On the Lax pairs of the sixth Painlevé equation

Robert Conte.
Service de physique de l'état condensé (CNRS URA 2464)
CEA Saclay, F–91191 Gif-sur-Yvette Cedex, France.
Robert.Conte@cea.fr

Abstract.

The sixth Painlevé equation P6 for $u(x)$ depends holomorphically on its four parameters $(2\alpha, -2\beta, 2\gamma, 1-2\delta) = (\theta_\infty^2, \theta_0^2, \theta_1^2, \theta_x^2)$, therefore one expects all Lax pairs of P6 to present this dependence. This is indeed the case for the second order scalar "Lax" pair of Richard Fuchs, whose singularities are four Fuchsian points of crossratio x, plus one apparent singularity located at $t = u$. The proof by Poincaré (1883) of the impossibility to remove this apparent singularity in the second order scalar isomonodromic deformation certainly motivated Jimbo and Miwa to consider matrix isomonodromy,

$$\partial_x \psi = L\psi, \quad \partial_t \psi = M\psi, \ [\partial_x - L, \partial_t - M] = 0. \tag{1}$$

With second order matrices, there exists a choice [3] whose only singularities of M are four Fuchsian points of crossratio x, but this choice presents the drawback to have a meromorphic dependence on one of the four monodromy exponents θ_j, see [6, Eq. (3.24)] and [9, Eqs. (4.18)–(4.22)]. The discrete Lax pair for $\mathrm{q}-\mathrm{P6}$ [5] displays the same unpleasant feature. The present work explores several directions in order to remove this drawback in matrix Lax pairs.

A first direction is to take a parametric representation of the four residues in the second order monodromy matrix M which is different from that of Jimbo and Miwa. This leads to a closed three-dimensional first order system

$$\frac{\mathrm{d}u_j}{\mathrm{d}x} = \frac{P_j(u_1, u_2, u_3, x)}{u_k - u_l}, \ j, k, l = 1, 2, 3, \tag{2}$$

in which P_j is a polynomial and j, k, l are all different. This system is currently under investigation for its explicit integration with, evidently, P6.

A second direction is to start from a third order monodromy matrix presenting one Fuchsian singularity and one nonFuchsian, then to convert it to a second order Lax pair either by a factorization method [4] or by a Laplace transform [8]. One good candidate to obtain the desired result is the three-wave resonant interaction

$$\left\{ \begin{array}{l} u_{j,t} + c_j u_{j,x} - i\bar{u}_k \bar{u}_l = 0, \\ \bar{u}_{j,t} + c_j \bar{u}_{j,x} + i u_k u_l = 0, \ c_j \in \mathcal{R}, \ i^2 = -1, \end{array} \right. \tag{3}$$

in which (j, k, l) denotes any permutation of $(1, 2, 3)$, c_j are the constant values of the group velocities, with $(c_2 - c_3)(c_3 - c_1)(c_1 - c_2) \neq 0$. One of its reductions can be integrated explicitly in terms of P6 [10], with a third order Lax pair of the above mentioned type, but the factorization of the residue seems to be algebraic, not rational like in [4], leading to technical difficulties not yet solved.

References.

[1] R. Fuchs, C. R. Acad. Sc. Paris **141** (1905) 555–558.

[2] R. Garnier, d'ordre

[3] M. Jimbo and T. Miwa, Physica D **2** (1981) 407–448.

[4] J. Harnad, Commun. Math. Phys. **166** (1994) 337–365.

[5] M. Jimbo and H. Sakai, Lett. Math. Phys. **38** (1996) 145–154.

[6] G. Mahoux, *The Painlevé property, one century later*, 35–76, ed. R. Conte, CRM series in mathematical physics (Springer, New York, 1999).

[7] R. Conte and M. Musette, Chaos, solitons and fractals **11** (2000) 41–52. http://arXiv.org/abs/solv-int/9803013 .

[8] M. Mazzocco, *The Kowalevski property* 219–238, CRM Proc. Lecture Notes **32** (Amer. Math. Soc., Providence, RI, 2002).

[9] Lin Run-liang, R. Conte, and M. Musette, J. Nonl. Math. Phys. **10**, Supp. 2, 107–118 (2003).
`http://www.sm.luth.se/~norbert/home_journal/10s2_9.pdf`

[10] R. Conte, A. M. Grundland, and M. Musette, J. Phys. A to appear (2006). Special issue on Painlevé VI. http://arXiv.org/abs/nlin.SI/0604nnn