MEROMORPHIC EXTENSION OF $L$-FUNCTIONS OF ANOSOV FLOWS AND PROFINITE GRAPHS

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Introduction

In the preceding paper [3] T. Sunada and the author introduced $L$-functions of certain classes of graphs and dynamical systems. Applying number theoretical argument we ([1], [3], [4]) obtained some results about homology classes of closed orbits of Anosov flows; the density theorem of Chebotarev's type, equidistribution theorem and so on. In this note we consider the meromorphic region of these $L$-functions.

Let $\varphi_t : X \to X$ be an Anosov flow on a compact Riemannian manifold $X$ whose non-wandering set coincides with the whole space. Given a unitary representation $\rho : \pi_1(X) \to U(N)$, we define the $L$-function of $\varphi_t$ associated with $\rho$ by

$$L_{\varphi_t}(s;\rho) = \prod_p \frac{1}{\det(1 - \rho(\langle p \rangle) \exp(-s \cdot l(p)))},$$

where $p$ runs over all $\varphi_t$-prime closed orbits, $l(p)$ is the period of $p$ and $\langle p \rangle$ is an element of $\pi_1(X)$ whose conjugacy class corresponds to the free homotopy class of $p$. When $\text{Res} > h$ is greater than the topological entropy $h = h(\varphi_t)$ of the flow $\varphi_t$, the right hand side of (1) converges absolutely, hence $L_{\varphi_t}(s;\rho)$ is holomorphic on $\text{Res} > h$. We know moreover that $L_{\varphi_t}(s;\rho)$ has a meromorphic extension to some neighborhood of $\text{Res} \geq h$ (see [3]).

We here give more information on the meromorphic domain of $L_{\varphi_t}(s;\rho)$. For the special case of the trivial representation $\rho = 1$, Pollicott [12] shows that the zeta function $\zeta_{\varphi_t}(s) = L_{\varphi_t}(s;1)$ extends meromorphically to a region $\text{Res} > h - \varepsilon$ for some $\varepsilon > 0$. We extend his proof to cases of general representations and show that there exists $\delta > 0$ such that for every unitary representation $\rho$ the $L$-function $L_{\varphi_t}(s;\rho)$ extends non-zero meromorphically to the domain $\text{Res} > h - \delta$.

When $\varphi_t$ is the geodesic flow on the unit tangent bundle of a manifold of constant

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negative curvature, the Selberg trace formula implies that \( L \)-functions extend to the entire complex plane. But in general, we have no idea to expalne \( L \)-functions by trace formulas. We therefore use twisted Ruelle operators acting Banach spaces of Lipschitz continuous sections of flat bundles over a profinite graph. Associated with a continuous unitary representation \( \rho : \pi_i(V,E) \rightarrow U(N) \) of the fundamental group of a profinite graph \((V,E)\) with the length function \( l : E \rightarrow R_+ \), the \( L \)-function is defined in the same way as (1):

\[
L_1(s;\rho) = \prod_c 1/\det(I - \rho(c)\exp(-s \cdot l(c)) ,
\]

where \( c \) runs over all prime cycles on \((V,E)\) and \( l(c) = \sum_{i=1}^k l(e_i) \) for \( c = (e_1, \ldots, e_n) \). Since it is interpreted by what we may call the trace of a twisted Ruelle operator, an estimate about the essential spectral radius implies that it has a non-zero meromorphic extension on \( \Re s > h - \delta \). We can get our assertion by use of Bowen's symbolic dynamics.

We use terminology and some notations given in [3] without any explanation, the reader should refer it for some basic definitions and results.

§1. Profinite graphs and twisted Ruelle operators

Given graphs \((V_n,E_n)\), \( n \geq 1 \), and morphisms of graphs \( \omega_n : (V_n,E_n) \rightarrow (V_{n-1},E_{n-1}) \) we define

\[
V = \lim_{n \to \infty} V_n = \left\{ \xi = (\xi_n) \in \prod_{n=1}^{\infty} V_n \mid \omega_n(\xi_n) = \xi_{n-1} \right\} ,
\]

\[
E = \lim_{n \to \infty} E_n = \left\{ (\xi,\eta) \in V \times V \mid (\xi_n,\eta_n) \in E_n \right\} .
\]

This new graph \((V,E)\) is called as a profinite graph if the following conditions hold;

P0) each \((V_n,E_n)\) is a finite graph,
P1) the map \( \omega_n : V_n \rightarrow V_{n-1} \) is surjective,
P2) the map of origine \( \sigma : E_n \rightarrow V_n \) and terminus \( \epsilon : E_n \rightarrow V_n \) are surjective,
P3) \( \omega_n : \sigma^{-1}(v) \rightarrow \sigma^{-1}(\omega_n(v)) \) is surjective for every \( v \in V_n \),
P4) if \( \omega_n(e) = \omega_n(e') \), \( e, e' \in E_n \) then \( \epsilon(e) = \epsilon(e') \).

Let \( \rho : \pi_i(V,E) = \lim_{n \to \infty} \pi_i(V_n,E_n) \rightarrow U(N) \) be a continuous unitary representation of the fundamental group of a profinite graph \((V,E)\). There are \( \pi_o \) and \( \rho_o : \pi_i(V_n,\epsilon) \rightarrow U(N) \)
→ \mathbb{U}(N) such that \( \rho \) is factored as \( \rho = \rho_{\circ} \circ \text{Proj} : \pi_{1}(V, E) \rightarrow \pi_{1}(V_{n_{0}}, E_{n_{0}}) \rightarrow \mathbb{U}(N) \). Without loss of generality we may assume \( n_{0} = 1 \). We can construct a locally distance preserving normal covering map \( \pi : (\hat{V}, \hat{E}, d_{\theta}) \rightarrow (V, E, d_{\theta}) \) with covering transformation group \( \pi_{1}(V, E) \); here \( d_{\theta} \) (0 < \( \theta < 1 \)) denotes the distance function defined by

\[
d_{\theta}(\xi, \eta) = \theta \sup \{ n | \xi_{n} = \eta_{n} \} \quad \text{on} \ V,
\]

\[
d_{\theta}(e, e') = \theta \sup \{ n | e_{n} = e'_{n} \} \quad \text{on} \ E,
\]

and so on. In this paper we suppose \((V, E, d_{\theta})\) is irreducible.

Let \( C_{d}(E) \) denote the Banach space of Lipschitz continuous complex valued functions with the norm \( \| f \|_{d} = \| f \|_{\infty} + \text{Lip}_{d}(f) \), here

\[
\text{Lip}_{d}(f) = \sup \left\{ \frac{f(e) - f(e')}{d_{\theta}(e, e')} \bigg| e, e' \right\}.
\]

Given \( f \in C_{d}(E) \) we define the operator \( \mathcal{L}_{f} : C^{n}(\hat{V}, C^{n}) \rightarrow C^{n}(\hat{V}, C^{n}) \) on the space of all \( C^{n} \)-valued continuous functions on \( \hat{V} \) by

\[
\mathcal{L}_{f} g(\xi) = \sum_{\xi \in \xi, \omega \in \omega} f(\pi_{1}(\xi)) g(\xi).
\]

Since it leaves invariant the subspace \( C_{d}(\hat{V}, \rho) \) of all Lipschitz continuous functions \( g \) on \( \hat{V} \) with \( g(\gamma \xi) = \rho(\gamma) g(\xi) \) for every \( \gamma \in \pi_{1}(V, E) \) and \( \xi \in \hat{V} \), we can set

\[
\mathcal{L}_{\rho} = \mathcal{L}_{f} \bigg|_{C_{d}(\hat{V}, \rho)} : C_{d}(\hat{V}, \rho) \rightarrow C_{d}(\hat{V}, \rho),
\]

and call the twisted Ruelle operator associated with \( \rho \). We [3] study its spectra and get the following result.

**PROPOSITION 1.** (1) If \( f \in C_{d}(E) \) is positive valued then the Ruelle operator \( \mathcal{L}_{\rho} \) associated with the trivial representation has a maximal simple positive eigenvalue \( \lambda(f) \) with a positive eigenfunction.

(2) If \( f \in C_{d}(E) \) is nowhere vanishing, every spectrum \( \lambda \) of \( \mathcal{L}_{\rho} \) satisfies \( | \lambda | \leq \lambda( | f | ) \).

For a positive valued function \( f \in C_{d}(E) \), we define the continuous transformation \( G_{f} \) on the compact space of all Borel probability measures on \( V \) by

\[
G_{f}(\mu) = \left( \int_{V} \mathcal{L}_{\rho} \, d\mu \right)^{-1} \cdot \left( \mathcal{L}_{\rho} \right)^{n} \mu.
\]
Using the theorem of Schauder-Tychonoff we get a measure \( \mu_f \) with \( G_f(\mu_f) = \mu_f \). If one choose a positive eigenfunction \( u \) of \( \mathcal{L}_{x,1} \) associated with \( \lambda(f) \), then the equality

\[
\lambda(f) \int u \, d\mu_f = \int \mathcal{L}_{x,1} u \, d\mu_f = \int \mathcal{L}_{x,1} 1 \, d\mu_f \cdot \int u \, d\mu_f
\]

leads us to \( (\mathcal{L}_{x,1})^* \mu_f = \lambda(f) \mu(f) \). In particular, \( \mu_f \) is a \( \mathcal{L}_{x,1} \)-invariant measure, provided \( f \in C_b(E) \) is positive valued and \( \mathcal{L}_{x,1} 1 = 1 \).

Let \( V = V^1 \cup \ldots \cup V^\nu \) denote the decomposition into primitive parts and for \( \nu \in V_m \) set

\[
C(\nu) = \{ \xi \in V \mid \xi_m = \nu \}.
\]

For every positive valued \( f \in C_b(E) \) with \( \mathcal{L}_{x,1} 1 = 1 \), we can conclude \( \mu_f(V') = \nu^{-1} \) and \( \mu_f(C(\nu)) > 0 \) in the following way. Permuting the indices we may assume that \( \mathcal{L}(e) \in V^{j+1} \) whenever \( \mathcal{L}(e) \in V^j \), \( j = 1, \ldots, \nu \), modulo \( \nu \). Since \( \mathcal{L}_{x,1} 1 = 1 \) we get

\[
\mu_f(V') = \int \delta_1 d\mu_f = \int \mathcal{L}_{x,1} \delta_1 d\mu_f = \int \delta_{j+1} d\mu_f = \mu_f(V^{j+1}),
\]

which leads us to the first assertion, where \( \delta_j \in C'(V) \) is the characteristic function of \( V' \). To show the second, we should notice that if \( V' \) contains \( C(\nu) \) then for sufficiently large \( n \) there is a \( n\nu \)-step path \( c \) with \( \mathcal{L}(c) = \xi \) and \( \mathcal{L}(c) \in C(\nu) \) for each \( \xi \in V' \). We therefore get

\[
\mu_f(C(\nu)) = \int \delta_0 d\mu_f = \int \mathcal{L}_{x,1} \delta_0 d\mu_f \geq (\inf \mathcal{L}^n) \nu \int \delta_0 d\mu_f > 0,
\]

where \( \delta_0 \) is the characteristic function of \( C(\nu) \).

§2. The essential spectral radius of twisted Ruelle operators

In this section we concern with isolated eigenvalues of twisted Ruelle operators. We first show the following.
PROPOSITION 2. If \( f \in C^1_c(E) \) is nowhere vanishing, the essential spectral radius of \( \mathcal{L}_{f,\omega} \) is not greater than \( \theta \lambda(\|f\|) \). Hence every spectrum \( \lambda \) of \( \mathcal{L}_{f,\omega} \) with \( \theta \lambda(\|f\|) \leq |\lambda| \) is an eigenvalue of finite multiplicity.

Given a positive valued \( f \in C^1_c(E) \) with \( \mathcal{D}_f \), we define the average \( E_m(g;f) : \hat{V} \to C^0 \) of \( g \in C^1(\hat{V},\rho) \) on \( m \)-stage by

\[
E_m(g;f) = \left( \int_{\hat{V}_m} \delta_{\pi \eta_m} d\mu_r \right)^{-1} \int g \epsilon_m d\mu_r.
\]

Here the function \( g_{\hat{V}} : V \to C^0 \), \( \hat{V}_m \) is defined by

\[
g_{\hat{V}}(\eta) = \begin{cases} 
g(\hat{\eta}) & \text{if } \eta_m = \pi_m(\hat{\eta}) \\
0 & \text{otherwise},
\end{cases}
\]

where \( \hat{\eta} \) is the element of \( \hat{V} \) with \( \hat{\eta}_m = \hat{\eta} \) and \( \pi(\hat{\eta}) = \eta \), and \( \pi_m : \hat{V}_m \to V_m \) is the covering map induced by \( \pi \). As \( E_m(g;f) \) is \( m \)-locally constant (i.e. \( E_m(g;f)(\xi) = E_m(g;f)(\xi') \) if \( \xi_m = \xi_m' \)), one can easily check that it is an element of \( C^0(\hat{V},\rho) \) and

\[
\|g - E_m(g;f)\|_\infty \leq \theta^m \text{ Lip}_\rho(g),
\]

\[
\text{Lip}_\rho(g - E_m(g;f)) \leq 2 \text{ Lip}_\rho(g).
\]

We now prove Proposition 2. By the scaling transformation of \( f \) by \( \lambda(\|f\|) \) and an associated positive eigenfunction we may assume \( \mathcal{L}_\omega \equiv 1 \). Since the image of \( C^1_c(\hat{V},\rho) \) by the operator \( E_m(\cdot;|f|) : C^1_c(\hat{V},\rho) \to C^0(\hat{V},\rho) \) is a finite dimensional subspace, the operator \( \mathcal{L}^m_{f,\omega} \circ E_m(\cdot;|f|) \) is compact. By using the estimate

\[
\|\mathcal{L}_{f,\omega} g\|_\infty \leq \|g\|_\infty
\]

(2)

\[
\text{Lip}_\rho(\mathcal{L}^m_{f,\omega} g) \leq C \|g\|_\infty + \theta^m \text{ Lip}_\rho(g)
\]

(see Lemma 3.4 [3]), we have

\[
\left\| \left( \mathcal{L}^m_{f,\omega} - \mathcal{L}^m_{f,\omega} \circ E_m(\cdot;|f|) \right) (g) \right\|_{\infty}
\leq \|g - E_m(g;|f|)\|_\infty \leq \theta^m \text{ Lip}_\rho(g),
\]

\[
\text{Lip}_\rho \left( \left( \mathcal{L}^m_{f,\omega} - \mathcal{L}^m_{f,\omega} \circ E_m(\cdot;|f|) \right) (g) \right)
\leq C \|g - E_m(g;|f|)\|_\infty + \theta^m \text{ Lip}_\rho(g - E_m(g;|f|))
\leq (C + 2\theta^m) \text{ Lip}_\rho(g).
\]
We therefore get the following bound of the essential spectral radius of $\mathcal{L}_{f,0}$

$$\text{ess.spec.rad.}(\mathcal{L}_{f,0}) = \limsup_{n \to \infty} \left[ \inf \left\{ \| \mathcal{L}_{f,0} - K \|_\theta \mid K \text{ is a compact operator} \right\} \right]^{1/n} \leq \limsup_{n \to \infty} \theta(C+3)^{1/n} = \theta,$$

which leads us to the conclusion.

In the case $f$ is locally constant we have

**PROPOSITION 3.** If $f \in C^1_\theta(E)$ is $n$-locally constant, an eigenvalue $\lambda$ of $\mathcal{L}_{f,0}$ with $\theta \lambda(|f|) < \lambda \leq \lambda(|f|)$ is that of $\mathcal{L}_{h,0} : C^1_\theta(\hat{V}_n,\rho) \to C^0_\theta(\hat{V}_n,\rho)$, where $h : E_n \to C$ is the function with $h(e) = h(e_n)$ for every $e \in E$.

Proof. The operator $\mathcal{L}_{f,0}$ leaves invariant the closed subspace $\Gamma_n(\rho)$ in $C^1_\theta(\hat{V}_n,\rho)$ consisting of $n$-locally constant functions, hence it induces an operator $\mathcal{L}_{f,0,n}$ : $C^1_\theta(\hat{V}_n,\rho)/\Gamma_n(\rho) \to C^1_\theta(\hat{V}_m,\rho)/\Gamma_n(\rho)$. The space $\Gamma_n(\rho)$ can be identified with $C^1_\theta(\hat{V}_m,\rho)$ so that the following diagram is commutative

$$\begin{array}{c}
\Gamma_n(\rho) \\
\downarrow \quad \downarrow \\
C^1_\theta(\hat{V}_m,\rho) \\
\mathcal{L}_{f,0} \quad \quad \quad \mathcal{L}_{f,0,n}
\end{array} \xrightarrow{\mathcal{L}_{f,0}} \Gamma_n(\rho)
$$

With respect to the induced norm on $C^1_\theta(\hat{V}_n,\rho)/\Gamma_n(\rho)$, we can show in the same way to prove (2) that

$$\left\| \mathcal{L}_{f,0,n}[g] \right\|_{\infty} = \inf \left\{ \| \mathcal{L}_{f,0}(g-k) \|_{\infty} \mid k \in \Gamma_n(\rho) \right\} \leq \theta^m \text{Lip}_\theta([g]),$$

$$\text{Lip}_\theta(\mathcal{L}_{f,0,n}[g]) \leq C \| [g] \|_{\infty} + \theta^m \text{Lip}_\theta([g]).$$

Therefore the essential spectral radius of $\mathcal{L}_{f,0,n}$ is not greater than $\theta$. Since a spectrum
of $\mathcal{L}_{r,o}$ is contained in one of the sets of spectra of $\mathcal{L}_{r,o}$ and of $\mathcal{L}_{r,an}$ (see p12 [5]), we get the conclusion.

§3. Meromorphic extension of $L$-functions of profinite graphs

Let $(V,E,d_E)$ be an irreducible profinite graph with a Lipschitz continuous function $l : E \to \mathbb{R}_+$. By the property P2) and P4) the cardinality of $V_n$ is not greater than $\#V_1 \cdot \mathcal{N}_{\alpha-1}$, $\mathcal{N}=\max_{v \in V} \#\alpha^{-1}(v)$, hence the dimension of $\Gamma_n(\rho)$ is not greater than $\mathcal{N} \cdot \#V_1 \cdot \mathcal{N}^{-1}$.

In order to get the meromorphic region of $L$-functions we shall treat the generalized $L$-function $L(f,\rho)$ defined for $f \in C_0^1(E)$, which is given by

$$L(f,\rho) = \exp \left( \sum_{c} \frac{1}{m} \sum_{\text{closed path}} \text{Trace } (\rho(c)) f(c) \right)$$

with $|c| = m$

This function is related to the $L$-function of $(V,E,d_E,l)$ by the equality $L(s;\rho) = L(e^{-s};\rho)$.

The generalized $L$-function converges on $U_\theta = \{ f \in C_0^1(E) \mid |f| > 0 \text{ and } \lambda(|f|) < 1 \}$ and is non-zero holomorphic for $f \in U_\theta$. We extend it meromorphically by using Proposition 2.

THEOREM 4. The generalized $L$-function $L(\cdot;\rho)$ can be meromorphically extended to the domain

$$U_\theta' = \{ f \in C_0^1(E) \mid |f| > 0 \text{ and } \lambda(|f|) < J \},$$

where $J = \exp \left\{ \frac{1}{2} \left( \log \theta - 2 \log \mathcal{N} - \log \theta^{-1} \right) \right\}$.

Proof. Let $g_0 \in U_\theta' \setminus U_\theta$ and $\theta = \sqrt{\theta}$. By Proposition 2 we can choose $\epsilon > 0$ so that the following conditions are satisfied:

1) $2 \log(\lambda(|f_0|) + \epsilon) \log \mathcal{N} < \log \theta \log(\sqrt{\theta} \lambda(|f_0|) + \epsilon)$,

2) the operator $\mathcal{L}_{\rho,g}: C_0^1(\hat{V},\rho) \to C_0^1(\hat{V},\rho)$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_M$ of finite multiplicity satisfying $|\lambda_j| \geq \theta^\epsilon \lambda(|f_0|) + 3\epsilon$,

3) the rest of the spectrum of $\mathcal{L}_{\rho,g}$ is contained in $\{ z \in \mathbb{C} \mid |z| \leq \theta^{\epsilon} \lambda(|f_0|) \}$.

By perturbation theory there exists positive $\delta < \min |f_0|$ such that the following five conditions hold for every $f \in B_{\rho}(f_0,4\delta) = \{ f \in C_0^1(E) \mid \| f-f_0 \|_\rho < 4\delta \}$:

1) $\lambda(|f|) < \lambda(|f_0|) + \epsilon$,

2) the operator $\mathcal{L}_{\rho,g}: C^1(\hat{V},\rho) \to C^1(\hat{V},\rho)$ has eigenvalues $\lambda_{ja}(f), j=1,\ldots,M, a=1,\ldots,$,
such that

\[ | \lambda_j^{\omega}(f) | > \theta | \lambda | | f_0 | + 2 \epsilon, \]

\[ \lambda_j^{\omega}(f) \neq \lambda_k^{\omega}(f) \text{ for } j \neq k. \]

3) the rest of the spectrum \( \lambda \) of \( \mathcal{L}_2, \omega \) satisfies \( | \lambda | < \theta | \lambda | | f | + \epsilon, \)

4) if we denote the associated decomposition by \( C^{\omega}(\tilde{V}, \rho) = V_1(f) \oplus \ldots \oplus V_k(f) \oplus V(f), \)
then the projection \( \pi_j(f) \) onto \( V_j(f) \) is analytic for \( f \in B_\delta(f_0; 4 \delta). \)

5) there exist transformations \( F_j(f) : C^{\omega}(\tilde{V}, \rho) \rightarrow C^{\omega}(V, \rho) \) with \( \pi_j(f) = F_j(f)^{-1} \pi_j(f_0)^{0} F_j(f), \) which are analytic for \( f \in B_\delta(f_0; 4 \delta). \)

We shall show that

\[
\exp \left\{ - \sum_{m=1}^{\infty} \frac{1}{m} \left( \sum_{c: \text{closed path}} \frac{\text{Trace } \rho(c)}{c} f(c) - \sum_{j=1}^{k} \sum_{a=1}^{n_j} \lambda_j^{\omega}(f)^m \right) \right\}
\]

converges absolutely and uniformly on some neighborhood of \( f_0. \) Define \( f_n : E_n \rightarrow C, \ n = 1, 2, \ldots, \) for each \( f \in C^{\omega}(E) \) by

\[
f_n(e) = \sup_{\tilde{\omega}_n(e) = e} A e f(e) + \sqrt{-1} \sup_{\tilde{\omega}_n(e) = e} J_m f(e),
\]

where \( \tilde{\omega}_n : E \rightarrow E_n \) is given by \( \tilde{\omega}_n(e) = e_n \) for \( e = (e_n) \in E. \) Since \( f^e \tilde{\omega}_n \) is \( n \)-locally constant, if we take sufficiently large \( n, \) we get

\[
\| f^e \tilde{\omega}_n - f_0 \|_\delta \leq \| f^e \tilde{\omega}_n - f \|_\delta + \| f - f_0 \|_\delta
\]

\[
\leq 3 \theta^n \| f \|_{\omega'} + 2 \delta
\]

\[
\leq 3 \theta^n (\| f_0 \|_{\omega'} + \| f - f_0 \|) + 2 \delta
\]

\[
< 3 \delta
\]

for every \( f \in B_\delta(f_0, 2 \delta) \) and \( n \geq n_0. \)

Choose positive \( \alpha \) so that

\[
2 \log(\theta | f_0 | + \epsilon) \log(\theta | f_0 | + \epsilon)^{-1} < \alpha < 2 \log(\sqrt{\theta} | f_0 | + \epsilon) \log(\sqrt{\theta} | f_0 | + \epsilon)^{-1},
\]
set \( n(m) = \lfloor am \rfloor \), where \([*]\) denotes the integer part of \(*\), and take \( m_1 \) so that \( n(m_1) > n_1 \). If we put \( R = (\| f_0 \| + \delta)(\min \| f_0 \| - \delta)^{-1} \), then for every \( f \in B_\theta(f_0; \delta) \) and \( m \geq m_1 \),

\[
\left| \frac{1}{m} \sum_{c \in m} \text{Trace } \rho(c) \left( f(c) - f_{nm} \circ \overline{\omega}_{nm} \right) \right| \\
\leq (1 + R \theta^m)^m \sum_{c \in m} |f(c)|.
\]

Therefore

\[
\limsup_m \left| \frac{1}{m} \sum_{c \in m} \text{Trace } \rho(c) \left( f(c) - f_{nm} \circ \overline{\omega}_{nm} \right) \right|^{1/m} = \theta^\lambda(\| f_0 \| + \epsilon) < 1,
\]

(see Lemma 5–2 of [3]), and we get the sum

\[
\exp \left\{ - \sum_{m=m_1} \frac{1}{m} \sum_{c \in m} \text{Trace } \rho(c) \left( f(c) - f_{nm} \circ \overline{\omega}_{nm} \right) \right\}
\]

converges absolutely and uniformly on \( B_\theta(f_0; \delta) \).

By Proposition 3 the set of eigenvalues of \( \mathfrak{L}_{f,a} : C'(\hat{\mathcal{V}}_{nm}) \rightarrow C'(\hat{\mathcal{V}}_{nm}) \) consists of \( \lambda_{i,a}(f_0 \circ \overline{\omega}_{nm}), j=1, \ldots, M, a=1, \ldots, m_n \), and \( \mu_i(f_0), i=1, \ldots, K \), with \( | \mu_i(f_0) | \leq \theta^\lambda(\| f_0 \| + \epsilon) \). Since the dimension of \( C'(\hat{\mathcal{V}}_{nm}) \) is not greater than \( N \# V_{n1} \) \( n_{n-1} \), we get

\[
\lim_{m \to \infty} \left| \frac{1}{m} \sum_{i=1}^K \mu_i(f_{nm})^m \right|^{1/m} \leq \epsilon \theta^\lambda(\| f_0 \| + \epsilon) < 1,
\]

hence the sum

\[
\exp \left\{ - \sum_{m=m_1} \sum_{i=1}^K \mu_i(f_{nm})^m \right\}
\]

also converges absolutely and uniformly on \( B_\theta(f_0; \delta) \).

The operator \( \mathfrak{L}_{f,a,j} = F_j \rho_a \mathfrak{L}_{f,a} \) \( F_j \rho_a \mathfrak{L}_{f,a} \) leaves invariant the space \( V_\lambda(f) \). By the analyticity for \( f \in B_\theta(f_0; \delta) \) it satifies

\[
\| \hat{\mathfrak{L}}_{f,a,j} - \hat{\mathfrak{L}}_{f,a,j} \|_{\ell^1} \leq C \| f - \rho \|_{\ell^1}
\]

with some positive \( C \) for every \( f, \rho \in B_\theta(f_0; \delta) \) and \( j=1, \ldots, M \). For each \( f \in B_\theta(f_0; \delta) \) and \( m \geq m_1 \), we can make the following estimate

\[
\left| \sum_{a=1}^m \lambda_j \omega(f)^m - \lambda_j \omega(f_{nm}) \circ \overline{\omega}_{nm} \right|^m
\]
\[
\begin{align*}
= & \left| \text{Trace} \left( \tilde{\mathcal{L}}_{f,p,j}^n \circ \tilde{\omega}_{\tilde{\mathcal{L}}_{f,p,j}} : V_j(f_0) \rightarrow V_j(f_0) \right) \right| \\
\leq & (\text{Const}) \left\| \left( \tilde{\mathcal{L}}_{f,p,j}^n \circ \tilde{\omega}_{\tilde{\mathcal{L}}_{f,p,j}} \right) \circ \pi_j(f_0) \right\|_\delta \\
\leq & (\text{Const}) \sum_{k=1}^{m-1} \left\| \left( \tilde{\mathcal{L}}_{f,p,j}^n \circ \pi_j(f_0) \right) \right\|_\delta \\
\leq & (\text{Const}) \left\| f - f_{\text{nom}} \circ \tilde{\omega}_{\text{nom}} \right\|_\delta \\
\times & \sum_{k=1}^{m-1} \left\| \tilde{\mathcal{L}}_{f,p,j}^n \circ \pi_j(f_0) \right\|_\delta \\
\leq & (\text{Const}) \theta_{\text{nom}} \left( \| f \|_\delta + \delta(\lambda(\| f_0 \|) + \epsilon)^m \right) \\
\leq & (\text{Const}) \theta_{\text{nom}} (\| f \|_\delta + \delta(\lambda(\| f_0 \|) + \epsilon)^m, \\
\text{hence} \quad \limsup_{m \to \infty} \left| \frac{1}{m} \sum_{j=1}^{m} \sum_{q=1}^{n} \lambda^q(f)^m \left( f_{\text{nom}} \circ \tilde{\omega}_{\text{nom}} \right)^m \right|^{1/m} \\
\leq & \theta^\epsilon (\lambda(\| f_0 \|) + \epsilon) < 1.
\end{align*}
\]

We therefore get the sum

\[
\exp \left\{ - \sum_{m=n}^\infty \frac{1}{m} \sum_{j=1}^{m} \sum_{q=1}^{n} \lambda^q(f)^m \left( f_{\text{nom}} \circ \tilde{\omega}_{\text{nom}} \right)^m \right\}
\]

converges absolutely and uniformly on $B_0(f_0;\delta)$. The property P2) and P4) imply that the set of all $m$-step closed paths on $(V,E)$ can be identified with that of all $m$-step closed paths on $(V_n,E_n)$ by the map induced by $\tilde{\omega}_n$, hence

\[
\sum_{c \text{: closed path in } (V,E) \text{ with } |c| = m} \text{Trace } \rho(c) f_n \circ \tilde{\omega}_n(c)
\]

\[
= \sum_{c \text{: closed path in } (V_n,E_n) \text{ with } |c| = m} \text{Trace } \rho_n(c) f_n(c)
\]
Meromorphic extension of $L$-Functions of Anosov Flows and profinite Graphs

$$= \text{Trace} \left( L_{f_{\mu}} : C^1(\hat{V}_n \rho) \rightarrow C^1(\hat{V}_n \rho) \right)$$

$$= \sum_{j=1}^n \sum_{a=1}^{n_j} \lambda_{j}^{a}(f_{a} \hat{\omega}_{j})^{m} + \sum_{i=1}^{1} \mu_{i}(f_{i})^{m}.$$ 

Summarising up we can now conclude that

$$\exp \left\{ - \sum_{m=m_1}^{1} \left( \sum_{c: \text{closed path}} \text{Trace} \rho(c) \text{ } f(c) - \sum_{j=1}^{n} \sum_{a=1}^{n_j} \lambda_{j}^{a}(f_{a})^{m} \right) \right\}$$

$$= \exp \left\{ - \sum_{m=m_1}^{1} \left( \sum_{c: \text{closed path}} \text{Trace} \rho(c) \text{ } f(c) - \sum_{j=1}^{n} \sum_{a=1}^{n_j} \lambda_{j}^{a}(f_{a})^{m} \right) \right\} \times \exp \left\{ - \sum_{m=m_1}^{1} \left( \sum_{c: \text{closed path}} \text{Trace} \rho(c) \text{ } f(c) - \sum_{j=1}^{n} \sum_{a=1}^{n_j} \lambda_{j}^{a}(f_{a})^{m} \right) \right\}$$

$$\times \exp \left\{ - \sum_{m=m_1}^{1} \left( \sum_{c: \text{closed path}} \text{Trace} \rho(c) \text{ } f(c) - \sum_{j=1}^{n} \sum_{a=1}^{n_j} \lambda_{j}^{a}(f_{a})^{m} \right) \right\}$$

converges absolutely and uniformly on $B_{\delta}(f_{\delta}; \delta)$, hence it is non-zero analytic for $f \in B_{\delta}(f_{\delta}; \delta)$.

On the other hand $\prod_{j=1}^{n} \prod_{a=1}^{n_j} (1 - \lambda_{j}^{a}(f))$ is analytic for $f \in B_{\delta}(f_{\delta}; \delta)$, we can get out conclusion.

By Proposition 1, if $(V,E)$ is not circuit then the continuous function $R \ni s \mapsto \lambda(e^{-st}) \in R$ is monotone decreasing and there exists $h = h(l) > 0$ with $\lambda(e^{-ht}) = 1$ (see Lemma 3-11 [3]). As was shown in [3], if $R_{e} s > h$ then $e^{-at} \in U_{a}$ and $L_{d}(s; \rho)$ is holomorphic. By using Theorem 4 we get

**COROLLARY.** If $(V,E)$ is not circuit then there is $\delta > 0$ such that for every continuous unitary representation $\rho : \pi_{1}(V,E) \rightarrow U(N)$ the $L$-function $L_{d}(s; \rho)$ extends meromorphically to the region $R_{e} s > h(l) - \delta$.

**REMARK.** If $l \in \bigcap_{s \in C_{l}} C_{l}(E)$, in particular if $l$ is locally constant, then $L$-functions extends meromorphically to the entire complex plane.
§4. \textit{L}-functions of Anosov flows

A flow $\varphi_t X \to X$ on a Riemannian manifold $X$ is called of Anosov type if the following conditions hold: The tangent bundle of $X$ splits into three $d\varphi_t$-invariant bundles $TX = E^s \oplus E^\alpha \oplus E^u$ such that

1) $E^s$ is the line bundle tangent to orbits of the flow,
2) there are $C > 0$ and $\lambda > 0$ with

$$\|d\varphi_t(u)\| \leq Ce^{-\lambda t} \|u\| \quad \text{for} \quad u \in E^s, t \geq 0,$$

$$\|d\varphi_{-t}(u)\| \leq Ce^{-\lambda t} \|u\| \quad \text{for} \quad u \in E^u, t \geq 0.$$ 

The non-wandering set consists of points $x$ such that for every neighborhood $U$ of $x$ and $t > 0$ there is $t_s > t$ with $\varphi_{t_s}(U) \cap U \neq \emptyset$. When the non-wandering set coincides with $X$, which is in the case when there is a $\varphi_t$-invariant measure, by use of Bowen's Symbolic dynamics, $L$-functions of an Anosov flow is related to those of profinite graphs in the following manner. There are irreducible non-circuit profinite graphs $(V^\omega, E^\omega), a = 0, \ldots, M$, with Lipschitz continuous length functions $l^a: E^\omega \to \mathbb{R}$, and continuous homomorphisms $\psi_a^\omega: \pi_1(V^\omega, E^\omega) \to \pi_1(X)$ such that for every unitary representation $\rho: \pi_1(X) \to U(N)$

$$L_{\varphi_a}(s; \rho) = L_{\psi_a^\omega}(s; \rho^\omega \psi_a) \prod_{\omega = 1}^{M} (s; \rho^\omega \psi_a)^{-1/q(a)},$$

$$h(\varphi_a) = h(l^a) > h(l^a), \quad a = 1, \ldots, M,$$

with some integer $q(a)$ associated to $(V^\omega, E^\omega)$. Combining this and Theorem 4 we get

**Theorem 5.** Let $\varphi_t X \to X$ be an Anosov flow on a compact manifold. If the non-wandering set of $\varphi_t$ coincides with $X$, then there exists $\delta > 0$ such that for every unitary representation $\rho: \pi_1(X) \to U(N)$ the $L$-function $L_{\varphi}(s; \rho)$ has a non-zero meromorphic extension to the domain $\Re s > h(\varphi) - \delta$.

Remark. We[3] give a piece of information about poles on the line $\Re s = h$. The $L$-function $L_{\varphi}(s; \rho)$ associated with an irreducible unitary representation has a pole at $s = h + \sqrt{-1}u$ if and only if $\rho$ is a character (i.e., dim $\rho = 1$) and $\rho(p) = \exp(\sqrt{-1}u p)$ for every closed orbit $p$. In this case $L_{\varphi}(s; \rho) = \zeta_{\varphi}(s - \sqrt{-1}u)$ and every poles on $\Re s$ are simple. In particular, if $\varphi_t$ is weak-mixing and the image of $\rho$ is a finite set, then $L_{\varphi}(s; \rho)$ does not have poles on $\Re s = h$ except a simple pole of the zeta function on $s = h$. 

Remark. Pollicott[11] show that if the correlation function decays exponentially fast for all Hölder continuous functions then the zeta function of weak-mixing flow has an analytic extension to a strip $h - \varepsilon < \Re s < h$.

§5. Examples of profinite graphs

Finite graphs and one-sided shifts of finite type are typical examples of profinite graphs. We here give some other examples which are not isomorphic to both of these.

Let $(V,E)$ be a finite graph. By blowing up $(V,E)$ at $v_0 \in V$ we construct a new graph $(V',E')$ in the following way. Given a surjective map $A: \tau^{-1}(v_0) \to S$ onto some finite set $S$, we put $V' = (V \setminus \{v_0\}) \cup S$, the disjoint union, and

$$E' = \{e \in E \mid \sigma(e) \neq v_0, \tau(e) \neq v_0\}$$

$$\cup \{(u, A(v)) \mid v \in \tau^{-1}(v_0) \setminus \{v_0\}\}$$

$$\cup \{(s, w) \mid s \in S \text{ and } w = \tau e \text{ for some } e \in \sigma^{-1}(v_0) \text{ with } \tau e \neq v_0\}$$

$$\cup \{(A(v_0), A(u))\},$$

where in case $v_0 \notin \tau^{-1}(v_0)$ the set $\{(A(v_0), A(u))\}$ means empty.

Figure 1 (blowing up at $v_0$)
one can easily check that

1) if \((V,E)\) is irreducible and \(u \not\in \xi^{-1}(u)\) then \((V',E')\) is also irreducible,

2) if \(\xi : E \to V\) is surjective so is \(\xi' : E' \to V'\).

If we define \(\omega : V' \to V\) by

\[
\omega(v') = \begin{cases} 
  v & \text{if } v' \in V \setminus \{u\} \\
  u & \text{if } v' \in S,
\end{cases}
\]

then it is a surjective morphism of graphs and satisfies the following two conditions;

(i) \(\omega \circ \xi^{-1}(v') \to \xi^{-1}(\omega(v'))\) is surjective for every \(v' \in V',\)

(ii) if \(\omega(e) = \omega(e'), e,e' \in E'\), then \(e = e'.\)

We can construct many profinite graphs by using this operation. One of the easiest one is the following; we construct \((V_{n+1},E_{n+1})\) by blowing up \((V_n,E_n)\) at the encircled vertex.

Figure 2
We can also construct product profinite graphs and quotient profinite graphs. The product of two profinite graphs \((V,E)\) and \((V',E')\) is given by \((\lim V_n \times V'_n, \lim E_n \times E'_n)\) with the morphism \(\omega \times \omega'\). The quotient graph \((V,E)/\sim=(\lim V_n/\sim_n, \lim E_n/\sim_n)\) is defined in the following way. Let \(\sim_n\) be an equivalence relation on the \(n\)-stage \(V_n\) of a profinite graph \((V,E)\), which inherits on \(E_n\) by \(e\sim e'\) if and only if \(\sigma(e)\sim \sigma(e')\) and \(\xi(e)\sim_n \xi(e)\). One can easily check the properties P1) – P4) when the following conditions hold:

(i) if \(v\sim_n w\) then \(\omega_n(v)\sim_{n-1} \omega_n(w)\),
(ii) if \(\omega_n(e)\sim_{n-1} \omega_n(e')\) then \(\xi(e)\sim_n \xi(e')\),
(iii) if \(e \in E_{n-1}\) and \(v \in V_n\) satisfy \(\sigma(e)\sim_{n-1} \omega_n(v)\),
then there is \(e' \in E_{n-1}\) with \(\sigma(e')\sim \omega_n(e')\sim_{n-1} v\) and \(\omega_n(e')\sim_{n-1} e\).

In particular, we can glue two profinite graphs \((V,E), (V',E')\) at given points \(\xi \in V, \eta \in V'\) and get a new profinite graph \((V'',E'')\) by putting \(V''=V \cup V', E''=E \cup E'\) and defining the relation \(v\sim_w w\) when \(v=w\) or \(v=\xi, w=\eta\).

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