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Almost Periodic Solutions of Hyperbolic Systems of Impulsive Differential Equations

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ABSTRACT. In the present paper sufficient conditions are found for existence of almost periodic solutions of hyperbolic systems of impulsive differential equations. The impulse effects are realizing at fixed moments of time.

1. INTRODUCTION

In the recent years the impulsive differential equations are an object of various investigations [1]–[5].

In the present paper is studied the problem of existence of almost periodic solutions of hyperbolic systems of impulsive differential equations in the case when the impulse effects are realized in fixed moments of time.

2. PRELIMINARY NOTES. STATEMENT OF THE PROBLEM

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with elements $z = \mathrm{col}(z_1, z_2, \ldots, z_n)$ and with the norm $\|z\| = \left(\sum_{i=1}^{n} z_i^2\right)^{1/2}$, and let $\mathbb{R} = (-\infty, \infty)$, $B_h = \{z \in \mathbb{R}^n : \|z\| \leq h\}$. Let the points $t_i \in \mathbb{R}$, $(i = \pm 1, \pm 2, \ldots)$ be fixed and such that $t_i < t_{i+1}$, $\lim_{i \to \pm\infty} t_i = \pm\infty$.

In the present paper we consider the following systems of impulsive differential equations:

$$\dot{z} = A(t)z + f(t), \quad t \neq t_i, \tag{1}$$

$$\Delta z(t_i) = b_i, \quad i = \pm 1, \pm 2, \ldots \tag{2}$$

and

$$\dot{z} = A(t)z + F(t, z), \quad t \neq t_i, \tag{3}$$

$$\Delta z(t_i) = I_i(z(t_i)), \quad i = \pm 1, \pm 2, \ldots, \tag{4}$$

where $t \in \mathbb{R}$, $z \in \mathbb{R}^n$, $A : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^n$, $f : \mathbb{R} \to \mathbb{R}^n$, $\{b_i\}$, $i = \pm 1, \pm 2, \ldots$, is a sequence of constant n-dimensional vectors, $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $I_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = \pm 1, \pm 2, \ldots$, $\Delta z(t_i) = z(t_i + 0) - z(t_i - 0)$, $i = \pm 1, \pm 2, \ldots$.

We denote by $z(t) = z(t; t_0, z_0)$ the solution of the problem (1), (2) ((3), (4)) with the initial condition $z(t_0) = z_0$.

Let us note that the solution $z(t)$ is piecewise continuous function with points of discontinuity at the moments t_i, $i = \pm 1, \pm 2, \ldots$, at which it is continuous from the left, i.e., the solution $z(t)$ of the system (1), (2) satisfies the relations

$$\begin{aligned} z(t_i - 0) &= z(t_i), \\ z(t_i + 0) &= z(t_i) + \Delta z(t_i) = z(t_i) + b_i, \end{aligned}$$

while for the solution $z(t)$ of the system (3), (4) the following relations hold true:

$$\begin{aligned} z(t_i - 0) &= z(t_i), \\ z(t_i + 0) &= z(t_i) + \Delta z(t_i) = z(t_i) + I_i(z(t_i)). \end{aligned}$$

For $t \neq t_i$, the solution $z(t)$ of the system (1), (2) is defined by the system (1). At the moments t_i, $i = \pm 1, \pm 2, \ldots$, the solution $z(t)$ of the system (1), (2) is an object of impulsive effect. At the intervals $(t_i, t_{i+1}]$, $i = \pm 1, \pm 2, \ldots$, the solution $z(t)$ of the system (1), (2) coincides with the solution of the system

$$\begin{aligned} \dot{v} &= A(t)v + f(t), \\ v(t_i) &= z(t_i) + b_i. \end{aligned}$$

The solution $z(t)$ of the system (3), (4) is described in analogous way.

Together with the systems (1) and (3) we will consider the corresponding homogeneous system

$$\dot{z} = A(t)z. \tag{5}$$

Definition 1. [4] The system (5) is said to be *hyperbolic*, if there exist constants $a > 0$, $\lambda > 0$, and for each $t \in \mathbb{R}$ there exist linear spaces $M^+(t)$ and $M^-(t)$ whose external direct sum is $M^+(t) \oplus M^-(t) = \mathbb{R}^n$ and such that if $z_0 \in M^+(t_0)$ then for all $t \geq t_0$ the inequality

$$\|z(t; t_0, z_0)\| \leq a\|z_0\|e^{-\lambda(t-t_0)},$$

holds true, while if $z_0 \in M^-(t_0)$ then for all $t \leq t_0$ we have

$$\|z(t; t_0, z_0)\| \leq a\|z_0\|e^{\lambda(t-t_0)}.$$

In the present paper we shall consider the problem of existence of almost periodic solutions of the systems (1), (2) and (3), (4) assuming that the corresponding homogeneous system (5) is hyperbolic.

Since the solutions of (1)–(4) are piecewise continuous functions, we adopt the following definition of almost periodic piecewise continuous function.

Definition 2. [6] A piecewise continuous function $\varphi : \mathbb{R} \to \mathbb{R}^n$ with discontinuities of the first kind at the points t_i, $i = \pm 1, \pm 2, \ldots$, is called *almost periodic*, if:

a) The set of sequences $\bar{t}_j = \{t_{i+j} - t_i\}_{i=-\infty}^{\infty}$ is uniformly almost periodic.

b) For any $\varepsilon > 0$ there exists a real number $\delta > 0$ such that if the points t' and t'' belong to one and the same interval of continuiuty and they satisfy the inequality $|t' - t''| < \delta$, then $\|\varphi(t') - \varphi(t'')\| < \varepsilon$.

c) For any $\varepsilon > 0$ there exists a relatively dense set T of ε-almost periods such that if $\tau \in T$, then $\|\varphi(t + \tau) - \varphi(t)\| < \varepsilon$ for all $t \in \mathbb{R}$ satisfying the condition $|t - t_i| > \varepsilon$, $i = \pm 1, \pm 2, \ldots$.

We introduce the following conditions:

H1. The matrix-valued function $A(t)$ is continuous and almost periodic in the sense of Bohr for $t \in \mathbb{R}$.

H2. The system (5) is hyperbolic.

H3. The function $f(t)$ is piecewise continuous with points of discontinuity of first kind t_i, $i = \pm 1, \pm 2, \ldots$, and it is almost periodic.

H4. The sequence $\{b_i\}$, $i = \pm 1, \pm 2, \ldots$, is almost periodic.

H5. The set of sequences $\{E_j\}$, $j = \pm 1, \pm 2, \ldots$, is uniformly almost periodic, and there exists $\theta > 0$ such that

$$\inf \{t_{i+1} - t_i > \theta : \ i = \pm 1, \pm 2, \ldots\}.$$

H6. The function $F(t, z)$ is almost periodic with respect to t, and uniformly continuous with respect to z, $z \in \Omega$, $\Omega \subset \mathbb{R}^n$.

H7. The functions $I_i(z)$ are continuous and they form almost periodic sequence uniformly with respect to $z \in \Omega$, $\Omega \subset \mathbb{R}^n$.

In the paper we shall use the following lemmas.

Lemma 1. [7] *Let the set of sequences $\{E_j\}$ be uniformly almost periodic.*

Then for each $p > 0$ there exists a positive integer N such that on each interval of length p there are no more than N elements of the sequence $\{t_i\}$.

Lemma 2. [7] *Let the condition* **H5** *be fulfilled.*

Then, if $\psi(t)$, $\psi : \mathbb{R} \to \mathbb{R}^n$, is almost periodic function, then the sequence $\{\psi(t_i)\}$, $i = \pm 1, \pm 2, \ldots$, is almost periodic.

Lemma 3. [7] *Let the following conditions be fulfilled:*

1. *Condition* H5 *is met.*

2. *$\psi(t)$, $\psi : \mathbb{R} \to \Omega$, $\Omega \subset \mathbb{R}^n$, is piecewise continuous function with points of discontinuity of the first kind t_i, $i = \pm 1, \pm 2, \ldots$, and it is almost periodic.*

3. *$F(y)$ is uniformly continuous function, defined in G.*

Then $F(\psi(t))$ is almost periodic function.

Lemma 4. [7] *Let conditions* H1, H3–H5 *be fulfilled.*

Then for each $\varepsilon > 0$ there exist ε_1, $0 < \varepsilon_1 < \varepsilon$ and relatively dense sets T of real numbers, and Q of whole numbers, such that the following relations are fulfilled:

(i) $\|A(t+\tau) - A(t)\| < \varepsilon, \quad t \in \mathbb{R},\ \tau \in T,$

(ii) $\|f(t+\tau) - f(t)\| < \varepsilon, \quad t \in \mathbb{R},\ \tau \in T,\ |t - t_i| > \varepsilon,\ i = \pm 1, \pm 2, \ldots$

(iii) $\|b_{i+q} - b_i\| < \varepsilon, \quad q \in Q,\ i = \pm 1, \pm 2, \ldots$

(iv) $|E_q - \tau| < \varepsilon_1, \quad q \in Q,\ \tau \in T,\ i = \pm 1, \pm 2, \ldots.$

3. Main results

Lemma 5. *Let conditions* H1, H2 *and* H5 *be fulfilled.*

Then there exists a non-singular transformation defined by almost periodic matrix $S(t)$, which reduces the system (5) *into the next ones*

$$\dot{x} = Q^+(t)\,x \tag{6}$$

and

$$\dot{y} = Q^-(t)y, \tag{7}$$

where $x \in \mathbb{R}^k$, $y \in \mathbb{R}^{n-k}$, and the following assertions hold true:

1. *The matrices $Q^+(t)$ and $Q^-(t)$ are almost periodic matrix-valued functions.*

2. *If $\Phi^+(t,s)$ and $\Phi^-(t,s)$ are the corresponding fundamental matrices of the systems* (6) *and* (7) *then following inequalities hold true*

$$\|\Phi^+(t,s)\| \le \bar{a}e^{-\lambda(t-s)}, \quad t \ge s, \tag{8}$$

$$\|\Phi^-(t,s)\| \le \bar{a}e^{\lambda(t-s)}, \quad t \le s, \tag{9}$$

where $\bar{a} > 0$, $\lambda > 0$.

3. For each $\varepsilon > 0$, $t \in \mathbb{R}$, $s \in \mathbb{R}$ there exists relatively dense set T of ε-almost periods, such that for each $\tau \in T$ the fundamental matrices $\Phi^+(t,s)$ and $\Phi^-(t,s)$ satisfy

$$\|\Phi^+(t+\tau, s+\tau) - \Phi^+(t,s)\| \le \varepsilon K e^{-\frac{\lambda}{2}(t-s)}, \quad t \ge s, \tag{10}$$

$$\|\Phi^-(t+\tau, s+\tau) - \Phi^-(t,s)\| \le \varepsilon K e^{\frac{\lambda}{2}(t-s)}, \quad t \le s, \tag{11}$$

where $K = const. > 0$, and λ is the constant defined at the assertion 2 of Lemma 5.

Proof. Assertions 1 and 2 are immediate consequences of Theorem 1 in [4]. In fact, following the ideas used in [4], we define the matrix $S(t)$ to be formed by the vector-columns which are the solutions of (5). It follows from the conditions of Lemma 5 that $S(t)$ consists of almost periodic functions.

On the other hand, the transformation $z = S(t)u$ rewrites (5) in the form

$$\dot{u} = Q(t)u,$$

where

$$Q(t) = S^{-1}(t)\left(A(t)S(t) - \dot{S}(t)\right).$$

Hence, $Q(t)$ is almost periodic function. The estimates (8) and (9) are direct consequences of Theorem 1 in [4].

To prove assertion 3, we let $\Phi^+(t,s)$ and $\Phi^-(t,s)$ to be the fundamental matrices of the systems (6) and (7), respectively. Then for each $\varepsilon > 0$ the following relations hold true:

$$\frac{\partial \Phi^+(t,s)}{\partial t} = Q^+(t)\Phi^+(t+\tau, s+\tau) + \left[Q^+(t+\tau) - Q^+(t)\right]\Phi^+(t+\tau, s+\tau),$$
$$\frac{\partial \Phi^-(t,s)}{\partial t} = Q^-(t)\Phi^-(t+\tau, s+\tau) + \left[Q^-(t+\tau) - Q^-(t)\right]\Phi^-(t+\tau, s+\tau),$$

and

$$\begin{aligned}\Phi^+(t+\tau, s+\tau) &= \Phi^+(t,s) + \\ &+ \int_s^t \Phi^+(t,v)\left(Q^+(v+\tau) - Q^+(v)\right)\Phi^+(v+\tau, s+\tau)dv,\\ \Phi^-(t+\tau, s+\tau) &= \Phi^-(t,s) + \\ &+ \int_s^t \Phi^-(t,v)\left(Q^-(v+\tau) - Q^-(v)\right)\Phi^-(v+\tau, s+\tau)dv.\end{aligned}$$

Therefore,

$$\left\|\Phi^+(t+\tau, s+\tau) - \Phi^+(t,s)\right\| \le$$

$$\int_s^t \left\|\Phi^+(t,s)\right\| \left\|Q^+(v+\tau) - Q^+(v)\right\| \left\|\Phi^+(v+\tau, s+\tau)\right\| dv,$$

$$\left\|\Phi^-(t+\tau, s+\tau) - \Phi^-(t,s)\right\| \leq$$

$$\int_t^s \left\|\Phi^-(t,v)\right\| \left\|Q^-(v+\tau) - Q^-(v)\right\| \left\|\Phi^-(v+\tau, s+\tau)\right\| dv.$$

It follows from (8)

$$\left\|\Phi^+(t+\tau, s+\tau) - \Phi^+(t,s)\right\| \leq \varepsilon K e^{-\frac{\lambda}{2}(t-s)},$$

where $K = (\bar{a})^2$. The proof of the estimate (11) is analogous. ■

It follows from Lemma 5 that by the help of transformation with the matrix $S(t)$, the system (1), (2) takes on the form

$$\dot{x} = Q^+(t)x + f^+(t), \qquad t \neq t_i, \tag{12}$$

$$\Delta x(t_i) = b_i^+, \qquad i = \pm 1, \pm 2, \ldots, \tag{13}$$

$$\dot{y} = Q^-(t)y + f^-(t), \qquad t \neq t_i, \tag{14}$$

$$\Delta y(t_i) = b_i^-, \qquad i = \pm 1, \pm 2, \ldots, \tag{15}$$

where $x \in \mathbb{R}^k$, $y \in \mathbb{R}^{n-k}$, $f^+ : \mathbb{R} \to \mathbb{R}^k$, $f^- : \mathbb{R} \to \mathbb{R}^{n-k}$, and b_i^+ and b_i^- are k- and $n-k$-dimensional constant vectors, respectivelly.

In an analogous way, the system (3), (4) after the transformation with the matrix $S(t)$ takes on the form

$$\dot{x} = Q^+(t)x + F^+(t,x,y), \tag{16}$$

$$\dot{y} = Q^-(t)y + F^-(t,x,y), \tag{17}$$

$$\Delta x(t_i) = I_i^+(x(t_i), y(t_i)), \tag{18}$$

$$\Delta y(t_i) = I_i^-(x(t_i), y(t_i)), \qquad i = \pm 1, \pm 2, \ldots, \tag{19}$$

where $F^+ : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$, $F^- : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$, $I_i^+ : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$, $I_i^- : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$, $i = \pm 1, \pm 2, \ldots$.

Theorem 1. *Let the conditions* H1–H5 *be fulfilled.*

Then the system (1), (2) *has a unique almost periodic solution, which is asymptotically stable.*

Proof. We consider the following equations

$$x(t) = \int_{-\infty}^t \Phi^+(t,s) f^+(s) ds + \sum_{t_i < t} \Phi^+(t,t_i) b_i^+,$$

$$y(t) = -\int_t^{\infty} \Phi^-(t,s)f^-(s)ds - \sum_{t_i>t} \Phi^-(t,t_i)b_i^-,$$

which are equivalent to the systems (12)–(15).

Let $\varepsilon > 0$ be an arbitrary chosen constant. It follows from Lemma 4 that there exist sets T and Q such that for each $\tau \in T$ and $q \in Q$ the following estimates hold true:

$$\begin{aligned} \|x(t+\tau) - x(t)\| \le & \int_{-\infty}^{t} \left\|\Phi^+(t+\tau, s+\tau) - \Phi^+(t,s)\right\| \left\|f^+(s+\tau)\right\| ds + \\ & + \int_{-\infty}^{t} \left\|\Phi^+(t,s)\right\| \left\|f^+(s+\tau) - f^+(s)\right\| ds + \\ & + \sum_{t_i<t} \left\|\Phi^+(t+\tau, t_{i+q}) - \Phi^+(t,t_i)\right\| \left\|b_{i+q}^+\right\| + \\ & + \sum_{t_i<t} \left\|\Phi^+(t,t_i)\right\| \left\|b_{i+q}^+ - b_i^+\right\| \end{aligned} \tag{20}$$

and

$$\begin{aligned} \|y(t+\tau) - y(t)\| \le & \int_{t}^{\infty} \left\|\Phi^-(t+\tau, s+\tau) - \Phi^-(t,s)\right\| \left\|f^-(s+\tau)\right\| d\tau + \\ & + \int_{t}^{-\infty} \left\|\Phi^-(t,s)\right\| \left\|f^-(s+\tau) - f^-(s)\right\| ds + \\ & + \sum_{t<t_i} \left\|\Phi^-(t+\tau, t_{i+q}) - \Phi^-(t,t_i)\right\| \left\|b_{i+q}^-\right\| + \\ & + \sum_{t<t_i} \left\|\Phi^-(t,t_i)\right\| \left\|b_{i+q}^- - b_i^-\right\|. \end{aligned} \tag{21}$$

It follows from Lemma 5, (20) and (21) that

$$\|x(t+\tau) - x(t)\| \le R_1\varepsilon, \tag{22}$$

where

$$R_1 = \frac{2K}{\lambda}\sup_{t\in R}\left\|f^+(t)\right\| + \frac{\bar{a}}{\lambda} + \frac{2KN}{1-e^{-\lambda/2}}\sup_{i=\pm1,\pm2,\dots}\left\|b_i^+\right\| + \frac{2N\bar{a}}{1-e^{-\lambda}}.$$

In the same manner we obtain

$$\|y(t+\tau) - y(t)\| \le R_2\varepsilon, \tag{23}$$

where

$$R_2 = \frac{2K}{\lambda}\sup_{t\in R}\left\|f^-(t)\right\| + \frac{\bar{a}}{\lambda} + \frac{2KN}{1-e^{-\lambda/2}}\sup_{i=\pm1,\pm2,\dots}\left\|b_i^-\right\| + \frac{2N\bar{a}}{1-e^{-\lambda}}.$$

It follows from (22) and (23) that the solution $z(t) = (x(t), y(t))$ of the system (1), (2) is almost periodic function. On the other hand, each solution $(x(t), y(t))$ of the system (1), (2) can be written in the form

$$x(t) = \Phi^+(t, t_0)c + \int_{t_0}^{t} \Phi^+(t, s)f^+(s)ds + \sum_{s<t_i<t} \Phi^+(t, t_i)b_i^+,$$

$$y(t) = -\int_{t}^{\infty} \Phi^-(t, s)f^-(s)ds - \sum_{t_i>t} \Phi^-(t, t_i)b_i^-,$$

where c is a constant k-dimensional vector.

It follows that for two different solutions $z_1(t)$ and $z_2(t)$ of the system (1), (2) the estimate

$$\|z_1(t) - z_2(t)\| \le \bar{a}e^{-\lambda(t-t_0)} \|z_1(t_0) - z_2(t_0)\| \tag{24}$$

holds true.

Thus, (24) implies the uniqueness and almost periodicity of the solution of (1), (2). ■

Theorem 2. *Let the following conditions be fulfilled:*

1. *Conditions* H1, H2, H5–H7 *are met.*

2. *The functions $F(t, z)$ and $I_i(z)$, $i = \pm 1, \pm 2, \ldots$, satisfy the Lipschitz condition with a constant L with respect to the variable z in the domain $\mathbb{R} \times B_h$, i.e.,*

$$\|F(t, z_1) - F(t, z_2)\| + \|I_i(z_1) - I_i(z_2)\| \le L\|z_1 - z_2\|,$$

and they are bounded, i.e., there exists a constant $H > 0$ such that

$$\max\left\{\sup_{t\in R, z\in B_h} \|F(t, z)\|, \sup_{i=\pm1,\pm2,\ldots, z\in B_h} \|I_i(z)\|\right\} \le H.$$

3. *The inequalities*

$$H\frac{\bar{a}}{\lambda} + H\frac{2\bar{a}N}{1 - e^{-\lambda}} < h,$$

$$L\frac{\bar{a}}{\lambda} + L\frac{2\bar{a}N}{1 - e^{-\lambda}} < 1$$

are valid.

Then the system (3), (4) has a unique almost periodic solution.

Proof. We denote by APC the set of all functions $\varphi(t)$, $\varphi : \mathbb{R} \to \mathbb{R}^n$, which are piecewise continuous with points of discontinuity of the first kind t_i, $i = \pm 1, \pm 2, \ldots$, and which are almost periodic. We introduce in APC a norm as follows: if $\varphi \in APC$ then

$|\varphi| = \sup_{t\in R} \|\varphi(t)\|$. On the set D, $D \subset APC : D = \{\varphi \in APC : |\varphi| < h\}$, we define an operator R such that if $\varphi \in D$, then $\varphi = (\varphi^+, \varphi^-)$, where $\varphi^+ : \mathbb{R} \to \mathbb{R}^k$, $\varphi^- : \mathbb{R} \to \mathbb{R}^{n-k}$, $R\varphi = (R\varphi^+, R\varphi^-)$ and $u = R\varphi^+$ is almost periodic solution of the system

$$\begin{aligned} \dot{u} &= Q^+(t)u + F^+(t, \varphi(t)), \quad t \neq t_i, \\ \Delta u(t_i) &= I_i^+(\varphi(t_i)), \quad i = \pm 1, \pm 2, \dots, \end{aligned}$$

and $v = R\varphi^-$ is almost periodic solution of the system

$$\begin{aligned} \dot{v} &= Q^-(t)v + F^-(t, \varphi(t)), \quad t \neq t_i, \\ \Delta v(t_i) &= I_i^-(\varphi(t_i)), \quad i = \pm 1, \pm 2 \dots. \end{aligned}$$

Let us note that the existence of the almost periodic solutions $u(t)$ and $v(t)$ is guaranteed by Theorem 1. In fact, the almost periodicity of the sequence $\{\varphi(t_i)\}$, $i = \pm 1, \pm 2, \dots$ follows from Lemma 2, and if the method of finding of common almost periods [4] will be used, we obtain that the sequence $\{I_i(\varphi(t_i))\}$, $i = \pm 1, \pm 2, \dots$ is almost periodic also. The almost periodicity of the function $F(t, \varphi(t))$ follows from Lemma 3. Further on, conditions 2 and 3 imply that $R(D) \subset D$.

Let $\varphi(t)$, $\psi(t) \in D$. Then the estimate

$$\|R\varphi - R\psi\| \leq L\bar{a}\left(\frac{1}{\lambda} + \frac{2N}{1 - e^{-\lambda}}\right)|\varphi - \psi| \tag{25}$$

holds true.

It follows from condition 3 of Theorem 2, and from (25), that R is a contraction operator on D. Hence, the system (3), (4) has a unique almost periodic solution. ■

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References

[1] D. D. Bainov and P. S. Simeonov, *Systems with Impulse Effect: Stability, Theory and Applications*, Ellis Horwood, Chichester, 1989.

[2] D. D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations: Periodic Solutions and Applications*, Longman, Harlow, 1993.

[3] D. D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations: Asymptotic Properties of the Solutions and Applications*, World Scientific Publishers, Singapore, 1995.

[4] S. G. Hristova and D. D. Bainov, Integral surfaces for hyperbolic ordinary differential equations with impulses, *COMPEL* 4 (1985), 1–18.

[5] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific Publishers, Singapore, 1989.

[6] M. A. Hekimova and D. D. Bainov, Almost periodic solutions of singularly perturbed systems of differential equations with impulse effect, *Rendiconti di Matematica* **8**, Roma (1988), 19–30.

[7] A. M. Samoilenko and N. A. Perestyuk, *Differential Equations with Impulse Effect*, Višča Shkola, Kiev, 1987 (in Russian).

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