On the Lax pairs of the sixth Painlevé equation

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Abstract.

The sixth Painlevé equation $P_6$ for $u(x)$ depends holomorphically on its four parameters $(2\alpha, -2\beta, 2\gamma, 1 - 2\delta) = (\theta_\infty^2, \theta_0^2, \theta_1^2, \theta_2^2)$, therefore one expects all Lax pairs of $P_6$ to present this dependence. This is indeed the case for the second order scalar “Lax” pair of Richard Fuchs, whose singularities are four Fuchsian points of crossratio $x$, plus one apparent singularity located at $t = u$. The proof by Poincaré (1883) of the impossibility to remove this apparent singularity in the second order scalar isomonodromic deformation certainly motivated Jimbo and Miwa to consider matrix isomonodromy,

$$\partial_x \psi = L\psi, \quad \partial_t \psi = M\psi, \quad [\partial_x - L, \partial_t - M] = 0. \quad (1)$$

With second order matrices, there exists a choice [3] whose only singularities of $M$ are four Fuchsian points of crossratio $x$, but this choice presents the drawback to have a meromorphic dependence on one of the four monodromy exponents $\theta_j$, see [6, Eq. (3.24)] and [9, Eqs. (4.18)–(4.22)]. The discrete Lax pair for $q - P_6$ [5] displays the same unpleasant feature. The present work explores several directions in order to remove this drawback in matrix Lax pairs.

A first direction is to take a parametric representation of the four residues in the second order monodromy matrix $M$ which is different from that of Jimbo and Miwa. This leads to a closed three-dimensional first order system

$$\frac{du_j}{dx} = \frac{P_j(u_1, u_2, u_3, x)}{u_k - u_l}, \quad j, k, l = 1, 2, 3, \quad (2)$$

in which $P_j$ is a polynomial and $j, k, l$ are all different. This system is currently under investigation for its explicit integration with, evidently, $P_6$.

A second direction is to start from a third order monodromy matrix presenting one Fuchsian singularity and one nonFuchsian, then to convert it to a second order Lax pair either by a factorization method [4] or by a Laplace transform [8]. One good candidate to obtain the desired result is the three-wave resonant interaction

$$\begin{cases} u_{j,t} + c_j u_{j,x} - i\bar{u}_k \bar{u}_l = 0, \\ \bar{u}_{j,t} + c_j \bar{u}_{j,x} + iu_k u_l = 0, \end{cases}, \quad c_j \in \mathcal{R}, \quad t^2 = -1, \quad (3)$$

in which $(j, k, l)$ denotes any permutation of $(1, 2, 3)$, $c_j$ are the constant values of the group velocities, with $(c_2 - c_3)(c_3 - c_1)(c_1 - c_2) \neq 0$. One of its reductions can be integrated explicitly in terms of $P_6$ [10], with a third order Lax pair of the above mentioned type, but the factorization of the residue seems to be algebraic, not rational like in [4], leading to technical difficulties not yet solved.
References.