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# MEROMORPHIC EXTENSION OF $L$-FUNCTIONS OF ANOSOV FLOWS AND PROFINITE GRAPHS

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## Introduction

In the preceding paper [3] T. Sunada and the author introduced $L$-functions of certain classes of graphs and dynamical systems. Applying number theoretical argument we ([1], [3], [4]) obtained some results about homology classes of closed orbits of Anosov flows; the density theorem of Chebotarev's type, equidistribution theorem and so on. In this note we consider the meromorphic region of these $L$-functions.

Let $\varphi_t : X \to X$ be an Anosov flow on a compact Riemannian manifold $X$ whose non-wondering set coincides with the whole space. Given a unitary representation $\rho:\pi_1(X) \to U(N)$, we define the $L$-function of $\varphi_t$ associated with $\rho$ by

(1) $$L_{\varphi_t}(s;\rho)=\prod_p 1/\det\{I-\rho(\langle p\rangle)\exp(-s\, l(p))\},$$

where $p$ runs over all $\varphi_t$-prime closed orbits, $l(p)$ is the period of $p$ and $\langle p\rangle$ is an element of $\pi_1(X)$ whose conjugacy class corresponds to the free homotopy class of $p$. When $\mathscr{R}e\, s$ is greater than the topological entropy $h=h(\varphi_t)$ of the flow $\varphi_t$, the right hand side of (1) converges absolutely, hence $L_{\varphi_t}(s;\rho)$ is holomorphic on $\mathscr{R}e\, s>h$. We know moreover that $L_{\varphi_t}(s;\rho)$ has a meromorphic extension to some neighborhood of $\mathscr{R}e\, s \geqq h$(see[3]).

We here give more information on the meromorphic domain of $L_{\varphi_t}(s;\rho)$. For the special case of the trivial representation $\rho=\mathbb{1}$, Pollicott[12] shows that the zeta function $\zeta_{\varphi_t}(s)=L_{\varphi_t}(s;\mathbb{1})$ extends meromorphically to a region $\mathscr{R}e\, s>h-\varepsilon$ for some $\varepsilon>0$. We extend his proof to cases of general representations and show that there exists $\delta>0$ such that for every unitary representation $\rho$ the $L$-function $L_{\varphi_t}(s;\rho)$ extends non-zero meromorphically to the domain $\mathscr{R}e\, s>h-\delta$.

When $\varphi_t$ is the geodesic flow on the unit tangent bundle of a manifold of constant

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negative curvature, the Selberg trace formula implies that $L$-functions extend to the entire complex plane. But in general, we have no idea to explane $L$-functions by trace formulas. We therefore use twisted Ruelle operators acting Banach spaces of Lipschitz continuous sections of flat bundles over a profinite graph. Associated with a continuous unitary representation $\rho:\pi_1(V,E)\to U(N)$ of the fundamental group of a profinite graph $(V,E)$ with the length function $l:E\to R_+$, the $L$-function is defined in the same way as (1):

$$L_l(s;\rho)=\prod_c 1/\det\{I-\rho(\langle c\rangle)\exp(-s\ l(c)\},$$

where $c$ runs over all prime cycles on $(V,E)$ and $l(c)=\sum_{i=1}^{n} l(e_i)$ for $c=(e_1,\ldots,e_n)$. Since it is interpreted by what we may call the trace of a twisted Ruelle operator, an estimate about the essential spectral radius implies that it has a non-zero meromorphic extension on $\mathscr{R}es > h-\delta$. We can get our assertion by use of Bowen's symbolic dynamics.

We use terminology and some notations given in [3] without any explanation, the reader should refer it for some basic definitions and results.

## §1. Profinite graphs and twisted Ruelle operators

Given graphs $(V_n, E_n)$, $n\geq 1$, and morphisms of graphs $\omega_n:(V_n,E_n)\to(V_{n-1}, E_{n-1})$ we define

$$V=\varprojlim V_n=\left\{\ \xi=(\xi_n)\in\prod_{n=1}^{\infty}V_n\ \mid\ \omega_n(\xi_n)=\xi_{n-1}\ \right\},$$

$$E=\varprojlim E_n=\left\{\ (\xi,\eta)\in V\times V\ \mid\ (\xi_n,\eta_n)\in E_n\ \right\}.$$

This new graph $(V,E)$ is called as a profinite graph if the following conditions hold;

P0) each $(V_n,E_n)$ is a finite graph,
P1) the map $\omega_n:V_n\to V_{n-1}$ is surjective,
P2) the map of origine $o:E_n\to V_n$ and terminus $t:E_n\to V_n$ are surjective,
P3) $\omega_n:o^{-1}(v)\to o^{-1}(\omega_n(v))$ is surjective for every $v\in V_n$,
P4) if $\omega_n(e)=\omega_n(e')$, $e$, $e'\in E_n$, then $t(e)=t(e')$.

Let $\rho:\pi_1(V,E)=\varprojlim \pi_1(V_n,E_n)\to U(N)$ be a continuous unitary representation of the fundamental group of a profinite graph $(V,E)$. There are $n_o$ and $\rho_o:\pi_1(V_{n_o},E_{n_o})$

$\to U(N)$ such that $\rho$ is fuctored as $\rho=\rho_o \circ \mathrm{Proj} : \pi_1(V,E) \to \pi_1(V_{n_o},E_{n_o}) \to U(N)$. Without loss of generality we may assume $n_o=1$. We can construct a locally distance preserving normal covering map $\pi : (\hat{V},\hat{E},d_\theta) \to (V,E,d_\theta)$ with covering transformation group $\pi_1(V_1,E_1)$; here $d_\theta$ $(0<\theta<1)$ denotes the distance function defined by

$$d_\theta(\xi,\eta)=\theta^{\sup\{n \mid \xi_n=\eta_n\}} \text{ on } V,$$

$$d_\theta(\mathrm{e},\mathrm{e}')=\theta^{\sup\{n \mid \mathrm{e}_n=\mathrm{e}'_n\}} \text{ on } E,$$

and so on. In this paper we suppose $(V,E,d_\theta)$ is irreducible.

Let $C^1_\theta(E)$ denote the Banach space of Lipschitz continuous complex valued functions with the norm $\|f\|_\theta=\|f\|_\infty+\mathrm{Lip}_\theta(f)$, here

$$\mathrm{Lip}_\theta(f)=\sup\left\{ \frac{f(\mathrm{e})-f(\mathrm{e}')}{d_\theta(\mathrm{e},\mathrm{e}')} \;\middle|\; \mathrm{e}\neq\mathrm{e}' \right\}.$$

Given $f \in C^1_\theta(E)$ we define the operator $\mathscr{L}_f : C^0(\hat{V},C^N) \to C^0(\hat{V},C^N)$ on the space of all $C^N$—valued continuous functions on $\hat{V}$ by

$$\mathscr{L}_f g(\xi)=\sum_{\mathrm{e}\in\hat{E}:o(\mathrm{e})=\xi} f(\pi\mathrm{e})g(t\mathrm{e}).$$

Since it leaves invariant the subspace $C^1_\theta(\hat{V},\rho)$ of all Lipschitz continuous functions $g$ on $\hat{V}$ with $g(\gamma\xi)=\rho(\gamma)g(\xi)$ for every $\gamma \in \pi_1(V_1,E_1)$ and $\xi \in \hat{V}$, we can set

$$\mathscr{L}_{f,\rho}=\mathscr{L}_f\Big|_{C^1_\theta(\hat{V},\rho)} : C^1_\theta(\hat{V},\rho) \to C^1_\theta(\hat{V},\rho),$$

and call the twisted Ruelle operator associated with $\rho$. We[3] study its spectra and get the following result.

PROPOSITION 1. (1) *If $f \in C^1_\theta(E)$ is positive valued then the Ruelle operator $\mathscr{L}_{f,1}$ associated with the trivial representation has a maximal simple positive eigenvalue $\lambda(f)$ with a positive eigenfunction.*

(2) *If $f \in C^1_\theta(E)$ is nowhere vanishing, every spectrum $\lambda$ of $\mathscr{L}_{f,\rho}$ satisfies $|\lambda| \leq \lambda(|f|)$.*

For a positive valued function $f \in C^1_\theta(E)$, we define the continuous transformation $G_f$ on the compact space of all Borel probability measures on $V$ by

$$G_f(\mu)=\left(\int_V \mathscr{L}_{f,1} 1 \, d\mu\right)^{-1} \cdot \left(\mathscr{L}_{f,1}\right)^* \mu.$$

Using the theorem of Schauder-Tychonoff we get a measure $\mu_f$ with $G_f(\mu_f)=\mu_f$. If one choose a positive eigenfunction $u$ of $\mathscr{L}_{f,1}$ associated with $\lambda(f)$, then the equality

$$\lambda(f)\int u\ d\mu_f=\int \mathscr{L}_{f,1}\ u\ d\mu_f\ =\int \mathscr{L}_{f,1}\ 1\ d\mu_f\cdot\int u\ d\mu_f$$

leads us to $\left(\mathscr{L}_{f,1}\right)^*\mu_f=\lambda(f)\ \mu(f)$. In particular, $\mu_f$ is a $\mathscr{L}_{f,1}$—invariant measure, provided $f\in C^1_\theta(E)$ is positive valued and $\mathscr{L}_f\ 1=1$.

Let $V=V^1\ \ldots\cup V^\nu$ denote the decomposition into primitive parts and for $\upsilon\in V_m$ set

$$C(\upsilon)=\{\xi\in V\ \mid\ \xi_m=\upsilon\}.$$

For every positive valued $f\in C^1_\theta(E)$ with $\mathscr{L}_f\ 1=1$, we can conclude $\mu_f(V^j)=\nu^{-1}$ and $\mu_f(C(\upsilon))>0$ in the following way. Permuting the indices we may assume that $t(e)\in V^{j+1}$ whenever $o(e)\in V^j$, $j=1,\ldots,\nu$, modulo $\nu$. Since $\mathscr{L}_f\ 1=1$ we get

$$\mu_f(V^j)=\int\delta_j d\mu_f=\int\mathscr{L}_{f,1}\ \delta_j d\mu_f\ =\int\ \delta_{j+1}\ d\mu_f=\mu_f(V^{j+1}),$$

which leads us to the first assertion, where $\delta_j\in C^1(V)$ is the characteristic function of $V^j$. To show the second, we should notice that if $V^j$ contains $C(\upsilon)$ then for sufficiently large $n$ there is a $n\nu$-step path $c$ with $o(c)=\xi$ and $t(c)\in C(\upsilon)$ for each $\xi\in V^j$. We therefore get

$$\mu_f(C(\upsilon))=\int\ \delta_\upsilon d\mu_f=\int\mathscr{L}^{n\nu}_{f,1}\ \ \delta_\upsilon d\mu_f\ \geqq(\inf\ f)^{n\nu}\int\ \delta_j d\mu_f>\ 0,$$

where $\delta_\upsilon$ is the characteristic function of $C(\upsilon)$.

## §2. The essential spectral radius of twisted Ruelle operators

In this section we concern with isolated eigenvalues of twisted Ruelle operators. We first show the following.

PROPOSITION 2. *If $f \in C^1_\theta(E)$ is nowhere vanishing, the essential spectral radius of $\mathscr{L}_{f,\rho}$ is not greater than $\theta\lambda(|f|)$. Hence every spectrum $\lambda$ of $\mathscr{L}_{f,\rho}$ with $\theta\lambda(|f|) < |\lambda| \leqq \lambda(|f|)$ is an eigenvalue of finite multiplicity.*

Given a positive valued $f \in C^1_\theta(E)$ with $\mathscr{L}_f\, 1=1$ we define the avarage $E_m(g;f)$ : $\hat{V} \to C^N$ of $g \in C^1_\theta(\hat{V},\rho)$ on $m$-stage by

$$E_m(g;f)= \left( \int \delta_{(\pi\xi)_m}\, d\mu_f \right)^{-1} \int g_{\xi_m}\, d\mu_f.$$

Here the function $g_{\hat{v}} : V \to C^N$, $\hat{v} \in \hat{V}_m$ is defined by

$$g_{\hat{v}}(\eta)= \begin{cases} g(\hat{\eta}) & \text{if } \eta_m = \pi_m(\hat{v}) \\ 0 & \text{otherwize,} \end{cases}$$

where $\hat{\eta}$ is the element of $\hat{V}$ with $\hat{\eta}_m=\hat{v}$ and $\pi(\hat{\eta})=\eta$, and $\pi_m : \hat{V}_m \to V_m$ is the covering map induced by $\pi$. As $E_m(g;f)$ is $m$-locally constant (i.e. $E_m(g;f)(\xi)=E_m(g;f)(\xi')$ if $\xi_m=\xi_m'$), one can easily check that it is an element of $C^1_\theta(\hat{V},\rho)$ and

$$\| g-E_m(g;f) \|_\infty \leqq \theta^m \operatorname{Lip}_\theta(g),$$

$$\operatorname{Lip}_\theta(g-E_m(g;f)) \leqq 2 \operatorname{Lip}_\theta(g).$$

We now prove Proposition 2. By the scaling transformation of $f$ by $\lambda(|f|)$ and an associated positive eigenfunction we may assume $\mathscr{L}_{|f|}\, 1=1$. Since the image of $C^1_\theta(\hat{V},\rho)$ by the operator $E_m(\cdot;|f|): C^1_\theta(\hat{V},\rho) \to C^1_\theta(\hat{V},\rho)$ is a finite dimensional subspace, the operator $\mathscr{L}^m_{f,\rho}\circ E_m(\cdot;|f|)$ is compact. By using the estimate

$$\| \mathscr{L}_{f,\rho}\, g \|_\infty \leqq \| g \|_\infty$$

(2)

$$\operatorname{Lip}_\theta(\mathscr{L}^m_{f,\rho}\, g) \leqq C \| g \|_\infty + \theta^m \operatorname{Lip}_\theta(g)$$

(see Lemma3-4 [3]), we have

$$\left\| \left( \mathscr{L}^m_{f,\rho} - \mathscr{L}^m_{f,\rho}\circ E_m(\cdot;|f|) \right) (g) \right\|_\infty$$

$$\leqq \| g-E_m(g;|f|) \|_\infty \leqq \theta^m \operatorname{Lip}_\theta(g),$$

$$\operatorname{Lip}_\theta\left( \left( \mathscr{L}^m_{f,\rho} - \mathscr{L}^m_{f,\rho} \circ E_m(\cdot;|f|) \right) (g)\right)$$

$$\leqq C \| g-E_m(g;|f|) \|_\infty + \theta^m \operatorname{Lip}_\theta(g-E_m(g;|f|))$$

$$\leqq (C+2)\theta^m \operatorname{Lip}_\theta(g).$$

We therefore get the following bound of the essential spectral radius of $\mathscr{L}_{f,\rho}$

$$\begin{aligned}&\text{ess.spec.rad.}(\mathscr{L}_{f,\rho})\\ &=\limsup_{m\to\infty}\left[\inf\{\|\mathscr{L}^m_{f,\rho}-K\|_\theta \;\middle|\; K \text{ is a compact operator}\}\right]^{1/m}\\ &\leqq \limsup_{m\to\infty}\theta(C+3)^{1/m}=\theta,\end{aligned}$$

which leads us to the conclusion.

In the case $f$ is locally constant we have

PROPOSITION 3. *If $f\in C^1_\theta(E)$ is n-locally constant, an eigenvalue $\lambda$ of $\mathscr{L}_{f,\rho}$ with $\theta\lambda(|f|)<\lambda\leqq\lambda(|f|)$ is that of $\mathscr{L}_{h,\rho}: C^1_\theta(\hat{V}_n,\rho)\to C^1_\theta(\hat{V}_n,\rho)$, where $h: E_n\to C$ is the function with $f(e)=h(e_n)$ for every $e\in E$.*

Proof. The operator $\mathscr{L}_{f,\rho}$ leaves invariant the closed subspace $\Gamma_n(\rho)$ in $C^1_\theta(\hat{V},\rho)$ consisting of $n$-locally constant functions, hence it induces an operator $\mathscr{L}_{f,\rho,n}: C^1_\theta(\hat{V},\rho)/\Gamma_n(\rho)\to C^1_\theta(\hat{V},\rho)/\Gamma_n(\rho)$. The space $\Gamma_n(\rho)$ can be identified with $C^1_\theta(\hat{V}_n,\rho)$ so that the following diagram is comutative

$$\begin{array}{ccc}\Gamma_n(\rho) & \xrightarrow{\;\mathscr{L}_{f,\rho}\;} & \Gamma_n(\rho)\\ \updownarrow & & \updownarrow\\ C^1_\theta(\hat{V}_n,\rho) & \xrightarrow[\;\mathscr{L}_{h,\rho}\;]{} & C^1_\theta(\hat{V}_n,\rho)\,.\end{array}$$

With respect to the induced norm on $C^1_\theta(\hat{V},\rho)/\Gamma_n(\rho)$, we can show in the same way to prove (2) that

$$\left\|\mathscr{L}^m_{f,\rho,n}[g]\right\|_\infty=\inf\{\|\mathscr{L}^m_{f,\rho}(g-k)\|_\infty \mid k\in\Gamma_n(\rho)\}$$
$$\leqq \theta^m\,\mathrm{Lip}_\theta([g]),$$

$$\mathrm{Lip}_\theta(\mathscr{L}^m_{f,\rho,n}[g])\leqq C\,\|[g]\|_\infty+\theta^m\,\mathrm{Lip}_\theta([g]).$$

Therefore the essential spectral radius of $\mathscr{L}_{f,\rho,n}$ is not greater than $\theta$. Since a spectrum of $\mathscr{L}_{f,\rho}$ is containd in one of the sets of spectra of $\mathscr{L}_{f,\rho} \Big| \Gamma_n(\rho)$ and of $\mathscr{L}_{f,\rho,n}$ (see p12 [5]), we get the conclusion.

## §3. Meromorphic extension of $L$-functions of profinite graphs

Let $(V,E,d_\theta)$ be an irreducible profinite graph with a Lipschitz continuous function $l : E \to R_+$. By the property P2) and P4) the cardinarity of $V_n$ is not greater than $\# V_1 \cdot \mathscr{N}^{n-1}$, $\mathscr{N} = \max_{v \in V_1} \# \sigma^{-1}(v)$, hence the dimension of $\Gamma_n(\rho)$ is not greater than $N \cdot \# V_1 \cdot \mathscr{N}^{n-1}$.

In order to get the meromorphic region of $L$-functions we shall treat the generalized $L$-function $L(f;\rho)$ defined for $f \in C^1_\theta(E)$, which is given by

$$L(f;\rho) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} \sum_{\substack{c:\text{closed path} \\ \text{with } |c| = m}} \text{Trace } \rho(\langle c \rangle)\, f(c)\right)$$

This function is related to the $L$-function of $(V,E,d_\theta,l)$ by the equality $L_l(s;\rho) = L(e^{-sl};\rho)$ . The generalized $L$-function converges on $U_\theta = \{f \in C^1_\theta(E) \mid |f| > 0 \text{ and } \lambda(|f|) < 1\}$ and is non-zero holomorphic for $f \in U_\theta$. We extend it meromorphically by using Proposition 2.

THEOREM 4. *The generalized L-function $L(\cdot;\rho)$ can be meromorphically extended to the domain*

$$U'_\theta = \{f \in C^1_\theta(E) \mid |f| > 0 \text{ and } \lambda(|f|) < J\},$$

*where* $J = \exp\{\frac{1}{2}(\log\theta)^2(2\log\mathscr{N} - \log\theta)^{-1}\}$.

Proof. Let $f_0 \in U'_\theta \setminus U_\theta$ and $\theta' = \sqrt{\theta}$. By Proposition 2 we can choose $\varepsilon > 0$ so that the following conditions are satisfied:

1) $2\log(\lambda(|f_0|) + \varepsilon)\log\mathscr{N} < \log\theta \log(\sqrt{\theta}\,\lambda(|f_0|) + \varepsilon)$,

2) the operator $\mathscr{L}_{f_0,\rho} : C^1_{\theta'}(\hat{V},\rho) \to C^1_{\theta'}(\hat{V},\rho)$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_M$ of finite multiplicity satisfying $|\lambda_j| \geqq \theta'\lambda(|f_0|) + 3\varepsilon$,

3) the rest of the spectrum of $\mathscr{L}_{f_0,\rho}$ is contained in $\{z \in C \mid |z| \leqq \theta'\lambda(|f_0|)\}$.

By perturbation theory there exists positive $\delta < \min |f_0|$ such that the following five conditions hold for every $f \in B_{\theta'}(f_0, 4\delta) = \{f \in C^1_{\theta'}(E) \mid \|f - f_0\|_{\theta'} < 4\delta\}$;

1) $\lambda(|f|) < \lambda(|f_0|) + \varepsilon$,

2) the operator $\mathscr{L}_{f_0,\rho} : C^1(\hat{V},\rho) \to C^1_{\theta'}(\hat{V},\rho)$ has eigenvalues $\lambda_j^{(a)}(f), j = 1, \ldots, M,\ a = 1, \ldots,$

$m_j$ such that

$$|\lambda_j^{(a)}(f)| > \theta'\lambda(|f_0|)+2\varepsilon,$$

$$\lambda_j^{(a)}(f) \neq \lambda_k^{(b)}(f) \text{ for } j \neq k,$$

3) the rest of the spectrum $\lambda$ of $\mathscr{L}_{f,\rho}$ satisfies $|\lambda| < \theta'\lambda(|f|)+\varepsilon$,

4) if we denote the associated decomposition by $C^1_{\theta'}(\hat{V},\rho)=V_1(f)\oplus\ldots\oplus V_M(f)\oplus V(f)$, then the projection $\pi_j(f)$ onto $V_j(f)$ is analytic for $f \in B_{\theta'}(f_0;4\delta)$,

5) there exist transformations $F_j(f) : C^1_\theta(\hat{V},\rho) \to C^1_\theta(\hat{V},\rho)$ with $\pi_j(f)=F_j(f)^{-1}\circ\pi_j(f_0)\circ F_j(f)$, which are analytic for $f \in B_{\theta'}(f_0;4\delta)$.

We shall show that

$$\exp\{-\sum_{m=1}^{\infty}\frac{1}{m}\Big(\sum_{\substack{c:\text{closed path}\\ \text{with } |c|=m}}\text{Trace } \rho(\langle c\rangle)\, f(c) - \sum_{j=1}^{M}\sum_{a=1}^{m_j}\lambda_j^{(a)}(f)^m\Big)\}$$

converges absolutely and uniformly on some neighborhood of $f_0$. Define $f_n : E_n \to C$, $n=1,2,\ldots$, for each $f \in C^1_{\theta'}(E)$ by

$$f_n(e)=\sup_{\tilde{\omega}_n(\mathfrak{e})=e}\mathscr{R}e\, f(\mathfrak{e}) + \sqrt{-1}\sup_{\tilde{\omega}_n(\mathfrak{e})=e}\mathscr{I}m\, f(\mathfrak{e}),$$

where $\tilde{\omega}_n : E \to E_n$ is given by $\tilde{\omega}_n(\mathfrak{e})=e_n$ for $\mathfrak{e}=(e_n)\in E$. Since $f_n\circ\tilde{\omega}_n$ is $n$-locally constant, if we take sufficiently large $n_1$, we get

$$\begin{aligned}\|f_n\circ\tilde{\omega}_n - f_0\|_{\theta'} &\leq \|f_n\circ\tilde{\omega}_n - f\|_{\theta'} + \|f-f_0\|_{\theta'}\\ &\leq 3\theta'^n\|f\|_{\theta'} + 2\delta\\ &\leq 3\theta'^n(\|f_0\|_{\theta'} + \|f-f_0\|\ )+2\delta\\ &< 3\delta\end{aligned}$$

for every $f \in B_\theta(f_0;2\delta)$ and $n \geq n_1$.

Choose positive $\alpha$ so that

$$2\log(\lambda(|f_0|)+\varepsilon)\,|\log\theta|^{-1} < \alpha < |\log(\sqrt{\theta}\,\lambda(|\ \ _0|)+\varepsilon)|\,(\log_e\ \ )^{-1},$$

set $n(m)=[\alpha m]$, where $[*]$ denotes the integer part of $*$, and take $m_1$ so that $n(m_1) > n_1$. If we put $R=(\|f_0\|_\theta+\delta)(\min|f_0|-\delta)^{-1}$, then for every $f\in B_\theta(f_0;\delta)$ and $m\geqq m_1$,

$$|\frac{1}{m}\sum_{|c|=m} \text{Trace } \rho(\langle c\rangle)\left(f(c)-f_{n(m)}\circ\tilde{\omega}_{n(m)}\right)|$$

$$\leqq(1+R\theta^{n(m)})^m R\theta^{n(m)}\sum_{|c|=m}|f(c)|.$$

Therefore

$$\limsup_{m\to\infty}|\frac{1}{m}\sum_{c}\text{Trace } \rho(\langle c\rangle)\left(f(c)-f_{n(m)}\circ\tilde{\omega}_{n(m)}\right)|^{1/m}$$

$$\leqq\theta^\alpha\lambda(|f|)\leqq\theta^\alpha(\lambda(|f_0|)+\varepsilon)<1,$$

(see Lemma 5—2 of [3]), and we get the sum

$$\exp\left\{-\sum_{m=m_1}^{\infty}\frac{1}{m}\sum_{|c|=m}\text{Trace } \rho(\langle c\rangle)\left(f(c)-f_{n(m)}\circ\tilde{\omega}_{n(m)}\right)\right\}$$

converges absolutely and uniformly on $B_\theta(f_0;\delta)$.

By Proposition 3 the set of eigenvalues of $\mathscr{L}_{f_n,\rho}: C^1(\hat{V}_n,\rho)\to C^1(\hat{V}_n,\rho)$ consists of $\lambda_j^{(a)}(f_n\circ\tilde{\omega}_n), j=1,\ldots,M,\ a=1,\ldots,m_j$, and $\mu_i(f_n),\ i=1,\ldots,K$, with $|\mu_i(f_n)|\leqq\theta'\lambda(|f_0|)+\varepsilon$. Since the dimension of $C^1(\hat{V}_n,\rho)$ is not greater than $N\#V_{1,e}\ell^{n-1}$, we get

$$\lim_{m\to\infty}|\frac{1}{m}\sum_{i=1}^{K}\mu_i(f_{n(m)})^m|^{1/m}\leqq{}_e\ell^\alpha(\theta'\lambda(|f_0|)+\varepsilon)<1,$$

hence the sum

$$\exp\{-\sum_{m=m_1}^{\infty}\sum_{i=1}^{K}\mu_i(f_{n(m)})^m\}$$

also converges absolutely and uniformly on $B_\theta(f_0;\delta)$.

The operator $\mathscr{L}_{f,\rho,j}=F_j(f)\circ\mathscr{L}_{f,\rho}\circ F_j(f)^{-1}$ leaves invariant the space $V_j(f)$. By the analyticity for $f\in B_{\theta'}(f_0;4\delta)$ it saticefies

$$\|\hat{\mathscr{L}}_{f,\rho,j}-\hat{\mathscr{L}}_{f',\rho,j}\|_{\theta'}\leqq C\|f-f'\|_{\theta'}$$

with some positive $C$ for every $f,f'\in B_{\theta'}(f_0;3\delta)$ and $j=1,\ldots,M$. For each $f\in B_\theta(f_0;\delta)$ and $m\geqq m_1$ we can make the following estimate

$$|\sum_{a=1}^{m_j}\lambda_j^{(a)}(f)^m-\lambda_j^{(a)}(f_{n(m)}\circ\tilde{\omega}_{n(m)})^m|$$

$$= \left| \text{Trace} \left( \hat{\mathscr{L}}^m_{f,\rho;j} - \hat{\mathscr{L}}_{f_{n(m)}}{}^m \circ \tilde{\omega}_{n(m),\rho;j} : V_j(f_0) \to V_j(f_0) \right) \right|$$

$$\leqq (\text{Const}) \cdot \left\| \left( \hat{\mathscr{L}}^m_{f,\rho;j} - \hat{\mathscr{L}}^m_{f_{n(m)}} \circ \tilde{\omega}_{n(m),\rho;j} \right) \circ \pi_j(f_0) \right\|_{\theta'}$$

$$\leqq (\text{Const}) \sum_{k=1}^{m-1} \left\| \left( \hat{\mathscr{L}}_{f,\rho;j} \circ \pi_j(f_0) \right)^k \right.$$

$$\circ \left( \hat{\mathscr{L}}_{f,\rho;j} - \hat{\mathscr{L}}_{f_{n(m)}} \circ \tilde{\omega}_{n(m),\rho;j} \right) \circ \pi_j(f_0)$$

$$\left. \circ \left( \hat{\mathscr{L}}_{f_{n(m)}} \circ \tilde{\omega}_{n(m),\rho;j} \circ \pi_j(f_0) \right)^{m-k-1} \right\|_{\theta'}$$

$$\leqq (\text{Const}) \ \| f - f_{n(m)} \circ \tilde{\omega}_{n(m)} \|_{\theta'}$$

$$\times \sum_{k=1}^{m-1} \left\| \hat{\mathscr{L}}_{f,\rho;j} \circ \pi_j(f_0) \right\|^k_{\theta'} \left\| \hat{\mathscr{L}}^m_{f_{n(m)}} \circ \tilde{\omega}_{n(m),\rho;j}\, \pi_j(f_0) \right\|^{m-k-1}_{\theta'}$$

$$\leqq (\text{Const}) \ \theta'^{n(m)} \| f \|_\theta \sum_{k=1}^{m-1} \lambda(\,|\,f\,|\,)^k \lambda(\,|\,f_{n(m)} \circ \tilde{\omega}_{n(m)}\,|\,)^{m-k-1}$$

$$\leqq (\text{Const}) m \theta'^{n(m)} (\| f_0 \| + \delta)(\lambda(\,|\,f_0\,|\,) + \varepsilon)^m,$$

hence

$$\limsup_{m \to \infty} \left| \frac{1}{m} \sum_{j=1}^{M} \sum_{\alpha=1}^{m_j} \lambda_j^{(\alpha)}(f)^m - \lambda_j^{(\alpha)}(f_{n(m)} \circ \tilde{\omega}_{n(m)})^m \right|^{1/m}$$

$$\leqq \theta^{\alpha/2} (\lambda(\,|\,f_0\,|\,) + \varepsilon) < 1.$$

We therefore get the sum

$$\exp \left\{ - \sum_{m=m_1}^{\infty} \frac{1}{m} \sum_{j=1}^{M} \sum_{\alpha=1}^{m_j} \lambda_j^{(\alpha)}(f)^m - \lambda_j^{(\alpha)}(f_{n(m)} \circ \tilde{\omega}_{n(m)})^m \right\}$$

converges absolutely and uniformly on $B_\theta(f_0;\delta)$.

The property P2) and P4) imply that the set of all $m$-step closed paths on $(V,E)$ can be identified with that of all $m$-step closed paths on $(V_n,E_n)$ by the map induced by $\tilde{\omega}_n$, hence

$$\sum_{\substack{c:\text{closed path in} \\ (V,E) \text{ with } |c| = m}} \text{Trace } \rho(\langle c \rangle)\, f_n \circ \tilde{\omega}_n(c)$$

$$= \sum_{\substack{c:\text{closed path in} \\ (V_n,E_n) \text{ with } |c| = m}} \text{Trace } \rho_0(\langle c \rangle)\, f_n(c)$$

$$=\text{Trace}\left(\mathscr{L}^{m}_{f_n,\rho} : C^1(\hat{V}_n,\rho)\rightarrow C^1(\hat{V}_n,\rho)\right)$$

$$=\sum_{j=1}^{M}\sum_{a=1}^{m_j}\lambda_j^{(a)}(f_n\circ\tilde{\omega}_n)^m+\sum_{i=1}^{K}\mu_i(f_n)^m.$$

Summarising up we can now conclude that

$$\exp\{-\sum_{m=1}^{\infty}\frac{1}{m}\Big(\sum_{\substack{c:\text{closed path}\\ \text{with } |c|=m}}\text{Trace }\rho(\langle c\rangle)\,f(c)-\sum_{j=1}^{M}\sum_{a=1}^{m_j}\lambda_j^{(a)}(f)^m\Big)\}$$

$$=\exp\{-\sum_{m=1}^{m_1-1}\frac{1}{m}\Big(\sum_{c}\text{Trace }\rho(\langle c\rangle)\,f(c)-\sum_{j=1}^{M}\sum_{a=1}^{m_j}\lambda_j^{(a)}(f)^m\Big)\}$$

$$\times\exp\{-\sum_{m=m_1}^{\infty}\frac{1}{m}\sum_{|c|=m}\text{Trace }\rho(\langle c\rangle)\Big(f(c)-f_{n(m)}\circ\tilde{\omega}_{n(m)}\Big)\}$$

$$\times\exp\Big\{-\sum_{m=m_1}^{\infty}\frac{1}{m}\sum_{j=1}^{M}\sum_{a=1}^{m_j}\lambda_j^{(a)}(f)^m-\lambda_j^{(a)}(f_{n(m)}\circ\tilde{\omega}_{n(m)})^m\Big\}$$

$$\times\exp\{-\sum_{m=m_1}^{\infty}\frac{1}{m}\sum_{i=1}^{K}\mu_i(f_n)^m\}$$

converges absolutely and uniformly on $B_\theta(f_0;\delta)$, hence it is non-zero analytic for $f\in B_\theta(f_0;\delta)$.

On the other hand $\prod_{j=1}^{M}\prod_{a=1}^{m_j}(1-\lambda_j^{(a)}(f))$ is analytic for $f\in B_\theta(f_0;\delta)$, we can get out conclusion.

By Proposition 1, if $(V,E)$ is not circuit then the continuous function $R\ni s\mapsto\lambda(e^{-sl})\in R_+$ is monotone decreasing and there exists $h=h(l)>0$ with $\lambda(e^{-hl})=1$ (see Lemma 3-11 [3]). As was shown in [3], if $\mathscr{R}e\ s>h$ then $e^{-sl}\in U_\theta$ and $L_l(s;\rho)$ is holomorphic. By using Theorem 4 we get

COROLLARY. *If $(V,E)$ is not circuit then there is $\delta>0$ such that for every continuous unitary representation $\rho:\pi_1(V,E)\rightarrow U(N)$ the L-function $L_l(s;\rho)$ extends meromorphically to the region $\mathscr{R}e\ s>h(l)-\delta$.*

REMARK. If $l\in\bigcap_{0<\theta<1}C^1_\theta(E)$, in particular if $l$ is locally constant, then $L$-functions extends meromorphically to the entire complex plane.

## §4. $L$-functions of Anosov flows

A flow $\varphi_t: X \to X$ on a Riemannian manifold $X$ is called of Anosov type if the following conditions hold: The tangent bundle of $X$ splits into three $d\varphi_t$-invariant bundles $TX = E^t \oplus E^s \oplus E^u$ such that

1) $E^t$ is the line bundle tangent to orbits of the flow,
2) there are $C>0$ and $\lambda>0$ with

$$\| d\varphi_t(v) \| \leqq Ce^{-\lambda t} \| v \| \qquad \text{for } v \in E^s,\ t \geqq 0,$$

$$\| d\varphi_{-t}(v) \| \leqq Ce^{-\lambda t} \| v \| \qquad \text{for } v \in E^u,\ t \geqq 0,$$

The non-wondering set consists of points $x$ such that for every neighborhood $U$ of $x$ and $t>0$ there is $t_0>t$ with $\varphi_{t_0}(U) \cap U \neq \phi$. When the non-wondering set coincides with $X$, which is in the case when there is a $\varphi_t$-invariant measure, by use of Bowen's Symbolic dynamics, $L$-functions of an Anosov flow is related to those of profinite graphs in the following manner. There are irreducible non-circuit profinite graphs $(V^{(a)}, E^{(a)}), a = 0, \ldots, M$, with Lipschitz continuous length functions $l^{(a)}: E^{(a)} \to R_+$ ,and continuous homomorphisms $\psi_a: \pi_1(V^{(a)}, E^{(a)}) \to \pi_1(X)$ such that for every unitary representation $\rho: \pi_1(X) \to U(N)$

$$L_{\varphi_t}(s;\rho) = L_{l^{(0)}}(s;\rho\circ\psi_0) \prod_{a=1}^{M} (s;\rho\circ\psi_a)^{(-1)^{q(a)}},$$

$$h(\varphi_t) = h(l^{(0)}) > h(l^{(a)}), \quad a = 1, \ldots M,$$

with some integer $q(a)$ associated to $(V^{(a)}, E^{(a)})$. Combining this and Theorem 4 we get

THEOREM 5. *Let $\varphi_t: X \to X$ be an Anosov flow on a compact manifold. If the non-wondering set of $\varphi_t$ coincides with $X$, then there exists $\delta>0$ such that for every unitary representation $\rho: \pi_1(X) \to U(N)$ the L-function $L_{\varphi_t}(s;\rho)$ has a non-zero meromorphic extension to the domain $\mathcal{R}e\ s > h(\varphi_t) - \delta$.*

Remark. We[3] give a piece of information about poles on the line $\mathcal{R}e\ s = h$. The $L$-function $L_{\varphi_t}(s;\rho)$ associated with a irreducible unitary representation has a pole at $s = h + \sqrt{-1}u$ if and only if $\rho$ is a character (i.e. dim $\rho = 1$) and $\rho(\langle p \rangle) = \exp(\sqrt{-1}ul(p))$ for every closed orbit $p$. In this case $L_{\varphi_t}(s;\rho) = \zeta_{\varphi_t}(s - \sqrt{-1}u)$ and every poles on $\mathcal{R}e\ s$ are simple. In particular, if $\varphi_t$ is weak-mixing and the image of $\rho$ is a finite set, then $L_{\varphi_t}(s;\rho)$ does not have poles on $\mathcal{R}e\ s = h$ except a simple pole of the zeta function on $s = h$.

Remark. Pollicott[11] show that if the correlation function decays exponentially fast for all Hölder continuous functions then the zeta function of weak-mixing flow has an analytic extension to a strip $h-\varepsilon < \mathcal{R}e\ s < h$.

## §5. Examples of profinite graphs

Finite graphs and one-sided shifts of finite type are typical examples of profinite graphs. We here give some other examples which are not isomorphic to both of these.

Let $(V,E)$ be a finite graph. By blowing up $(V,E)$ at $v_0 \in V$ we construct a new graph $(V',E')$ in the following way. Given a surjective map $A: t^{-1}(v_0) \to S$ onto some finite set $S$, we put $V' = (V \setminus \{v_0\}) \cup S$, the disjoint union, and

$$E' = \{e \in E \mid o(e) \neq v_0, t(e) \neq v_0\}$$

$$\cup \{(v, A(v)) \mid v \in t^{-1}(v_0) \setminus \{v_0\}\}$$

$$\cup \{(s,w) \mid s \in S \text{ and } w = te \text{ for some } e \in o^{-1}(v_0) \text{ with } te \neq v_0\}$$

$$\cup \{(A(v_0), A(v_0))\},$$

where in case $v_0 \notin t^{-1}(v_0)$ the set $\{(A(v_0), A(v_0))\}$ means empty.

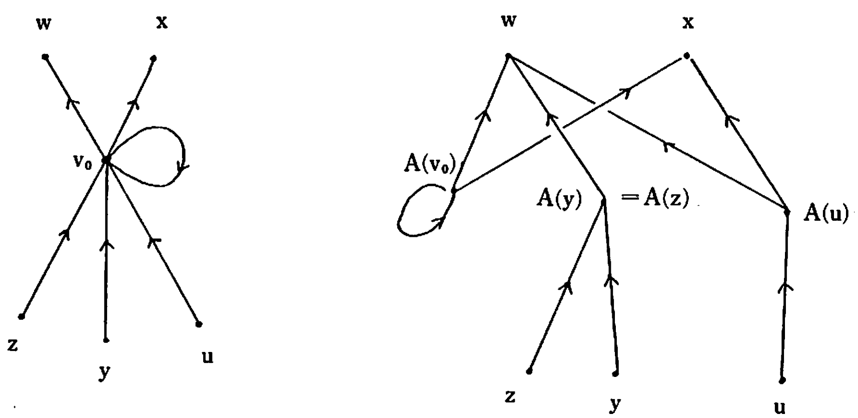


Figure 1 (blowing up at $v_0$)

one can easily check that

1) if $(V,E)$ is irreducible and $v_0 \notin \ell^{-1}(v_0)$ then $(V',E')$ is also irreducible,

2) if $\ell\colon E \to V$ is surjective so is $\ell\colon E' \to V'$.

If we define $\omega : V' \to V$ by

$$\omega(v') = \begin{cases} v & \text{if } v' \in V \setminus \{v_0\} \\ v_0 & \text{if } v' \in S, \end{cases}$$

then it is a surjective morphism of graphs and satisfies the following two conditions;

(i) $\omega\colon o^{-1}(v') \to o^{-1}(\omega(v'))$ is surjective for every $v' \in V'$,

(ii) if $\omega(e) = \omega(e')$, $e,e' \in E'$, then $\ell e = \ell e'$.

We can construct many profinite graphs by using this operation. One of the easiest one is the following; we construct $(V_{n+1},E_{n+1})$ by blowing up $(V_n,E_n)$ at the encircled vertex.

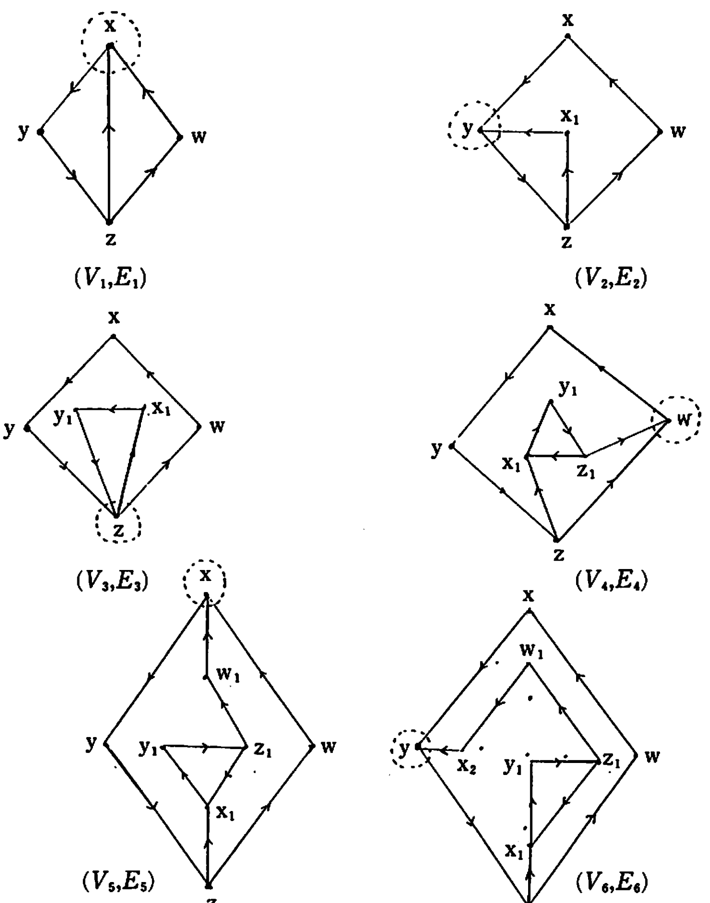


Figure 2

We can also construct product profinite graphs and quotient profinite graphs. The product of two profinite graphs $(V,E)$ and $(V'E')$ is given by $(\varprojlim V_n \times V'_n, \varprojlim E_n \times E'_n)$ with the morphism $\omega_n \times \omega'_n$. The quotient graph $(V,E)/\sim = (\varprojlim V_n/\sim_n, \varprojlim E_n/\sim_n)$ is defined in the following way. Let $\sim_n$ be an equivalence relation on the $n$-stage $V_n$ of a profinite graph $(V,E)$, which inherites on $E_n$ by $e \sim_n e'$ if and only if $o(e) \sim_n o(e')$ and $t(e) \sim_n t(e)$. One can easily check the properties P1) — P4) when the follwing conditions hold;

(i) if $v \sim_n w$ then $\omega_n(v) \sim_{n-1} \omega_n(w)$,

(ii) if $\omega_n(e) \sim_{n-1} \omega_n(e')$ then $t(e) \sim_n t(e')$,

(iii) if $e \in E_{n-1}$ and $v \in V_n$ satisfy $o(e) \sim_{n-1} \omega_n(v)$,

then there is $e' \in E_n$ with $o(e') \sim_n v$ and $\omega_n(e') \sim_{n-1} e$.

In particular, we can glue two profinite graphs $(V,E)$, $(V',E')$ at given points $\xi \in V$, $\eta \in V'$ and get a new profinite graph $(V'',E'')/\sim$ by putting $V'' = V \cup V'$, $E'' = E \cup E'$ and defining the relation $v \sim_n w$ when $v = w$ or $v = \xi_n$, $w = \eta_n$.

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