Holomorphic and singular solutions of q-analogue of the Briot-Bouquet type equations

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In this talk let $q > 1$: for a function $f(t,x)$ we define the q-difference operator D_q by

$$D_q f(t,x) = \frac{f(qt,x) - f(t,x)}{qt - t}.$$

We will study the following type of nonlinear difference-differential equations:

$$tD_q u = F(t, x, \{\partial_x^\alpha u\}_{|\alpha| \le m}) \tag{0.1}$$

where $(t,x) = (t, x_1, \dots, x_n) \in \mathbb{C}_t \times \mathbb{C}_x^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial_x^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$, $\partial_i = \partial/\partial_{x_i}$ for $i = 1, \dots, n$, $F(t,x,Z)$ $(Z = \{Z_\alpha\}_{|\alpha| \le m})$ is a function defined in a polydisk Δ centered at the origin of $\mathbb{C}_t \times \mathbb{C}_x^n \times \mathbb{C}_Z^\delta$ and δ is the cardinal of $\{\alpha \in \mathbb{N}^n : \ |\alpha| \le m\}$. Let us denote $\Delta_0 = \Delta \cap \{t = 0, Z = 0\}$.

The assumption are follows;

(A1) $F(t,x,Z)$ is holomorphic in Δ,

(A2) $F(0,x,0) = 0$ in Δ_0,

(A3) $\frac{\partial F}{\partial Z_\alpha}(0,x,0) = 0$ in Δ_0 for all $|\alpha| \le m$.

Definition 0.1 *If the equation (0.1) satisfies* (A1), (A2) *and* (A3) *we say that (0.1) is of q-analogue of the Briot-Bouquet type with respect to t.*

Definition 0.2 *([1]) Let us define*

$$\rho(x) = \frac{\partial F}{\partial u}(0,x,0,0),$$

then the holomorphic function $\rho(x)$ is called the characteristic exponent of the equation (0.1).

Let us denote by

1. $\mathcal{R}(\mathbb{C}\backslash\{0\})$ the universal covering space of $\mathbb{C}\backslash\{0\}$,
2. $S_\theta = \{t \in \mathcal{R}(\mathbb{C}\backslash\{0\});\ |\arg t| < \theta\}$,
3. $S\,(\epsilon(s)) = \{t \in \mathcal{R}(\mathbb{C}\backslash\{0\});\ 0 < |t| < \epsilon(\arg t)\}$ for some positive-valued function $\epsilon(s)$ defined and continuous on $\mathbb{R}$,

4. $D_R = \{x \in \mathbb{C}^n;\ |x_i| < R \text{ for } i = 1, \dots, n\}$,

5. $\mathbb{C}\{x\}$ the ring of germs of holomorphic functions at the origin of $\mathbb{C}^n$.

Definition 0.3 *We define the set $\widetilde{\mathcal{O}}_+$ of all functions $u(t,x)$ satisfying the following conditions;*
1. *$u(t,x)$ is holomorphic in $S(\epsilon(s)) \times D_R$ for some $\epsilon(s)$ and $R > 0$,*
2. *there is an $a > 0$ such that for any $\theta > 0$ and any compact subset K of D_R*

$$\max_{x \in K} |u(t,x)| = O(|t|^a) \quad \text{as} \quad t \to 0 \quad \text{in} \quad S_\theta.$$

Gérard-Tahara [1] studied the following type of nonlinear singular first order partial differential equations:

$$t\partial_t u = F(t, x, u, \partial_x u). \tag{0.2}$$

We concern the following result. Gérard-Tahara studied in [1] the structure of holomorphic and singular solutions of (0.2) and proved the following result;

Theorem 0.4 (Gérard R. and Tahara H.) *If* (0.2) *is of the Briot-Bouquet type and* $\rho(0) \notin \mathbf{N}^* = \{1, 2, 3, \dots\}$ *then we have;*
(1) (*Holomorphic solutions*) (0.2) *has a unique solution $u_0(t,x)$ holomorphic near the origin of $\mathbf{C} \times \mathbf{C}^n$ satisfying $u_0(0,x) \equiv 0$.*
(2) (*Singular solutions*) *Denote by S_+ the set of all $\widetilde{\mathcal{O}}_+$-solutions of* (0.2).

$$S_+ = \begin{cases} \{u_0(t,x)\} & \text{when} \quad Re\rho(0) \le 0, \\ \{u_0(t,x)\} \cup \{U(\varphi); 0 \ne \varphi(x) \in \mathbf{C}\{x\}\} & \text{when} \quad Re\rho(0) > 0, \end{cases}$$

where $U(\varphi)$ is an $\widetilde{\mathcal{O}}_+$-solution of (0.2) *having an expansion of the following form:*

$$U(\varphi) = \sum_{i=1}^{\infty} u_i(x) t^i + \sum_{k \le i+2(j-1), j \ge 1} \varphi_{i,j,k}(x) t^{i+j\rho(x)} (\log t)^k, \quad \varphi_{0,1,0}(x) = \varphi(x).$$

In this talk we will try to q-discrete the equation (0.2) to the equation (0.1) and we have;

Theorem 0.5 *If* (0.1) *is of q-analogue of the Briot-Bouquet type and $\rho(0) \ne (q^i - 1)/(q-1)$ for $i = 1, 2, \dots$ then we have:*
(1) (Holomorphic solutions) (0.1) *has a unique solution $u_0(t,x)$ holomorphic near the origin of $\mathbb{C}_t \times \mathbb{C}^n_x$ satisfying $u_0(0,x) \equiv 0$.*
(2) (Singular solutions) Set $\rho_q(x) = \log_q\{1 + (q-1)\rho(x)\}$. When $\Re\rho(0) > 0$ and

$$q^i (\rho_q(x))^j - \rho_q(x) \ne 0 \quad \text{on } D_r \text{ for } i + j \ge 1,\ (i,j) \ne (0,1),$$

for any $\varphi(x) \in \mathbb{C}\{x\}$ there exists an $\widetilde{\mathcal{O}}_+$-solutions $U(\varphi)$ of (0.1) *having an expansion of the following form:*
(0.3)

$$U(\varphi) = \sum_{i=1}^{\infty} u_i(x) t^i + \sum_{k \le i+2m(j-1), j \ge 1} \varphi_{i,j,k}(x) t^{i+\rho_q(x)j} (\log t)^k, \quad \varphi_{0,1,0}(x) = \varphi(x).$$

References

[1] Gérard, R. and Tahara, H., *Holomorphic and Singular Solutions of Nonlinear Singular First Order Partial Differential Equations*, Publ. RIMS, Kyoto Univ., 26(1990), 979–1000.