\mathcal{J}_+, \mathcal{K}_+\) are not compatible with the orientation of \(N\). Suppose that \(N, h, \mathcal{I}_+, \mathcal{J}_+, \mathcal{K}_+\) form a neutral hyperKähler manifold, i.e., suppose that \(\mathcal{I}_+, \mathcal{J}_+, \mathcal{K}_+\) are parallel with respect to the Levi-Civita connection \(\nabla\) of \(h\) (see [12], [7] for neutral hyperKähler 4-manifolds). Then \(\hat{N}_+\) is a product manifold and each constant section is horizontal with respect to \(\nabla\). By Theorem 5.1, we obtain

**Corollary 6.1** Let \((N, h, \mathcal{I}_+, \mathcal{J}_+, \mathcal{K}_+)\) be as above. Let \(M\) be a connected Riemann surface and \(F : M \to N\) a space-like and conformal immersion. Then the following (a), (b) are equivalent to each other:

(a) there exist real numbers \(a, b, c\) with \(a^2 - b^2 - c^2 = 1\) such that \(F\) is a holomorphic immersion with respect to a complex structure \(a\mathcal{I}_+ + b\mathcal{J}_+ + c\mathcal{K}_+\);

(b) \(F\) is with zero mean curvature vector and strictly isotropic.

**Example** Let \(\mathcal{I}_+\) be an almost complex structure of \(E^4_2\) given by \(\mathcal{I}_+(e_1) = e_2\) and \(\mathcal{I}_+(e_3) = e_4\) for \(e_1 := (1, 0, 0, 0)\), \(e_2 := (0, 1, 0, 0)\), \(e_3 := (0, 0, 1, 0)\), \(e_4 := (0, 0, 0, 1)\). If we denote by \(h\) the metric of \(E^4_2\), then \((E^4_2, h, \mathcal{I}_+)\) is Kähler. Let \(\mathcal{J}_+\) be an almost paracomplex structure of \(E^4_2\) given by \(\mathcal{J}_+(e_1) = e_3\) and \(\mathcal{J}_+(e_4) = -e_2\). Then \(\mathcal{J}_+\) is \(h\)-reversing and not compatible with the natural orientation of \(E^4_2\). In addition, \(\mathcal{J}_+\) satisfies \(\mathcal{I}_+\mathcal{J}_+ = -\mathcal{J}_+\mathcal{I}_+\).

If we set \(\mathcal{K}_+ := \mathcal{I}_+\mathcal{J}_+\), then \((E^4_2, h, \mathcal{I}_+, \mathcal{J}_+, \mathcal{K}_+)\) is a neutral hyperKähler manifold. Let \(F : M \to E^4_2\) be a holomorphic immersion with respect to \(a\mathcal{I}_+ + b\mathcal{J}_+ + c\mathcal{K}_+\) for real numbers \(a, b, c\) with \(a^2 - b^2 - c^2 = 1\). Then \(F\) is with zero mean curvature vector and strictly isotropic. Let \(F : M \to E^3_2\) be a space-like and conformal immersion and suppose that \(F\) is with zero mean curvature vector and strictly isotropic. There exist real
numbers $a$, $b$, $c$ with $a^2 - b^2 - c^2 = 1$ such that $(a\mathcal{I}_+ + b\mathcal{J}_+ + c\mathcal{K}_+)(F_u) = F_v$ at a point of $M$ for a local complex coordinate $w = u + \sqrt{-1}v$ of $M$. Then, noticing the first proof of Theorem 5.1 or that $F$ is real-analytic, we see that $(a\mathcal{I}_+ + b\mathcal{J}_+ + c\mathcal{K}_+)(F_u) = F_v$ on a neighborhood of this point. Therefore $F$ is holomorphic with respect to $a\mathcal{I}_+ + b\mathcal{J}_+ + c\mathcal{K}_+$.

Let $M$ be a connected Riemann surface and $F : M \to E^4_2$ a space-like and conformal immersion. Let $T_+$ be a 3-tensor field on $M$ defined by

$$T_+(V_1, V_2, V_3) := h(\mathcal{J}_+ \circ dF(V_1), B(V_2, V_3)),$$

where $V_1$, $V_2$, $V_3$ are tangent vector of $M$ and $B$ is the second fundamental form of $F$. The following is an analogue of a result in [1] and we can prove it similarly.

**Theorem 6.2** Let $F : M \to E^4_2$ be a space-like and conformal immersion of $M$ into $E^4_2$. Let $g$ be the induced metric by $F$ and $K$ the curvature of $g$. Then the following (a) and (b) are equivalent:

(a) the composition of $F$ and an isometry of $E^4_2$ is holomorphic with respect to $\mathcal{I}_+$;

(b) there exists a holomorphic cubic differential $\Psi$ on $M$ satisfying $T_+ = \text{Re} \, \Psi$ and $K = 2|\psi|^2/e^{6\alpha}$,

where $g$ and $\Psi$ are locally represented as $g = e^{2\alpha}dwd\overline{w}$, $\Psi = \psi dw^3$, respectively by a local complex coordinate $w$ of $M$. In addition, for a holomorphic cubic differential $\Psi = \psi dw^3$ and a conformal metric $g = e^{2\alpha}dwd\overline{w}$ on $M$ satisfying $K = 2|\psi|^2/e^{6\alpha}$, there exists a space-like and holomorphic immersion $F$ of a neighborhood of each point of $M$ into $E^4_2$ with respect to $\mathcal{I}_+$ satisfying $g = F^*h$ and $\text{Re} \, \Psi = T_+$; such an immersion as $F$ is uniquely determined by $g$ and $\Psi$ up to a composition of isometries of $E^4_2$ induced by elements of $SU(1,1)$ and parallel translations in $E^4_2$.

Let $N$ be an oriented 4-dimensional neutral manifold and $h$ the metric of $N$. Let $\mathcal{I}_-$ be an $h$-preserving almost complex structure of $N$ and $\mathcal{J}_-$ an $h$-reversing almost paracomplex structure of $N$. Suppose $\mathcal{I}_- \mathcal{J}_- = -\mathcal{J}_- \mathcal{I}_-$ and set $\mathcal{K}_- := \mathcal{I}_- \mathcal{J}_-$. Suppose that $\mathcal{I}_-$ is not compatible with the orientation of $N$. Then $\mathcal{J}_-, \mathcal{K}_-$ are compatible with the orientation of $N$. Suppose that $N$, $h$, $\mathcal{I}_-$, $\mathcal{J}_-$, $\mathcal{K}_-$ form a neutral hyperKähler manifold. Then $\tilde{N}_-$ is a product manifold and each constant section is horizontal. By Theorem 5.2, we obtain

**Corollary 6.3** Let $(N, h, \mathcal{I}_-, \mathcal{J}_-, \mathcal{K}_-)$ be as above. Let $M$ be a connected Lorentz surface and $F : M \to N$ a time-like and conformal immersion. Then the following (a), (b) are equivalent to each other:

(a) there exist real numbers $a$, $b$, $c$ with $a^2 - b^2 - c^2 = -1$ such that $F$ is a holomorphic immersion with respect to a paracomplex structure $a\mathcal{I}_- + b\mathcal{J}_- + c\mathcal{K}_-$;
(b) $F$ is with zero mean curvature vector and strictly isotropic.

Example Let $\mathcal{J}_-$ be an almost paracomplex structure of $E_2^4$ given by $\mathcal{J}_-(e_1) = e_3$ and $\mathcal{J}_-(e_4) = e_2$. Then $(E_2^4, h, \mathcal{J}_-)$ is paraKähler. Let $\mathcal{I}_-$ be an almost complex structure of $E_2^4$ given by $\mathcal{I}_-(e_1) = e_2$ and $\mathcal{I}_-(e_3) = -e_4$. Then $\mathcal{I}_-$ is $h$-preserving and not compatible with the orientation of $E_2^4$. In addition, $\mathcal{I}_-$ satisfies $\mathcal{I}_-\mathcal{J}_- = -\mathcal{J}_-\mathcal{I}_-$. If we set $\mathcal{K}_- := \mathcal{I}_-\mathcal{J}_-$, then $(E_2^4, h, \mathcal{I}_-, \mathcal{J}_-, \mathcal{K}_-)$ is a neutral hyperKähler manifold. Let $F$ be a holomorphic immersion with respect to $a\mathcal{I}_- + b\mathcal{J}_- + c\mathcal{K}_-$ with $a^2 - b^2 - c^2 = -1$. Then $F$ is with zero mean curvature vector and strictly isotropic. Let $F$ be a time-like and conformal immersion and suppose that $F$ has zero mean curvature vector and that $F$ is strictly isotropic. Then $F_{uu} = F_{vv}$ for a local paracomplex coordinate $w = u + jv$ of $M$ and therefore we can represent $F$ as $F(u, v) = F^1(u + v) + F^2(u - v)$. Since $h(F_u, F_u) = -h(F_v, F_v)$ and $h(F_u, F_v) = 0$, we have

$$h(\hat{F}^1, \hat{F}^1) = h(\hat{F}^2, \hat{F}^2) = 0. \tag{6.1}$$

From the isotropicity of $F$, we have $h(F_{uu}, F_{uu}) = -h(F_{uv}, F_{uv})$ and $h(F_{uu}, F_{uv}) = 0$. These imply

$$h(\tilde{F}^1, \tilde{F}^1) = h(\tilde{F}^2, \tilde{F}^2) = 0. \tag{6.2}$$

By (6.1) together with (6.2), we see that if $(a, b, c)$ with $a^2 - b^2 - c^2 = -1$ satisfies $(a\mathcal{I}_- + b\mathcal{J}_- + c\mathcal{K}_-)(F_u) = F_v$ at a point of $M$, then $F$ is holomorphic with respect to $a\mathcal{I}_- + b\mathcal{J}_- + c\mathcal{K}_-$.

Let $M$ be a connected Lorentz surface and $F : M \rightarrow E_2^4$ a time-like and conformal immersion. Let $T_-$ be a 3-tensor field on $M$ defined by

$$T_-(V_1, V_2, V_3) := h(\pi_r dF(V_1), B(V_2, V_3)). \tag{6.3}$$

Referring to [1], we will prove

Theorem 6.4 Let $F : M \rightarrow E_2^4$ be a time-like and conformal immersion of $M$ into $E_2^4$. Let $g$ be the induced metric by $F$ and $K$ the curvature of $g$. Then the following (a) and (b) are equivalent:

(a) the composition of $F$ and an isometry of $E_2^4$ is holomorphic with respect to $\mathcal{J}_-$;

(b) there exists a holomorphic cubic differential $\Psi$ on $M$ satisfying $T_- = \text{Re} \Psi$ and $K = -2|\psi|^2/e^{6\alpha}$,

where $g$ and $\Psi$ are represented as $g = e^{2\alpha} dw d\overline{w}$, $\Psi = \psi dw^3$, respectively by a local paracomplex coordinate $w$ of $M$. In addition, for a holomorphic cubic differential $\Psi = \psi dw^3$ and a paraHermitian metric $g = e^{2\alpha} dw d\overline{w}$ on $M$ satisfying $K = -2|\psi|^2/e^{6\alpha}$, there
exists a time-like and holomorphic immersion $F$ of a neighborhood of each point of $M$ into $E^4_2$ with respect to $\mathcal{J}_-$ satisfying $g = F^* h$ and $\text{Re} \Psi = T_-$; such an immersion as $F$ is uniquely determined by $g$ and $\Psi$ up to a composition of isometries of $E^4_2$ induced by elements of $SU(1, 1)$ and parallel translations in $E^4_2$.

Proof Let $F : M \rightarrow E^4_2$ be a holomorphic immersion of $M$ into $E^4_2$ with respect to the paracomplex structure $\mathcal{J}_-$. Then the paracomplex structure $\mathcal{J}^M$ of $M$ satisfies $dF \circ \mathcal{J}^M = \mathcal{J}_- \circ dF$. Let $u, v$ be the real and the paraimaginary parts of $w$, respectively: $w = u + jv$. Then the induced metric $g$ by $F$ is locally represented as $g = e^{2\alpha}(du^2 - dv^2)$.

We set 
\[ T_1 := dF \left( \frac{\partial}{\partial u} \right), \quad T_2 := \mathcal{J}_- \circ dF \left( \frac{\partial}{\partial u} \right) = dF \left( \frac{\partial}{\partial v} \right), \]
\[ N_1 := \mathcal{J}_- \circ dF \left( \frac{\partial}{\partial u} \right), \quad N_2 := \mathcal{J}_- N_1. \]

Then we can find functions $\mu_1, \mu_2$ satisfying

\[
(\nabla_{T_1} T_1 \nabla_{T_2} T_2 \nabla_{T_1} N_1 \nabla_{T_1} N_2) = (T_1 T_2 N_1 N_2) \begin{pmatrix} \alpha_u & \alpha_v & -\mu_1 & \mu_2 \\ \alpha_v & \alpha_u & \mu_2 & -\mu_1 \\ \mu_1 & \mu_2 & \alpha_u & -\alpha_v \\ \mu_2 & \mu_1 & -\alpha_v & \alpha_u \end{pmatrix},
\]

\[
(\nabla_{T_1} T_1 \nabla_{T_2} T_2 \nabla_{T_2} N_1 \nabla_{T_2} N_2) = (T_1 T_2 N_1 N_2) \begin{pmatrix} \alpha_v & \alpha_u & -\mu_2 & \mu_1 \\ \alpha_u & \alpha_v & \mu_1 & -\mu_2 \\ \mu_2 & \mu_1 & \alpha_v & -\alpha_u \\ \mu_1 & \mu_2 & -\alpha_u & \alpha_v \end{pmatrix}.
\]

From (6.4), we obtain
\[
\alpha_{uu} - \alpha_{vv} - 2(\mu_1^2 - \mu_2^2) = 0, \\
(\mu_1)_v - (\mu_2)_u + 2(\mu_1 \alpha_v - \mu_2 \alpha_u) = 0, \\
(\mu_1)_u - (\mu_2)_v + 2(\mu_1 \alpha_u - \mu_2 \alpha_v) = 0.
\]

We set $\psi := e^{2\alpha}(\mu_1 + j\mu_2)$. Then the first equation of (6.5) means that the curvature $K$ of $(M, g)$ is represented as $K = -2|\psi|^2 e^{-6\alpha}$ and we see from the second and the third equations that $\psi$ is holomorphic with respect to a local paracomplex coordinate $w$. Let $T_-$ be a 3-tensor field on $M$ defined as in (6.3). We consider $T_-$ to be a paracomplex 3-linear function on the paracomplexification of the tangent plane of $M$. Then we obtain

\[
T_- \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right) = \frac{1}{2} \psi
\]
and

\[
T_- \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right) = T_- \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right) = T_- \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right) = 0.
\]
Therefore there exists a holomorphic cubic differential $\Psi$ on $M$ which is locally represented as $\Psi = \psi dw^3$ and satisfies $\text{Re} \, \Psi = T_\perp$. Hence we obtain (b) from (a). Let $\Psi = \psi dw^3$ and $g = e^{2\alpha} dw d\overline{w}$ be a holomorphic cubic differential and a paraHermitian metric on a Lorentz surface $M$, respectively. Suppose that $\Psi$ and $g$ satisfy $K = -2|\psi|^2 e^{-6\alpha}$. Then

$$A := \begin{pmatrix}
\alpha_u & \alpha_v & -e^{-2\alpha} \text{Re} \, \psi & e^{-2\alpha} \text{Im} \, \psi \\
\alpha_v & \alpha_u & e^{-2\alpha} \text{Im} \, \psi & -e^{-2\alpha} \text{Re} \, \psi \\
e^{-2\alpha} \text{Re} \, \psi & e^{-2\alpha} \text{Im} \, \psi & \alpha_u & -\alpha_v \\
e^{-2\alpha} \text{Im} \, \psi & e^{-2\alpha} \text{Re} \, \psi & -\alpha_v & \alpha_u
\end{pmatrix},$$

and

$$B := \begin{pmatrix}
\alpha_v & \alpha_u & -e^{-2\alpha} \text{Im} \, \psi & e^{-2\alpha} \text{Re} \, \psi \\
\alpha_u & \alpha_v & e^{-2\alpha} \text{Re} \, \psi & -e^{-2\alpha} \text{Im} \, \psi \\
e^{-2\alpha} \text{Im} \, \psi & e^{-2\alpha} \text{Re} \, \psi & \alpha_v & -\alpha_u \\
e^{-2\alpha} \text{Re} \, \psi & e^{-2\alpha} \text{Im} \, \psi & -\alpha_u & \alpha_v
\end{pmatrix}$$

satisfy $A_v - B_u = AB - BA$. This is the compatibility condition of the following linear over-determined system

$$(X_1 \, X_2 \, X_3 \, X_4)_u = (X_1 \, X_2 \, X_3 \, X_4)A,$$

$$(X_1 \, X_2 \, X_3 \, X_4)_v = (X_1 \, X_2 \, X_3 \, X_4)B,$$  \hspace{1cm} (6.6)

where $X_i \ (i = 1, 2, 3, 4)$ are $E_4^2$-valued functions and therefore we see that the system (6.6) has a unique local solution for a given initial value. Therefore we can find a time-like and holomorphic immersion $F$ of a neighborhood of each point of $M$ into $E_4^2$ with respect to $\mathcal{J}_-$ as in Theorem 6.4 and $F$ is uniquely determined by $g$ and $\Psi$ up to a composition of isometries of $E_4^2$ induced by elements of $SU(1, 1)$ and parallel translations in $E_4^2$. We will show (a) from (b) and suppose (b). Noticing that $\Psi$ is a holomorphic differential and that $T_- = \text{Re} \, \Psi$, we obtain

$$T_- \left( V, \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right) = T_- \left( V, \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right)$$

for any $V \in T_a(M)$ and any $a \in M$. This means that $F$ has zero mean curvature vector. We set $t_1 := dF(\partial/\partial u)$, $t_2 = dF(\partial/\partial v)$. The following hold:

$$\text{Re} \, \psi = T_- \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right) = h(n_1, \nabla_{t_1} t_1),$$

$$\text{Im} \, \psi = T_- \left( \frac{\partial}{\partial v}, \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right) = h(n_2, \nabla_{t_2} t_2),$$  \hspace{1cm} (6.7)

where $n_k$ is the normal component of $\mathcal{I}_-(t_k)$. We obtain

$$h(n_1, n_1) = -h(n_2, n_2), \quad h(n_1, n_2) = 0.$$  \hspace{1cm} (6.8)

Therefore, using (6.7) and (6.8), we obtain

$$\left( \nabla_{t_1} t_1 \right)^\perp = \frac{1}{h(n_1, n_1)} (\text{Re} \, \psi) n_1 - (\text{Im} \, \psi) n_2.$$  \hspace{1cm} (6.9)
Similarly, we obtain
\[
(\nabla_{t_1} t_2)^\perp = \frac{1}{h(n_1, n_1)}((\text{Im } \psi)n_1 - (\text{Re } \psi)n_2).
\] (6.10)

Therefore \( F \) is isotropic. Using (6.9) and (6.10), we obtain
\[
K = -\frac{2|\psi|^2}{e^{4\alpha} h(n_1, n_1)}.
\]

Since we suppose \( K = -\frac{2|\psi|^2}{e^{6\alpha}} \), we obtain \( h(n_1, n_1) = e^{2\alpha} \). This means \( n_k = T_-(t_k) \) for \( k = 1, 2 \). Considering the composition of \( F \) and a suitable isometry of \( E_1^2 \) if necessary, we can suppose \( J_1 t_1 = t_2 \) at a point of \( M \). Then \( J_1 n_1 = -n_2 \) at this point. Then \( F \) is holomorphic with respect to \( J_1 \). Hence we obtain Theorem 6.4.

**Remark** In the above proof, the part where we prove (a) from (b) needs a nontrivial revision of the proof of the corresponding result in [1]. We can show (a) from (b) of Theorem 6.2, without nontrivial revisions or referring to the proof of Theorem 6.4.

7 Time-like surfaces with zero mean curvature vector in pseudo-Riemannian space forms

7.1 A holomorphic line bundle for paracomplex quadratic differentials

This subsection is analogous to a section of [3].

Let \( M \) be a Lorentz surface and \( \{(U_\lambda, w_\lambda)\}_{\lambda \in \Lambda} \) a holomorphic system of paracomplex coordinate neighborhoods of \( M \). Let \( \mathcal{L} \) be a holomorphic line bundle on \( M \) such that a family of transition functions \( \{\theta_{\lambda\mu}\} \) for \( \{U_\lambda\} \) is given by \( \theta_{\lambda\mu} = (dw_\mu/dw_\lambda)^2 \) on \( U_\lambda \cap U_\mu \) for \( \lambda, \mu \) with \( U_\lambda \cap U_\mu \neq \emptyset \). A section of \( \mathcal{L} \) is just a paracomplex quadratic differential on \( M \).

Let \( g \) be a paraHermitian metric on \( M \). We represent \( g \) as \( g = e^{4\alpha} dw_\lambda d\overline{w}_\lambda \) on \( U_\lambda \) by a function \( \alpha_\lambda \). Let \( \tilde{\Phi}, \tilde{\Psi} \) be sections of the direct sum \( \mathcal{L}^n \) of \( n \) copies of \( \mathcal{L} \). Then \( \tilde{\Phi}, \tilde{\Psi} \) are represented by \( (\phi_{\lambda,1}, \ldots, \phi_{\lambda,n}), (\psi_{\lambda,1}, \ldots, \psi_{\lambda,n}) \) on \( U_\lambda \), respectively, where \( \phi_{\lambda,k}, \psi_{\lambda,k} \) are paracomplex-valued functions on \( U_\lambda \). We set
\[
\tilde{g}_\lambda(\tilde{\Phi}, \tilde{\Psi}) := \frac{4}{e^{4\alpha_\lambda}} \sum_{k=1}^n \varepsilon_k \phi_{\lambda,k} \overline{\psi}_{\lambda,k},
\]

where \( \varepsilon_k = 1 (k = 1, \ldots, p), -1 (k = p + 1, \ldots, p + q) \) and \( p \geq 0, q \geq 0, p + q = n \). Then we have \( \tilde{g}_\lambda(\tilde{\Phi}, \tilde{\Psi}) = \tilde{g}_\mu(\tilde{\Phi}, \tilde{\Psi}) \) on \( U_\lambda \cap U_\mu \) if \( U_\lambda \cap U_\mu \neq \emptyset \), and therefore we can define a non-degenerate paraHermitian metric \( \tilde{g} \) on \( \mathcal{L}^n \) by \( \tilde{g}(\tilde{\Phi}, \tilde{\Psi}) = \tilde{g}_\lambda(\tilde{\Phi}, \tilde{\Psi}) \) on \( U_\lambda \).
Let $\nabla$ be the Levi-Civita connection of $g$. Then for a paracomplex quadratic differential $\Phi = \phi_\lambda dw_\lambda \otimes dw_\lambda$, we obtain
\[
\nabla_{\partial_\lambda} \Phi = \left( \frac{\partial \phi_\lambda}{\partial w_\lambda} - 4 \phi_\lambda \frac{\partial \alpha_\lambda}{\partial w_\lambda} \right) dw_\lambda \otimes dw_\lambda, \qquad \nabla_{\overline{\partial}_\lambda} \Phi = \frac{\partial \phi_\lambda}{\partial w_\lambda} dw_\lambda \otimes dw_\lambda, \tag{7.1}
\]
where $\partial_\lambda := \partial/\partial w_\lambda$, $\overline{\partial}_\lambda := \partial/\partial \overline{w}_\lambda$. Noticing (7.1), for a section $\tilde{\Phi}$ of $L^n$, we define an $L^n$-valued 1-form $\tilde{\nabla} \tilde{\Phi}$ by
\[
\tilde{\nabla}_{\partial_\lambda} \tilde{\Phi} := \frac{\partial \tilde{\Phi}}{\partial w_\lambda} - 4 \phi_\lambda \frac{\partial \alpha_\lambda}{\partial w_\lambda}, \quad \tilde{\nabla}_{\overline{\partial}_\lambda} \tilde{\Phi} := \frac{\partial \tilde{\Phi}}{\partial \overline{w}_\lambda}, \tag{7.2}
\]
where $\partial \Phi/\partial w_\lambda$ and $\partial \Phi/\partial \overline{w}_\lambda$ are given by
\[
\left( \frac{\partial \phi_\lambda,1}{\partial w_\lambda}, \ldots, \frac{\partial \phi_\lambda,n}{\partial w_\lambda} \right), \quad \left( \frac{\partial \phi_\lambda,1}{\partial \overline{w}_\lambda}, \ldots, \frac{\partial \phi_\lambda,n}{\partial \overline{w}_\lambda} \right)
\]
on $U_\lambda$, respectively. Then $\tilde{\nabla}$ is a connection of $L^n$. In addition, we see by (7.2) that $\tilde{g}$ is parallel with respect to $\tilde{\nabla}$ and that the $(0,1)$-part of $\tilde{\nabla}$ is given by $\overline{\partial}$.

### 7.2 The equations of Gauss-Codazzi-Ricci

Let $N$ be an $(n+2)$-dimensional pseudo-Riemannian space form with signature $(p+1, q+1)$ and $M$ a Lorentz surface. Let $F : M \rightarrow N$ be a time-like and conformal immersion of $M$ into $N$ with zero mean curvature vector and let $\nu_1, \ldots, \nu_n$ form a local frame of the normal bundle on $M$ with respect to $F$ satisfying $h(\nu_k, \nu_l) = \varepsilon_k \delta_{kl}$, where $h$ is the metric on $N$. Let $w$ be a local paracomplex coordinate of $M$ and set $\partial_w := \partial/\partial w$, $\overline{\partial}_w := \partial/\partial \overline{w}$. Let $\Phi_1, \ldots, \Phi_n$ be paracomplex quadratic differentials on the domain of $\nu_k$ defined by
\[
\Phi_k := \phi_k dw \otimes dw, \quad \phi_k := h(\nabla_{\partial_w} dF(\partial_w), \nu_k) \tag{7.3}
\]
for $k = 1, \ldots, n$, where $\nabla$ denotes the Levi-Civita connection of $h$, and $h$, $\nabla$ and $dF$ are paracomplexified if necessary as above. Let $L$ be a holomorphic line bundle on $M$ such that each section is a paracomplex quadratic differential on $M$. We see that $\Phi_1, \ldots, \Phi_n$ form a local section $\tilde{\Phi}$ of the direct sum $L^n$ of $n$ copies of $L$. As we already saw in the previous subsection, the induced metric $g$ on $M$ by $F$ gives a non-degenerate paraHermitian metric $\tilde{g}$ on $L^n$ and the Levi-Civita connection of $g$ gives a connection $\tilde{\nabla}$ of the holomorphic paraHermitian vector bundle $(L^n, \tilde{g})$. Let $\Theta$ be an $n \times n$ matrix such that the $(k,l)$-component is given by a 2-form $2\varepsilon_k \phi_k \overline{\phi_l} e^{-2\alpha} dw \wedge d\overline{w}$, where $\alpha$ is a real-valued function given by $g = e^{2\alpha} dw d\overline{w}$.

Referring to [3], we can prove the following theorem:

**Theorem 7.1** There exists a $(0,1)$-form $\omega$ on the domain of $\Phi$, valued in the Lie algebra of $O(p, q, \tilde{C})$ satisfying
\[
\tilde{g}(\tilde{\Phi}, \tilde{\Phi}) = L_0 - K, \quad \tilde{\nabla}_{\overline{\partial}_w} \tilde{\Phi} = \tilde{\Phi} \cdot \omega(\overline{\partial}_w), \quad \text{Re } \tilde{\Omega} = \text{Re } \Theta, \tag{7.4}
\]
where \( K \) is the curvature of \( g \), \( L_0 \) denotes the constant sectional curvature of \( N \) and 
\[ \tilde{\Omega} := d\omega + \bar{\omega} \wedge \omega. \]
In addition, for a paraHermitian metric \( g \) on \( M \) and a section \( \tilde{\Phi} \) of \( \mathcal{L}^n \) satisfying (7.4) with \( \omega \) as above, there exists an isometric immersion of a neighborhood of each point of \( M \) into \( N \) with zero mean curvature vector such that \( \tilde{\Phi} \) gives paracomplex quadratic differentials with respect to a suitable local frame \( \{\nu_1, \ldots, \nu_n\} \) of the normal bundle satisfying \( h(\nu_k, \nu_l) = \varepsilon_k \delta_{kl} \). Such an immersion is unique up to an isometry of \( N \).

**Remark** Whether the relations in (7.4) hold does not depend on the choice of \( \nu_1, \ldots, \nu_n \).

### 7.3 Holomorphic quartic differentials

We can find holomorphic differentials on time-like surfaces in \( N \) with zero mean curvature vector.

Suppose \( n = p = 1 \). Then \( N \) is a 3-dimensional Lorentz space form. The first equation in (7.4) gives a relation between the induced metric and the analogue of the Hopf differential, and the second equation in (7.4) says that the analogue of the Hopf differential is holomorphic with respect to local paracomplex coordinates. The third equation in (7.4) is trivial.

Suppose \( n = p = 2 \). Then \( N \) is a 4-dimensional Lorentz space form. In this case, the second equation in (7.4) is given by

\[
\frac{\partial \phi_1}{\partial \bar{w}} = \psi \phi_2, \quad \frac{\partial \phi_2}{\partial \bar{w}} = -\psi \phi_1
\]

(7.5)

for a paracomplex-valued function \( \psi \). We set \( Q := \Phi_1 \otimes \Phi_1 + \Phi_2 \otimes \Phi_2 \). Then \( Q \) does not depend on the choice of a local orthonormal frame \( \nu_1, \nu_2 \) and we see by (7.5) that a paracomplex quartic differential \( Q \) on \( M \) is holomorphic. We see that \( Q \equiv 0 \) if and only if the second fundamental form of \( F : M \to N \) vanishes.

Suppose \( n = 2 \) and \( p = q = 1 \). Then \( N \) is a 4-dimensional neutral space form. In this case, the second equation in (7.4) is given by

\[
\frac{\partial \phi_1}{\partial \bar{w}} = \psi \phi_2, \quad \frac{\partial \phi_2}{\partial \bar{w}} = \psi \phi_1
\]

(7.6)

for a paracomplex-valued function \( \psi \). We set \( Q := \Phi_1 \otimes \Phi_1 - \Phi_2 \otimes \Phi_2 \). Then \( Q \) does not depend on the choice of a local frame \( \nu_1, \nu_2 \) satisfying \( h(\nu_k, \nu_l) = \varepsilon_k \delta_{kl} \) and (7.6) implies that a paracomplex quartic differential \( Q \) on \( M \) is holomorphic. The isotropicity of \( F \) is just the condition \( Q \equiv 0 \). We set \( \tilde{\iota} := 1/\sqrt{2} (\nu_2 - \nu_1) \), \( \tilde{\nu} := 1/\sqrt{2} (\nu_2 + \nu_1) \). Then \( \tilde{\iota}, \tilde{\nu} \) are light-like normal vector fields of \( F \) satisfying \( h(\tilde{\iota}, \tilde{\nu}) = -1 \). We set

\[
\Phi_{\tilde{\iota}} := h(\nabla_{\partial_w} dF(\partial_w), \tilde{\iota))dw \otimes dw, \quad \Phi_{\tilde{\nu}} := h(\nabla_{\partial_w} dF(\partial_w), \tilde{\nu))dw \otimes dw.
\]

Then we obtain

\[
Q = -2\Phi_{\tilde{\iota}} \otimes \Phi_{\tilde{\nu}}.
\]

(7.7)
Suppose $Q \equiv 0$. We suppose that for $T = T_1 + T_2$, $(\nabla_T T) \perp$ is zero or properly light-like and that $(\nabla_{T_1} T_1) \perp \neq (\nabla_{T_2} T_2) \perp$. If $\hat{\nabla}_{T_1} \hat{F}$ and $\hat{\nabla}_{T_2} \hat{F}$ are zero, then (b) of Theorem 4.3 says that $F$ is strictly isotropic and then both $\Phi_i$ and $\Phi_\nu$ in (7.7) are null. Suppose that $\hat{\nabla}_{T_1} \hat{F}$ and $\hat{\nabla}_{T_2} \hat{F}$ are properly light-like so that $\hat{\nabla}_T \hat{F} = 0$. Then (c) of Theorem 4.3 says that $F$ has properly light-like second fundamental form and then either $\Phi_i$ and $\Phi_\nu$ in (7.7) vanishes. This condition is equivalent to a condition that a light-like normal vector field of $F$ is contained in a constant direction.

**Example** Let $F : M \to E^4_2$ be a time-like and conformal immersion. Suppose that $F$ has zero mean curvature vector and that a light-like normal vector field $\tilde{\nu}$ of $F$ is contained in a constant direction. Then the normal components $F_{uu}^\perp$, $F_{uv}^\perp$ of $F_{uu}$, $F_{uv}$, respectively are contained in the direction given by $\tilde{\nu}$. Therefore $F$ is isotropic and if $F_{uu}^\perp \neq F_{uv}^\perp$, then we can choose the orientation of $E^4_2$ so that $F$ has properly light-like second fundamental form. Suppose that $F$ has zero mean curvature vector, isotropicity and properly light-like second fundamental form. Then we have

$$h(F_{uu}^\perp, F_{uv}^\perp) = h(F_{uu}^\perp, F_{uv}^\perp) = 0, \quad h(F_{uu}^\perp, F_{uv}^\perp) = 0.$$  \hfill (7.8)

We represent $F$ as $F(u, v) = F^1(u + v) + F^2(u - v)$. Then we see from (7.8) that $F^1$, $F^2$ satisfy

$$h(\hat{F}^1(u + v), \hat{F}^2(u - v))h(\hat{F}^1(u + v), \hat{F}^2(u - v)) = h(\hat{F}^1(u + v), \hat{F}^2(u - v))h(\hat{F}^1(u + v), \hat{F}^2(u - v)).$$  \hfill (7.9)

Using (6.1) and (7.9), we obtain

$$\left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}\right) \log h(F_u, F_u) = \left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}\right) \log h(\hat{F}^1(u + v), \hat{F}^2(u - v)) = 0.$$

Then we see that one of $\hat{F}^1$ and $\hat{F}^2$ is contained in a constant direction and that the other is parallel to a fixed light-like 2-plane in $E^4_2$. Therefore there exists a light-like normal vector field of $F$ contained in a constant direction.

Suppose $L_0 = 1$, i.e., $N = S^4_2$. In addition, suppose that the shape operator of $F$ with respect to $\tilde{i}$ is neither zero nor light-like. Then $\tilde{i}$ gives a time-like immersion $\iota$ of Willmore type of a neighborhood of each point of $M$ into each of $S^3_1$, $E^3_1$, $H^3_1$. The conformal Gauss map of a time-like surface in each of $S^3_1$, $E^3_1$ and $H^3_1$ is a map from the surface into $S^4_2$, and except umbilical points and light-likely umbilical points, it is a time-like and conformal immersion into $S^4_2$. A time-like surface in $S^3_1$, $E^3_1$ or $H^3_1$ is of Willmore type if and only if except umbilical points and light-likely umbilical points, the mean curvature vector of the conformal Gauss map vanishes. We can find a holomorphic quartic differential $\hat{Q}$ on a time-like surface of Willmore type in each of $S^3_1$, $E^3_1$ and $H^3_1$, and except umbilical points and light-likely umbilical points, the holomorphic quartic differential $Q$ given by
the conformal Gauss map of the surface as in (7.7) coincides with \( \tilde{Q} \) up to a nonzero constant, if \( Q \) and \( \tilde{Q} \) do not become null. In addition, \( \tilde{Q} \equiv 0 \) if and only if a light-like normal vector field of the conformal Gauss map is contained in a constant direction; \( \tilde{Q} \) is null if and only if the shape operator of the conformal Gauss map with respect to a light-like normal vector field is light-like. We can have analogous discussions for \( L_0 = 0, -1 \), referring to [3]. We will discuss the conformal Gauss maps of time-like surfaces in Section 8.

In general, we set

\[
Q := \sum_{k=1}^{n} \varepsilon_k \Phi_k \otimes \Phi_k = \left( \sum_{k=1}^{n} \varepsilon_k \phi_k^2 \right) dw \otimes dw \otimes dw \otimes dw. \tag{7.10}
\]

Noticing (7.3), we see that \( Q \) does not depend on the choice of a local frame \( \{ \nu_1, \ldots, \nu_n \} \) satisfying \( h(\nu_k, \nu_l) = \varepsilon_k \delta_{kl} \). Therefore we can define a paracomplex quartic differential \( Q \) on \( M \) by (7.10). By the second equation in (7.4), we see that \( \sum_{k=1}^{n} \varepsilon_k \phi_k^2 \) is holomorphic. Hence we see that a time-like and conformal immersion \( F : M \rightarrow N \) with zero mean curvature vector gives a holomorphic quartic differential \( Q \) on \( M \) by (7.10).

8 The conformal Gauss maps

8.1 The conformal Gauss maps of time-like surfaces in 3-dimensional Lorentz space forms

Let \( \iota \) be a time-like immersion of an oriented two-dimensional manifold \( M \) into \( N_1^3 = S_1^3 \), \( E_1^3 \) or \( H_1^3 \). Let \( \langle \, , \rangle \) be the metric of \( E_2^5 \), and set

\[
L := \{ x = (x^1, x^2, x^3, x^4, x^5) \in E_2^5 \mid \langle x, x \rangle = 0 \}.
\]

We identify \( S_1^3, E_1^3, H_1^3 \) with \( L \cap \{ x^5 = 1 \} \), \( L \cap \{ x^5 = x^1 + 1 \} \), \( L \cap \{ x^1 = 1 \} \), respectively and therefore we consider \( \iota : M \rightarrow N_1^3 \) to be an \( L \)-valued function. Let \( e_3 \) be a unit normal vector field of \( \iota \) in \( N_1^3 \). We can naturally choose \( e_3 \) for the orientations of \( M \) and \( N_1^3 \). Let \( H \) be the mean curvature of \( \iota \) with respect to \( e_3 \). Let \( \gamma_\iota : M \rightarrow S_2^4 \) be a map from \( M \) into \( S_2^4 \) defined by \( \gamma_\iota := e_3 + H \iota \). At an umbilical point of \( \iota \), the tangential component of \( d\gamma_\iota \) is zero. A light-like umbilical point of \( \iota \) is a point of \( M \) where the tangential component of \( d\gamma_\iota \) is light-like. The complement of the set of umbilical points and light-like umbilical points of \( \iota \) is denoted by \( \text{Reg} (\iota) \). We set \( \Lambda := H^2 - K^M + \delta \), where \( \delta = +1, 0 \) or \( -1 \) according to \( N_1^3 = S_1^3, E_1^3 \) or \( H_1^3 \), and \( K^M \) is the curvature of the induced metric \( g^M \) by \( \iota \). Then \( \text{Reg} (\iota) \) is the set of nonzero points of \( \Lambda \). The restriction of \( \gamma_\iota \) on \( \text{Reg} (\iota) \) is a time-like immersion which induces a Lorentz metric \( g \) given by \( g = \Lambda g^M \). The map \( \gamma_\iota : M \rightarrow S_2^4 \) is called the conformal Gauss map of a time-like immersion \( \iota : M \rightarrow N_1^3 \).

We see that \( \iota \) is an \( L \)-valued normal vector field of \( F := \gamma_\iota \mid_{\text{Reg} (\iota)} \) and that the trace of the
shape operator of $F$ with respect to $\iota$ vanishes. The shape operator of $F$ with respect to $\iota$ is neither zero nor light-like.

Let $(u, v)$ be local coordinates on a neighborhood $U$ of each point of $\text{Reg} (\iota)$ such that $g^M$ is locally represented as $g^M = e^{2\alpha}(du^2 - dv^2)$. We have

$$
(e_3)_u = -\frac{1}{e^{2\alpha}} \partial_u \left( \frac{\partial}{\partial u} \right) + \frac{m}{e^{2\alpha}} \partial_u \left( \frac{\partial}{\partial v} \right),
$$

$$
(e_3)_v = -\frac{m}{e^{2\alpha}} \partial_u \left( \frac{\partial}{\partial u} \right) + \frac{n}{e^{2\alpha}} \partial_v \left( \frac{\partial}{\partial v} \right),
$$

where $l := b(\partial/\partial u, \partial/\partial u)$, $m := b(\partial/\partial u, \partial/\partial v)$, $n := b(\partial/\partial v, \partial/\partial v)$ and $b$ is the second fundamental form of $\iota$ with respect to $e_3$. Let $e_4$ be an $L$-valued function on $U$ satisfying

$$
\left\langle du \left( \frac{\partial}{\partial u} \right), e_4 \right\rangle = 0, \quad \left\langle dv \left( \frac{\partial}{\partial v} \right), e_4 \right\rangle = 0, \quad \langle e_3, e_4 \rangle = 0, \quad \langle \iota, e_4 \rangle = -1.
$$

Let $\nu$ be an $L$-valued normal vector field of $F$ satisfying $\langle \nu, \iota \rangle = -1$. Then $\nu$ is locally represented as

$$
\nu = \frac{H_u}{e^{2\alpha} \Lambda} d\gamma_l \left( \frac{\partial}{\partial u} \right) - \frac{H_v}{e^{2\alpha} \Lambda} d\gamma_l \left( \frac{\partial}{\partial v} \right) + \frac{1}{2} \left( \frac{H_u^2}{e^{2\alpha} \Lambda} + \frac{H_v^2}{e^{2\alpha} \Lambda} \right) \iota + He_3 + e_4.
$$

Noticing (8.1), we see that the trace of the shape operator of $F$ with respect to $\nu$ is given by $- (1/\Lambda)(\Delta^M H + 2\Lambda H)$, where $\Delta^M$ is the Laplacian on $M$ with respect to the Lorentz metric $g^M$, i.e., an operator on $C^\infty(M)$ such that for each $f \in C^\infty(M)$, $\Delta^M f$ is the trace of the Hessian of $f$ with respect to $g^M$. Since $\Lambda \equiv 0$ implies $\Delta^M H = 0$, we obtain

**Theorem 8.1** The mean curvature vector of $F = \gamma|_{\text{Reg} (\iota)}$ vanishes if and only if $\iota : M \longrightarrow N^3_1$ satisfies $\Delta^M H + 2\Lambda H = 0$.

In the following, a time-like immersion $\iota : M \longrightarrow N^3_1$ is said to be of Willmore type if $\iota$ satisfies $\Delta^M H + 2\Lambda H = 0$.

Referring to [3], we obtain

**Theorem 8.2** Let $\gamma : M \longrightarrow S^3_2$ be a time-like immersion with zero mean curvature vector and neither zero nor light-like shape operator with respect to a light-like normal vector field $\iota$. Then $\gamma$ or $-\gamma$ is the conformal Gauss map of a time-like immersion of Willmore type of a neighborhood of each point of $M$ into each of $S^3_1$, $E^3_1$, $H^3_1$ given by $\iota$.

Let $M$ be a Lorentz surface and $\iota$ a time-like and conformal immersion of $M$ into $N^3_1 = S^3_1$, $E^3_1$ or $H^3_1$. Let $e_3$, $H$, $b$ be as above. Let $\Xi$ be a 4-tensor field on $M$ given by

$$
\Xi := 2b \otimes \text{Hess}_H^M + (H^2 + \delta)b \otimes b - 2dH \otimes \nabla^M b,
$$

where $\text{Hess}_H^M$ is the Hessian of $H$ with respect to the Levi-Civita connection $\nabla^M$ of $g^M$. We consider $\Xi$ to be a paracomplex 4-linear function on the paracomplexification
of the tangent plane of $M$ at each point. Let $w$ be a local paracomplex coordinate on a neighborhood $U$ of each point of $M$ and $u, v$ the real and the paraimaginary parts of $w$, respectively: $w = u + jv$. Then $g^M$ is represented as $g^M = e^{2\alpha}(du^2 - dv^2)$ on $U$. Referring to [3], we will prove

**Proposition 8.3** If $i : M \rightarrow N^3_1$ is of Willmore type, then a paracomplex quartic differential

\[
\tilde{Q} := \Xi \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right) \, dw \otimes dw \otimes dw \otimes dw
\]  

(8.2)

is holomorphic.

**Proof** The proof of Proposition 8.3 consists of five parts.

**Part 1** We set $e_0 := i$. We denote by $e_1, e_2$ not only $e^{-\alpha} \partial/\partial u, e^{-\alpha} \partial/\partial u$ but also $e^{-\alpha} dt(\partial/\partial u), e^{-\alpha} dt(\partial/\partial v)$, respectively. Let $\omega^1_0, \omega^2_0$ be 1-forms given by $de_0 = e^1_1 \omega^1_0 + e^2_2 \omega^2_0$. Then we have $\omega^1_0 = e^1_{\partial u}$, $\omega^2_0 = e^2_{\partial v}$. We set $b_{kl} := b(e_k, e_l)$ for $k, l \in \{1, 2\}$. Then we have

\[
\begin{align*}
\delta e_1 &= -\frac{\delta}{2} e_0 \omega^1_0 + e_2 \omega^2_0 + e_3 (b_{11} \omega^1_0 + b_{12} \omega^2_0) + e_4 \omega_0^1, \\
\delta e_2 &= \frac{\delta}{2} e_0 \omega^2_0 + e_1 \omega^1_0 + e_3 (b_{21} \omega^1_0 + b_{22} \omega^2_0) - e_4 \omega_0^2, \\
\delta e_3 &= -e_1 (b_{11} \omega^1_0 + b_{12} \omega^2_0) + e_2 (b_{21} \omega^1_0 + b_{22} \omega^2_0), \\
\delta e_4 &= -\frac{\delta}{2} (e_1 \omega^1_0 + e_2 \omega^2_0),
\end{align*}
\]  

(8.3)

where $\omega^1_0$ is a 1-form satisfying

\[
d\omega^1_0 = -\omega^2_0 \wedge \omega^1_0, \quad d\omega^2_0 = -\omega^1_0 \wedge \omega^1_0.
\]  

(8.4)

We set $\omega := \omega^1_0 + j \omega^2_0 (= e^\alpha dw)$. Then we can rewrite (8.4) into

\[
d\omega = -j \omega^1_0 \wedge \omega.
\]  

(8.5)

**Part 2** We set

\[
\begin{align*}
\hat{e}_0 &= e_0, \quad \hat{e}_1 := e_1, \quad \hat{e}_2 := e_2, \quad \hat{e}_3 := e_3 + He_0, \quad \hat{e}_4 := \frac{1}{2} H^2 e_0 + He_3 + e_4.
\end{align*}
\]

Then by (8.3), we obtain

\[
\begin{align*}
d\hat{e}_0 &= \hat{e}_1 \omega^1_0 + \hat{e}_2 \omega^2_0, \\
d\hat{e}_1 &= \hat{e}_0 \omega^1_0 + \hat{e}_2 \omega^2_0 + \hat{e}_3 \left( \frac{b_{11} + b_{22}}{2} \omega^1_0 + b_{12} \omega^2_0 \right) + \hat{e}_4 \omega^1_0, \\
d\hat{e}_2 &= \hat{e}_0 \omega^2_0 + \hat{e}_1 \omega^1_0 + \hat{e}_3 \left( b_{21} \omega^1_0 + \frac{b_{11} + b_{22}}{2} \omega^2_0 \right) - \hat{e}_4 \omega^2_0, \\
d\hat{e}_3 &= \hat{e}_0 dH - \hat{e}_1 \left( \frac{b_{11} + b_{22}}{2} \omega^1_0 + b_{12} \omega^2_0 \right) + \hat{e}_2 \left( b_{21} \omega^1_0 + \frac{b_{11} + b_{22}}{2} \omega^2_0 \right), \\
d\hat{e}_4 &= \hat{e}_1 \omega^1_0 + \hat{e}_2 \omega^2_0 + \hat{e}_3 dH,
\end{align*}
\]  

(8.6)
where
\[
\hat{\omega}_1^0 := \left( -\frac{\delta}{2} - Hb_{11} + \frac{1}{2}H^2 \right) \omega_0^1 - Hb_{12} \omega_0^2, \\
\hat{\omega}_2^0 := -Hb_{21} \omega_0^1 + \left( \frac{\delta}{2} - Hb_{22} - \frac{1}{2}H^2 \right) \omega_0^2.
\]

We set \( e := e_1 + je_2 \) and
\[
\hat{\omega} := \hat{\omega}_1^0 - j\hat{\omega}_2^0, \quad \psi := \frac{1}{2}b(e,e) = \frac{b_{11} + b_{22}}{2} + jb_{12}.
\]

Then by (8.6), we obtain
\[
d\omega_1^2 = -\frac{j}{2} \bar{\psi} \omega \wedge \bar{\omega} - \frac{j}{2}(\omega \wedge \bar{\omega} - \bar{\omega} \wedge \hat{\omega}). \tag{8.8}
\]

**Part 3** We set \( H_1 := e_1(H), \) \( H_2 := e_2(H). \) Then we have \( dH = H_1 \omega_0^1 + H_2 \omega_0^2. \) Therefore by (8.4) together with the fourth equation in (8.6), we see that there exist functions \( p_{kl} \) \( (k,l = 1,2) \) satisfying \( p_{12} = p_{21} \) and
\[
dH_1 = H_2 \omega_1^2 + \frac{b_{11} + b_{22}}{2} \hat{\omega}_1^0 - b_{12} \hat{\omega}_2^0 + p_{11} \omega_0^1 + p_{12} \omega_0^2, \\
dH_2 = H_1 \omega_1^2 + b_{12} \hat{\omega}_1^0 - \frac{b_{11} + b_{22}}{2} \hat{\omega}_2^0 + p_{21} \omega_0^1 + p_{22} \omega_0^2. \tag{8.9}
\]

We set
\[
\varphi := \frac{1}{2}dH(e) = \frac{1}{2}(H_1 + jH_2), \quad q := \frac{1}{2} \left( \frac{p_{11} + p_{22}}{2} + jp_{12} \right). \tag{8.10}
\]

Then we can rewrite (8.9) into
\[
d\varphi = j\varphi \omega_1^2 + \frac{1}{2} \bar{\psi} \hat{\omega} + q \omega + \frac{1}{4}(p_{11} - p_{22})\bar{\varphi}. \tag{8.11}
\]

Therefore we obtain
\[
q = \frac{1}{4} \text{Hess}_d^M(e,e) + \frac{1}{8}(H^2 + \delta)b(e,e).
\]

By (8.9), we obtain \( p_{11} - p_{22} = \Delta^M H + 2\Lambda H. \) In the following, we suppose that \( \iota : M \rightarrow N_1^3 \) is of Willmore type. Then by (8.11) together with \( p_{11} - p_{22} = \Delta^M H + 2\Lambda H, \) we obtain
\[
d\varphi = j\varphi \omega_1^2 + \frac{1}{2} \bar{\psi} \hat{\omega} + q \omega. \tag{8.12}
\]

**Part 4** We set \( b_{klm} := (\nabla_{ek}^M b)(e_k, e_l) \) \( (k,l,m \in \{1,2\}). \) Then \( b_{klm} \) is symmetric with respect to \( k, l, m. \) For \( k, l \in \{1,2\}, \) we obtain
\[
db_{kl} = \sum_{m=1}^{2} (b_{lm} \omega_k^m + b_{km} \omega_l^m + b_{klm} \omega_0^m), \tag{8.13}
\]

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where \( \omega_1^2 = 0, \omega_2^2 = 0, \omega_1^2 := \omega_2^1 \). In particular, we have \( H_k = (1/2)(b_{11k} - b_{22k}) \) for \( k = 1, 2 \). By (8.4) together with (8.13), we obtain
\[
d\hat{\omega} = -j\omega_1^2 \wedge \hat{\omega} - \varphi \psi \omega \wedge \overline{\omega}.
\] (8.14)

We set
\[
\zeta := \frac{1}{4}(\nabla^M b)(e, e, e) = \frac{b_{111} + 3b_{122}}{4} + j\frac{3b_{112} + b_{222}}{4}.
\] (8.15)

Then by (8.13) together with \( H_k = (1/2)(b_{11k} - b_{22k}) \), we obtain
\[
d\psi = 2j\psi \omega^1 + \varphi \overline{\omega} + \zeta \omega.
\] (8.16)

Using (8.5), (8.8), (8.12), (8.14) and (8.16), we obtain
\[
dq = 2jq\omega^2 + \varphi \omega^2 + \zeta \omega + a_q \omega,
\]
\[
d\zeta = 3j\zeta \omega^2 - \frac{3}{2} \psi \omega - \psi^2 \overline{\psi} \omega + q \omega + a_\zeta \omega,
\] (8.17)

where \( a_q, a_\zeta \) are paracomplex-valued functions.

**Part 5** By \( \psi, \varphi, q, \zeta \) as above, we obtain
\[
\tilde{Q} = (\psi q - \varphi \zeta) \omega \otimes \omega \otimes \omega \otimes \omega.
\] (8.18)

Using (8.12), (8.16) and (8.17), we obtain
\[
d(\psi q - \varphi \zeta) = 4j(\psi q - \varphi \zeta) \omega^2 + (\psi a_q - \varphi a_\zeta) \omega.
\]

Therefore we see by (8.5) that \((\psi q - \varphi \zeta)e^{4\alpha} \) is a holomorphic function of a paracomplex variable \( w \). Hence we see from (8.18) that \( \tilde{Q} \) is a holomorphic quartic differential on \( M \) given by a time-like and conformal immersion \( \iota : M \rightarrow N_1^3 \) of Willmore type. \( \square \)

**Remark** We can show that \( \Xi(\partial_w, \partial_w, \partial_w, \partial_w) \) with \( \partial_w := \partial/\partial w \) is holomorphic with respect to \( w \), using
\[
K^M = -\frac{4}{e^{2\alpha}} \partial_w, \quad \phi_w = e^{2\alpha} H_w, \quad \phi\overline{\phi} = \Lambda e^{4\alpha}, \quad H_{\overline{w}w} = -\frac{\Lambda e^{2\alpha}}{2} H
\]
and \( \phi := 2b(\partial_w, \partial_w) \). On the other hand, the rest of the current subsection needs contents which appeared in the above proof of Proposition 8.3.

Referring to [3], we will prove

**Theorem 8.4** Let \( \iota : M \rightarrow N_1^3 \) be a time-like and conformal immersion of Willmore type. Then on \( \text{Reg}(\iota) \), the following hold:

(a) the null points of the differential \( Q \) for \( F := \gamma|_{\text{Reg}(\iota)} \) coincide with the null points of \( \tilde{Q} \), and a null point of \( Q \) is just given by a condition that the shape operator of \( F \) with respect to \( \nu \) is light-like;

Referring to [3], we will prove
(b) except the null points, \( Q \) coincides with \( \tilde{Q} \) up to a nonzero constant;

(c) \( Q \equiv 0 \) if and only if a light-like normal vector field \( \nu \) of \( F \) is contained in a constant direction.

In order to prove Theorem 8.4, we will prove

**Proposition 8.5** Let \( \iota : M \longrightarrow N^3_1 \) be a time-like and conformal immersion of Willmore type. Then \( \Lambda^3 \langle dv, dv \rangle = 4|\psi q - \varphi \zeta|^2 g \) on \( \text{Reg} (\iota) \).

**Proof** We set

\[
W := 2\varphi \overline{\varphi} \hat{e}_0 - \varphi \psi e - \varphi \psi \overline{\varphi} + \psi \overline{\psi} \hat{e}_4. \tag{8.19}
\]

Then we have \( \langle W, W \rangle = 0 \). We set

\[
Z := 2\overline{\varphi} q \hat{e}_0 - \overline{\psi} q e - \overline{\varphi} \zeta + \overline{\psi} \zeta \hat{e}_4.
\]

Then we have

\[
\langle Z, Z \rangle = 0, \quad \langle Z, \overline{Z} \rangle = 2|\psi q - \varphi \zeta|^2.
\]

Using (8.6), (8.12) and (8.16), we obtain \( dW = Z\omega + \overline{Z}\overline{\omega} \). Therefore we obtain

\[
\langle W_u, W_u \rangle = -\langle W_v, W_v \rangle = 4|\psi q - \varphi \zeta|^2 e^{2\alpha}, \quad \langle W_u, W_v \rangle = 0. \tag{8.20}
\]

On \( \text{Reg} (\iota) \), we set

\[
\hat{e}_0 := \iota, \quad \hat{e}_1 := \frac{1}{\sqrt{|\Lambda|}|e^\alpha d\gamma_u} \left( \frac{\partial}{\partial u} \right), \quad \hat{e}_2 := \frac{1}{\sqrt{|\Lambda|}|e^\alpha d\gamma_v} \left( \frac{\partial}{\partial v} \right), \quad \hat{e}_3 := \gamma, \quad \hat{e}_4 := \nu.
\]

Then we obtain

\[
[e_0, e_1, e_2, e_3, e_4] = [\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4] \tilde{M}^{-1},
\]

where

\[
\tilde{M}^{-1} := \begin{pmatrix}
1 & P & -Q & -H & R \\
0 & -\frac{\hat{b}_{11} + \hat{b}_{22}}{2} & \hat{b}_{12} & 0 & -\hat{H}_1 \\
0 & \hat{b}_{12} & \frac{\hat{b}_{11} + \hat{b}_{22}}{2} & 0 & \hat{H}_2 \\
0 & 0 & 0 & 1 & -H \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

and

\[
P + jQ := 2\frac{\varphi \overline{\varphi}}{\Lambda}, \quad R := \frac{H^2}{2} + 2\frac{\varphi \overline{\varphi}}{\Lambda}, \quad \tilde{H}_i := \text{sgn} (\Lambda) \frac{H_i}{\sqrt{|\Lambda|}}, \quad \tilde{b}_{ij} := \text{sgn} (\Lambda) \frac{b_{ij}}{\sqrt{|\Lambda|}}.
\]
Since

\[ [\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4] = [e_0, e_1, e_2, e_3, e_4] \hat{M}, \quad \hat{M} := \begin{pmatrix}
1 & 0 & 0 & H & \frac{1}{2}H^2 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & H \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}, \]

we obtain

\[ [\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4] = [\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4] \tilde{M}^{-1} \tilde{M} \]

\[ = [\tilde{e}_0, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4] \left( \begin{array}{ccc}
1 & \frac{P}{2} & -Q & 0 & R - \frac{H^2}{2} \\
0 & -\frac{\tilde{b}_{11} + \tilde{b}_{22}}{2} & -\frac{\tilde{b}_{12}}{2} & 0 & -\hat{H}_1 \\
0 & \frac{\tilde{b}_{22}}{2} & \frac{\tilde{b}_{11} + \tilde{b}_{22}}{2} & 0 & \hat{H}_2 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array} \right). \]

Using this and (8.19), we obtain \( W = \Lambda \nu \). Therefore by (8.20), we obtain Proposition 8.5.

\[ \square \]

**Proof of Theorem 8.4** We set \( F := \gamma_i|_{\text{Reg}(\iota)}, \tilde{\iota} := \iota, \tilde{\nu} := \nu \). Then we obtain

\[ |h(\nabla_{\partial \omega} dF(\partial \omega), \iota)|^2 = \frac{\Lambda e^{4\alpha}}{4}. \] (8.21)

From Proposition 8.5, we obtain

\[ \langle \nu_u, \nu_u \rangle = -\langle \nu_v, \nu_v \rangle = \frac{4|\psi q - \varphi \zeta|^2 e^{2\alpha}}{\Lambda^2}, \quad \langle \nu_u, \nu_v \rangle = 0. \] (8.22)

We represent \( \nu_u, \nu_v \) as

\[ \nu_u = c_{11} \gamma_u + c_{12} \gamma_v + c_{1*} \nu, \quad \nu_v = c_{21} \gamma_u + c_{22} \gamma_v + c_{2*} \nu. \]

Then we have \( c_{21} = -c_{12} \). Since \( F \) has zero mean curvature vector, we have \( c_{22} = -c_{11} \). Then (8.22) yields

\[ |h(\nabla_{\partial \omega} dF(\partial \omega), \nu)|^2 = \frac{|\psi q - \varphi \zeta|^2 e^{4\alpha}}{\Lambda}. \] (8.23)

If we represent \( Q \) in (7.7) as \( Q = f dw \otimes dw \otimes dw \otimes dw \), then from (8.21) and (8.23), we obtain \( |f|^2 = |\psi q - \varphi \zeta|^2 e^{8\alpha} \). If we represent \( \tilde{Q} \) in (8.18) as \( \tilde{Q} = \tilde{f} dw \otimes dw \otimes dw \otimes dw \), then we have \( \tilde{f} = (\psi q - \varphi \zeta)e^{4\alpha} \) and therefore we obtain \( |\tilde{f}|^2 = |f|^2 \). Noticing (8.21), we see that at a point of \( \text{Reg}(\iota) \), \( Q \) is zero if and only if \( h(\nabla_{\partial \omega} dF(\partial \omega), \nu) \) is zero. A zero point of \( h(\nabla_{\partial \omega} dF(\partial \omega), \nu) \) is just a point where \( \nu_u \) and \( \nu_v \) are contained in the direction of \( \nu \). In addition, noticing \( dW = Z \omega + \overline{Z}Z \), we see that at a point, \( \nu_u \) and \( \nu_v \) are contained in the direction of \( \nu \) if and only if \( W \) and \( Z \) satisfy \( \zeta W = \psi Z \) and \( qW = \varphi Z \), which is

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Let $\langle \cdot, \cdot \rangle$ be a map defined by $\langle e_3, e_3 \rangle = -1$. We naturally choose $e_3$ for the orientations of $M$ and $N_2^3$. Let $H$ be the mean curvature of $\iota$ with respect to $e_3$. Let $\gamma_{\iota} : M \rightarrow H_2^2$ be a map defined by $\gamma_{\iota} := -e_3 + H\iota$. Then $F = \gamma_{\iota}|_{\text{Reg}(\iota)}$ is a time-like immersion which induces a metric $g$ conformal to $g^{\mathcal{M}}$ satisfying $g = \Lambda g^{\mathcal{M}}$, where $\Lambda := H^2 + K^M - \delta$ with $\delta = +1, 0$ or $-1$ according to $N_2^3 = S_2^3, E_2^3$ or $H_2^3$. The map $\gamma_{\iota} : M \rightarrow H_2^2$ is called the conformal Gauss map of a time-like immersion $\iota : M \rightarrow N_2^3$. We see that $\iota$ is an $L$-valued normal vector field of $F$ and that the trace of the shape operator of $F$ with respect to $\iota$ vanishes. Let $\nu$ be an $L$-valued normal vector field of $F$ satisfying $\langle \nu, \iota \rangle = -1$. Then the trace of the shape operator of $F$ with respect to $\nu$ vanishes if and only if $\Delta^M H - 2\Lambda H = 0$. We say that a time-like immersion $\iota : M \rightarrow N_2^3$ is of Willmore type if $\Delta^M H - 2\Lambda H = 0$. Then we obtain

**Theorem 8.6** The mean curvature vector of $F = \gamma_{\iota}|_{\text{Reg}(\iota)}$ vanishes if and only if $\iota : M \rightarrow N_2^3$ is of Willmore type.

In addition, we obtain

**Theorem 8.7** Let $\gamma : M \rightarrow H_2^2$ be a time-like immersion with zero mean curvature vector and neither zero nor light-like shape operator with respect to a light-like normal vector field of a time-like immersion $\iota : M \rightarrow N_2^3$. Then $\nu$ is contained in a constant direction.

Therefore we obtain (c) in Theorem 8.4. □

Remark See [4] for the conformal Gauss maps of zero mean curvature surfaces of mixed type in $E_1^3$. 

### 8.2 The conformal Gauss maps of time-like surfaces in 3-dimensional space forms with signature (1,2)

Referring to the previous subsection, we can define the conformal Gauss maps of time-like surfaces in $N_2^3 = S_2^3, E_2^3$ or $H_2^3$. Let $\nu$ be a time-like immersion of an oriented two-dimensional manifold $M$ into $N_2^3$. Let $\langle \cdot, \cdot \rangle$ be the metric of $E_3^5$, and set $L := \{ x = (x^1, x^2, x^3, x^4, x^5) \in E_3^5 \mid \langle x, x \rangle = 0 \}$. We identify $S_2^3, E_2^3, H_2^3$ with $L \cap \{x^5 = 1\}$, $L \cap \{x^5 = x^1 + 1\}$, $L \cap \{x^1 = 1\}$, respectively and therefore we consider $\nu : M \rightarrow N_2^3$ to be an $L$-valued function. Let $e_3$ be a normal vector field of $\nu$ in $N_2^3$ satisfying $\langle e_3, e_3 \rangle = -1$. We identify $e_3$ for the orientations of $M$ and $N_2^3$. Let $H$ be the mean curvature of $\iota$ with respect to $e_3$. Let $\gamma_{\iota} : M \rightarrow H_2^2$ be a map defined by $\gamma_{\iota} := -e_3 + H\iota$. Then $F = \gamma_{\iota}|_{\text{Reg}(\iota)}$ is a time-like immersion which induces a metric $g$ conformal to $g^{\mathcal{M}}$ satisfying $g = \Lambda g^{\mathcal{M}}$, where $\Lambda := H^2 + K^M - \delta$ with $\delta = +1, 0$ or $-1$ according to $N_2^3 = S_2^3, E_2^3$ or $H_2^3$. The map $\gamma_{\iota} : M \rightarrow H_2^2$ is called the conformal Gauss map of a time-like immersion $\iota : M \rightarrow N_2^3$. We see that $\iota$ is an $L$-valued normal vector field of $F$ and that the trace of the shape operator of $F$ with respect to $\iota$ vanishes. Let $\nu$ be an $L$-valued normal vector field of $F$ satisfying $\langle \nu, \iota \rangle = -1$. Then the trace of the shape operator of $F$ with respect to $\nu$ vanishes if and only if $\Delta^M H - 2\Lambda H = 0$. We say that a time-like immersion $\iota : M \rightarrow N_2^3$ is of Willmore type if $\Delta^M H - 2\Lambda H = 0$. Then we obtain

**Theorem 8.6** The mean curvature vector of $F = \gamma_{\iota}|_{\text{Reg}(\iota)}$ vanishes if and only if $\iota : M \rightarrow N_2^3$ is of Willmore type.

In addition, we obtain

**Theorem 8.7** Let $\gamma : M \rightarrow H_2^2$ be a time-like immersion with zero mean curvature vector and neither zero nor light-like shape operator with respect to a light-like normal vector field of a time-like immersion $\iota : M \rightarrow N_2^3$. Then $\nu$ is contained in a constant direction. □
vector field \( \iota \). Then \( \gamma \) or \(-\gamma\) is the conformal Gauss map of a time-like immersion of Willmore type of a neighborhood of each point of \( M \) into each of \( S_2^3, E_2^3, H_2^3 \) given by \( \iota \).

Let \( M \) be a Lorentz surface and \( \iota \) a time-like and conformal immersion of \( M \) into \( N_2^3 = S_2^3, E_2^3 \) or \( H_2^3 \). Let \( b \) be the second fundamental form of \( \iota \) with respect to \( e_3 \). Let \( \Xi \) be a 4-tensor field on \( M \) given by

\[
\Xi := 2b \otimes \text{Hess}_H - (H^2 - \delta) b \otimes b - 2dH \otimes \nabla^M b. \tag{8.24}
\]

Referring to the proof of Proposition 8.3, we can prove

**Proposition 8.8** If \( \iota : M \rightarrow N_2^3 \) is of Willmore type, then a paracomplex quartic differential \( \tilde{Q} \) as in (8.2) with (8.24) is holomorphic.

Referring to the proof of Theorem 8.4, we can prove

**Theorem 8.9** Let \( \iota : M \rightarrow N_2^3 \) be a time-like and conformal immersion of Willmore type. Then on \( \text{Reg} (\iota) \), (a), (b), (c) in Theorem 8.4 hold.

### 8.3 The conformal Gauss map of a time-like surface in \( L \subset E_2^4 \)

Let \( \langle , \rangle \) be the metric of \( E_2^4 \), and set

\[
L := \{ x = (x^1, x^2, x^3, x^4) \in E_2^4 \mid \langle x, x \rangle = 0 \}.
\]

Let \( \iota \) be a time-like immersion of an oriented two-dimensional manifold \( M \) into \( L \). Then \( \iota \) is a normal vector field of a time-like immersion \( \iota \). Let \( \xi \) be an \( L \)-valued normal vector field of \( \iota \) in \( E_2^4 \) satisfying \( \langle \xi, \iota \rangle = -1 \). Let \( H \) be the mean curvature of \( \iota : M \rightarrow L \) with respect to a normal vector field \( \iota \). Let \( \gamma_\iota : M \rightarrow E_2^4 \) be a map from \( M \) into \( E_2^4 \) defined by \( \gamma_\iota := -\xi + H \iota \). Then \( F = \gamma_\iota|_{\text{Reg}(\iota)} \) is a time-like immersion which induces a metric \( g \) conformal to the induced metric \( g^M \) by \( \iota \) satisfying \( g = \Lambda g^M \), where \( \Lambda := H^2 + K^M \) and \( K^M \) is the determinant of the shape operator of \( \iota \) with respect to a normal vector field \( \iota \). The map \( \gamma_\iota : M \rightarrow E_2^4 \) is called the conformal Gauss map of a time-like immersion \( \iota : M \rightarrow L \). We see that \( \iota \) is an \( L \)-valued normal vector field of \( F \) and that the trace of the shape operator of \( F \) with respect to \( \iota \) vanishes.

On a neighborhood \( U \) of each point of \( \text{Reg}(\iota) \), there exist local coordinates \( (u, v) \) and a function \( \alpha \) such that \( g^M \) is locally represented as \( g^M = e^{2\alpha}(du^2 - dv^2) \). Let \( \nu \) be an \( L \)-valued normal vector field of \( F \) satisfying \( \langle \nu, \iota \rangle = -1 \). Then \( \nu \) is locally represented as

\[
\nu = \frac{H_u}{e^{2\alpha}\Lambda} d\gamma_\iota \left( \frac{\partial}{\partial u} \right) - \frac{H_v}{e^{2\alpha}\Lambda} d\gamma_\iota \left( \frac{\partial}{\partial v} \right) + \xi - \frac{1}{2e^{2\alpha}\Lambda}(H_u^2 - H_v^2) \iota.
\]

The trace of the shape operator of \( F \) with respect to \( \nu \) is given by \(- (1/\Lambda)(\Delta^M H - 2\Lambda)\). Hence we obtain
Theorem 8.10 The mean curvature vector of $F = \gamma|_{\text{Reg}(\iota)}$ vanishes if and only if $\iota$ satisfies $\Delta^M H - 2\Lambda = 0$.

Referring to [3], we obtain

**Theorem 8.11** Let $\gamma : M \rightarrow E_2^4$ be a time-like immersion with zero mean curvature vector and neither zero nor light-like shape operator with respect to a light-like normal vector field $\iota$. Suppose $\langle \iota, \gamma \rangle \neq 0$. Then $\gamma$ or $-\gamma$ is the conformal Gauss map of a time-like immersion of $M$ into $L$ given by $\iota$ satisfying $\Delta^M H - 2\Lambda = 0$.

Let $M$ be a Lorentz surface and $\iota$ a time-like and conformal immersion of $M$ into $L \subset E_2^4$. Let $H$ be as above and $b$ the second fundamental form of $\iota : M \rightarrow L$ with respect to a normal vector field $\iota$. Let $\Xi$ be a 4-tensor field on $M$ given by

$$\Xi := b \otimes \text{Hess}^M_H - Hb \otimes b - dH \otimes \nabla^M b. \quad (8.25)$$

Then we obtain

**Proposition 8.12** If $\iota : M \rightarrow L \subset E_2^4$ satisfies $\Delta^M H - 2\Lambda = 0$, then a paracomplex quartic differential $\tilde{Q}$ as in (8.2) with (8.25) is holomorphic and represented as in (8.18), where $\psi, \varphi, \zeta$ are as in (8.7), (8.10), (8.15), respectively, and $q := (1/4)(\text{Hess}^M_H(e,e) - Hb(e,e))$.

Referring to [3], we can prove

**Proposition 8.13** Suppose that $\iota : M \rightarrow L \subset E_2^4$ satisfies $\Delta^M H - 2\Lambda = 0$. Then $\Lambda^3 \langle dv, dv \rangle = 4|\psi q - \varphi \zeta|^2g$ on $\text{Reg}(\iota)$.

By Proposition 8.13, we obtain

**Theorem 8.14** Let $\iota : M \rightarrow L \subset E_2^4$ be a time-like and conformal immersion satisfying $\Delta^M H - 2\Lambda = 0$. Then on $\text{Reg}(\iota)$, (a), (b), (c) in Theorem 8.4 hold.

**Acknowledgements**

The author is grateful to Professor Shin Kato and Doctor Kaname Hashimoto for helpful discussions. The author is also grateful to Professor Kazuyuki Hasegawa for valuable comments.

**A Lorentz surfaces**

**A.1 Paracomplex functions and holomorphic functions**

For $w_1 = (u_1, v_1), w_2 = (u_2, v_2) \in \mathbb{R}^2$, we set $w_1 \cdot w_2 := (u_1u_2 + v_1v_2, u_1v_2 + v_1u_2)$. Then we have an operation and we call this operation the *paracomplex product*. We call $w_1 \cdot w_2$
the paracomplex product of \( w_1 \) and \( w_2 \). We consider \( \mathbb{R}^2 \) equipped with the paracomplex product to be a commutative algebra. In this paper, this algebra is denoted by \( \mathcal{C} \). Each element of \( \mathcal{C} \) is called a paracomplex number. We denote \((1, 0) \in \mathcal{C}\) by 1 and therefore we consider \( \mathbb{R} \) to be a subset of \( \mathcal{C} \). We denote \((0, 1) \in \mathcal{C}\) by \( j \). Then each \( w = (u, v) \in \mathcal{C} \) is also denoted by \( u + jv \) and we have \( j^2 = j \cdot j = 1 \). For \( w = u + jv, u, v \) are denoted by \( \text{Re} w, \text{Im} w \) and called the real and the paraimaginary parts of \( w \), respectively. The conjugate number \( \overline{w} \) of \( w \) is given by \( \overline{w} = u - jv \). We set \( |w|^2 := w \cdot \overline{w} = u^2 - v^2 \). We say that \( w \) is null or a zero divisor if \( w \) is not zero and if \( |w|^2 = 0 \).

Let \( f \) be a smooth function of two variables \( u, v \) defined on an open set \( O \) of \( \mathcal{C} \) and valued in \( \mathcal{C} \). Then there exist real-valued functions \( a, b \) on \( O \) satisfying \( f = a + jb \). Such a function as \( f \) is called a paracomplex-valued function. We say that \( f \) is holomorphic with respect to a paracomplex variable \( w = u + jv \) if \( a, b \) satisfy \( \partial a/\partial u = \partial b/\partial v \) and \( \partial a/\partial v = \partial b/\partial u \) on \( O \). For a paracomplex-valued function \( f = a + jb \), we set

\[
\frac{\partial f}{\partial w} := \frac{1}{2} \left( \frac{\partial a}{\partial u} + \frac{\partial b}{\partial v} \right) + \frac{j}{2} \left( \frac{\partial b}{\partial u} + \frac{\partial a}{\partial v} \right),
\]

\[
\frac{\partial f}{\partial \overline{w}} := \frac{1}{2} \left( \frac{\partial a}{\partial u} - \frac{\partial b}{\partial v} \right) + \frac{j}{2} \left( \frac{\partial b}{\partial u} - \frac{\partial a}{\partial v} \right).
\]

Then \( f \) is holomorphic with respect to \( w \) if and only if \( \partial f/\partial \overline{w} = 0 \). If \( f \) is holomorphic with respect to \( w \), then \( f(u, v) \) is also denoted by \( f(w) \).

Suppose that \( f \) is holomorphic with respect to \( w \). Then

\[
\frac{\partial f}{\partial w} = \frac{\partial a}{\partial u} + j \frac{\partial b}{\partial v} = \frac{\partial b}{\partial u} + \frac{\partial a}{\partial v}.
\]

In addition, we obtain \( \partial^2 a/\partial u^2 = \partial^2 a/\partial v^2 \) and \( \partial^2 b/\partial u^2 = \partial^2 b/\partial v^2 \). Therefore there exist real-valued functions \( a^1, a^2, b^1, b^2 \) of one real variable satisfying

\[
a(u, v) = a^1(u + v) + a^2(u - v), \quad b(u, v) = b^1(u + v) + b^2(u - v).
\]

In addition, using \( \partial a/\partial u = \partial b/\partial v \) and \( \partial a/\partial v = \partial b/\partial u \), we see that \( a^1 - b^1 \) and \( a^2 + b^2 \) are constant. Therefore we obtain

\[
f(w) = (1 + j)a^1(u + v) + (1 - j)a^2(u - v) + j r \quad \text{(A.1)}
\]

for a real number \( r \). Conversely, such a paracomplex-valued function is holomorphic with respect to \( w \): if \( a^1, a^2 \) are real-valued functions of one real variable and if \( f \) is a paracomplex-valued function defined by (A.1), then \( f \) satisfies \( \partial f/\partial \overline{w} = 0 \) and therefore \( f \) is holomorphic with respect to \( w \).

Let \( f = a + jb \) be a holomorphic function with respect to \( w = u + jv \). We say that \( f \) is nondegenerate if \( |\partial f/\partial w|^2 \) does not vanish; we say that \( f \) is orientation-preserving if \( |\partial f/\partial w|^2 > 0 \). We represent \( a \) as \( a(u, v) = a^1(u + v) + a^2(u - v) \). Then \( f \) is nondegenerate (respectively, orientation-preserving) if and only if \( \dot{a}(u + v)a^2(u - v) \neq 0 \) (respectively, \( > 0 \)), where \( \dot{a}^k \) is the derivative of \( a^k \) for \( k \in \{1, 2\} \).
A.2 Lorentz surfaces, paracomplex structures and paraHermitian metrics

Let $M$ be a two-dimensional smooth manifold and $\{(U_\lambda, w_\lambda)\}_{\lambda \in \Lambda}$ its system of coordinate neighborhoods of class $C^\infty$. We consider $w_\lambda(U_\lambda)$ to be an open subset of $\mathcal{C}$ and we call $(U_\lambda, w_\lambda)$ a paracomplex coordinate neighborhood. For $\lambda, \mu \in \Lambda$ with $U_\lambda \cap U_\mu \neq \emptyset$, we consider a map $f_{\mu \lambda} := w_\mu \circ w_\lambda^{-1}|_{w_\lambda(U_\lambda \cap U_\mu)}$ to be a paracomplex-valued function on $w_\lambda(U_\lambda \cap U_\mu) \subset \mathcal{C}$. We say that $\{(U_\lambda, w_\lambda)\}_{\lambda \in \Lambda}$ is holomorphic if $f_{\mu \lambda}$ is an orientation-preserving holomorphic function with respect to a paracomplex variable on $w_\lambda(U_\lambda \cap U_\mu)$ for each pair $(\lambda, \mu)$ with $U_\lambda \cap U_\mu \neq \emptyset$. If $\{(U_\lambda, w_\lambda)\}_{\lambda \in \Lambda}$ is holomorphic, then $M$ equipped with $\{(U_\lambda, w_\lambda)\}_{\lambda \in \Lambda}$ is called a Lorentz surface and each $w_\lambda$ is called a local paracomplex coordinate of a Lorentz surface $M$. A Lorentz surface is orientable. We choose the orientation given by $(\partial/\partial u_\lambda, \partial/\partial v_\lambda)$ for each local paracomplex coordinate $w_\lambda$.

Let $M$ be a Lorentz surface. Let $\mathcal{J}$ be a tensor field on $M$ of type $(1,1)$ such that $\mathcal{J}(\partial/\partial u_\lambda) = \partial/\partial v_\lambda$ and $\mathcal{J}(\partial/\partial v_\lambda) = \partial/\partial u_\lambda$ for each local paracomplex coordinate $w_\lambda = u_\lambda + jv_\lambda$. We see that $\mathcal{J}$ is well-defined on $M$. We call $\mathcal{J}$ the paracomplex structure of a Lorentz surface $M$.

Let $M$ be an orientable two-dimensional manifold. Let $\mathcal{J}$ be a tensor field of type $(1,1)$ satisfying

- $\mathcal{J}$ is different from the identity map of the tangent space at each point of $M$;
- $\mathcal{J}^2$ gives the identity map of the tangent space at any point.

The eigenvalues of $\mathcal{J}$ are given by $\pm 1$. Noticing the corresponding eigenspaces, we can obtain a system of paracomplex coordinate neighborhoods $\{(U_\lambda, w_\lambda)\}_{\lambda \in \Lambda}$ such that if $U_\lambda \cap U_\mu \neq \emptyset$, then $f_{\mu \lambda} := w_\mu \circ w_\lambda^{-1}$ is an orientation-preserving holomorphic function. Then $M$ equipped with $\{(U_\lambda, w_\lambda)\}_{\lambda \in \Lambda}$ is a Lorentz surface such that $\mathcal{J}$ is the paracomplex structure of $M$.

A Lorentz metric (pseudo-Riemannian metric of signature $(1, 1)$) $g$ on a Lorentz surface $M$ is said to be paraHermitian if $g(\mathcal{J}V, \mathcal{J}W) = -g(V, W)$ for arbitrary tangent vectors $V, W$. Let $g$ be a paraHermitian metric on $M$. Then $g$ is locally represented as $g = e^{2\alpha \lambda} dw_\lambda d\overline{w}_\lambda$ or $-e^{2\alpha \lambda} dw_\lambda d\overline{w}_\lambda$ on $U_\lambda$ for a real-valued function $\alpha_\lambda$.

Let $M$ be an orientable two-dimensional manifold and $g$ a Lorentz metric on $M$. Then noticing one-dimensional distributions constructed by light-like directions of $M$ with respect to $g$, we see that there exist local coordinates $(u, v)$ on a neighborhood of each point of $M$ satisfying $g = e^{2\alpha}(du^2 - dv^2)$. If we set $w := u + jv$, then we can consider $M$ to be a Lorentz surface such that $w$ is a local paracomplex coordinate.

**Remark** Let $M$ be a Lorentz surface and $\mathcal{J}$ the paracomplex structure of a Lorentz surface $M$. Then for any paraHermitian metric $g$ on $M$, $\mathcal{J}$ is parallel with respect to the Levi-Civita connection of $g$. 

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References


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