Kumamoto J. Math.
Vol. 10, 11~29 March (1997)

ON THE HOMOLOGY GROUP ASSOCIATED WITH THE GENERAL AIRY INTEGRAL

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(Received Octorber 5, 1996)

1. INTRODUCTION

In [GRS], I.M. Gel'fand et al. introduced the general Airy system which generalizes the classical Airy differential equation:

$$d^2y/dx^2 - xy = 0. \tag{1.1}$$

It is known and is easy to show that (1.1) admits a solution given by the integral of the form of Laplace transform

$$y(x) = \int_{\Delta} e^{xu - \frac{1}{3}u^3} du, \tag{1.2}$$

where the path of integration Δ is given as follows. Let L_k $(0 \le k \le 2)$ be the half straight line in the complex u-plane defined by

$$L_k := \{ s e^{2k\pi i/3} \mid 0 \le s < +\infty \}$$

with an orientation which directs from 0 to ∞. Let Δ be any linear combination of $\Delta_{kl} := -L_k + L_l$ $(0 \le k < l \le 2)$. Then, if x is in any compact set of $\mathbb{C}$, the integral (1.2) converges uniformly in x and gives an entire solution to (1.1). If we choose any two paths of $\{\Delta_{kl}\}$, the functions defined by the integral (1.2) over these paths give a fundamental system of solutions to (1.1). This fact can be seen, for example, by computing their Wronskian determinant.

As for the general Airy system, Gel'fand et al. gave a solution by the integral of the form similar to (1.2):

$$\int_{\Delta} e^{P(u,z)} du_1 \wedge \cdots \wedge du_r,$$

where $P(u,z)$ is a polynomial in $u=(u_1,\dots,u_r)$ with coefficients depending rationally on the parameters z ([GRS], see also §2.2 (2.10)). They mentioned few remarks about the domain of integration Δ. But it seems to me that the discussion on a domain of integration Δ given in [GRS] is not sufficient, because they didn't give a rigorous proof to some of their statements. So I think it is worth to give in this note some supplementary remarks to make clear this point for further study, for example, the study of asymptotic behavior of the function as z tends to the singularity of the general Airy system. As is indicated in [GRS] §4, we understand Δ as a cycle of a certain locally finite homology group and study its structure. The definition of such homology group will be given in §2.2.

This note is organized as follows. In Section 2, we review the definition of general Airy integrals and state the main result. Section 3 is devoted to a study of polynomial maps depending on several parameters, which will be used in proving the main theorem in Section 4.

2. Definitions and Main results

2.1 General Airy function.

Let r and n be positive integers such that $r+1<n+1$. Let $M(r+1,n+1)$ denote the set of $(r+1)\times(n+1)$ complex matrices. An element $z\in M(r+1,n+1)$ is written as $z=(z_0,\dots,z_n)$ with the i-th column vector z_i. We also write $z=(z',z'')$, where $z'=(z_0,\dots,z_r)$, $z''=(z_{r+1},\dots,z_n)$. Put

$$Z=\{z\in M(r+1,n+1)\mid \det z'\neq 0\},$$

which is a Zariski open subset of $M(r+1,n+1)$ and will be the domain of definition of the general Airy functions.

Let J be the *Jordan group* of size $n+1$, namely,

$$J=\left\{\sum_{0\le i\le n} h_i\Lambda^i \mid h_i\in\mathbb{C}, h_0\neq 0\right\}\subset GL(n+1),$$

where $\Lambda=(\delta_{i+1,j})_{0\le i,j\le n}$ is the shift matrix of size $n+1$. An element $h=\sum_i h_i\Lambda^i\in J$ is also denoted by $h=[h_0,\dots,h_n]$. Notice that J is a maximal abelian subgroup of $GL(n+1)$. It is shown in [GRS] that there is an isomorphism

$$J\simeq \mathbb{C}^\times\times\mathbb{C}^n, \tag{2.1}$$

where $\mathbb{C}^\times$ is the complex torus and $\mathbb{C}^n$ is a vector group. This isomorphism is established as follows. Define a series of rational functions $\theta_k(x)$ in $x = (x_0, x_1, \dots, x_n)$ by

$$\log\left(1 + \frac{x_1}{x_0}T + \cdots + \frac{x_n}{x_0}T^n\right) = \sum_{k=1}^{\infty} \theta_k(x)T^k, \tag{2.2}$$

where T is an indeterminate. Then the correspondence

$$h = [h_0, \dots, h_n] \mapsto (h_0, \theta_1(h), \dots, \theta_n(h)) \tag{2.3}$$

establishes the isomorphism (2.1). Here, for $h = [h_0, \dots, h_n]$, we denote by $\theta_k(h)$ the value obtained by substituting $(h_0, \dots, h_n)$ into $\theta_k(x)$, by the abuse of notation. We introduce also a biholomorphic map $\iota : J \to \mathbb{C}^\times \times \mathbb{C}^n$ by

$$\iota([h_0, \dots, h_n]) = (h_0, h_1, \dots, h_n).$$

Let $\chi : J \to \mathbb{C}^\times$ be a character of J. Then the isomorphism (2.1) realized by the correspondence (2.3) tells that there is a vector $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{C}^{n+1}$, $\alpha_0 \in \mathbb{Z}$ such that

$$\chi(h) = h_0^{\alpha_0} \exp\left(\sum_{i=1}^{n} \alpha_i \theta_i(h)\right).$$

This character will be denoted by $\chi(\cdot\,, \alpha)$ to indicate its dependence on the parameters α. In the following, we consider only the characters of J satisfying the assumption:

$$\alpha_0 = -r - 1, \quad \alpha_n \neq 0. \tag{2.4}$$

The set of α satisfying the condition (2.4) is a complex manifold biholomorphic to $\mathbb{C}^{n-1} \times \mathbb{C}^\times$. Let Ω be the complex manifold of pairs (z, α) with $z \in Z$ and α satisfying (2.4). Let $\mathbb{P}^r$ be a projective space with the homogeneous coordinates $t = (t_0, \dots, t_r)$. Take a character $\chi(\cdot\,; \alpha)$ of J satisfying (2.4) and put

$$\omega(t, z, \alpha) := \chi(\iota^{-1}(tz); \alpha) \cdot \tau, \tag{2.5}$$

where

$$\tau = \sum_{i=0}^{r} (-1)^i t_i dt_0 \wedge \cdots \widehat{dt_i} \cdots \wedge dt_r, \tag{2.6}$$

the symbol $\widehat{dt_i}$ implying that the term dt_i does not appear in the expression. Note that for $w = (z, \alpha) \in \Omega$ fixed, ω is a holomorphic r-form on $\mathbb{P}^r \setminus H_w$, where $H_w = \{t \in \mathbb{P}^r \mid tz_0 = 0\}$. In fact, by virtue of the condition (2.4), the form ω is invariant under the homothety $t \mapsto ct$ ($c \in \mathbb{C}^\times$); it implies ω is a well defined r-form on the projective space $\mathbb{P}^r$.

Definition 2.1. The integral

$$\int_{\Delta(z,\alpha)} \omega(t,z,\alpha) \tag{2.7}$$

in called the *general Airy integral.*

The problem considered in this paper is the following.

Problem 2.2. Determine a possible chain $\Delta(z,\alpha)$ which depend continuously on (z,α) so that the integral (2.7) converges for each (z,α) and defines an analytic function of z and α.

Once a chain $\Delta(z,\alpha)$ is determined as desired we call the function defined by the integral (2.7) *a general Airy function.*

Remark 2.3. Any homology cycle with a compact support gives only zero value to the integral, since $\mathbb{P}^r \setminus H_w$ is homeomorphic to $\mathbb{C}^r$ and is contractible.

2.2 Main theorem.

Put

$$E = \{(t,z,\alpha) \in \mathbb{P}^r \times \Omega \mid tz_0 \neq 0\}.$$

Let $\pi : E \to \Omega$ be a fiber bundle with the projection $\pi(t,z,\alpha) = (z,\alpha)$. The fiber over $w = (z,\alpha) \in \Omega$ is denoted by $E_w := \mathbb{P}^r \setminus H_w$. Put

$$f(t,z,\alpha) = \sum_{i=1}^{n} \alpha_i \theta_i(tz), \tag{2.8}$$

which is a rational function on $\mathbb{P}^r \times \Omega$ with the pole on $H = \{(t,z) \in \mathbb{P}^r \times \Omega \mid tz_0 = 0\}$. Then the differential form ω in (2.5) can be written in the form

$$\omega = e^{f(t,z,\alpha)} \cdot (tz_0)^{-r-1} \cdot \tau. \tag{2.9}$$

Remark that if the 0-th column vector of z has the form ${}^t(1,0,\ldots,0)$ the defining equation of H_w has the form $t_0 = 0$, and if we take an affine coordinates $(u_1,\ldots,u_r)$ of E_w defined by $u_1 = t_1/t_0, \ldots, u_r = t_r/t_0$, then $(tz_0)^{-r-1} \cdot \tau = du_1 \wedge \cdots \wedge du_r$ and f turns into a polynomial in u with coefficients depending on $w = (z,\alpha)$, which we denote by $P_w(u)$. Thus the differential form ω is written as

$$\omega = e^{P_w(u)} du_1 \wedge \cdots \wedge du_r. \tag{2.10}$$

Therefore the integral (2.7) can be a considered as a kind of oscillatory integral with a polynomial phase $P_w(u)$.

We want to take a cycle in some locally finite homology group as a domain of integration $\Delta = \Delta(z, \alpha)$ in the integral (2.7) (cf. Remark at the end of §2.1). To get an idea, we make here a heuristic consideration. Take $w = (z, \alpha)$ such that $z_0 = {}^t(1, 0, \dots, 0)$ and fix it. Then by the above remark, the differential form ω has the form (2.10) and the support $|\Delta|$ of Δ cannot be compact. In order to make the integral of ω converge, we must take Δ so that the term e^{P_w} become exponentially small as $u \in \Delta$ tends to infinity and this decrease dominates the eventual increase of the measure of Δ.

Keeping in mind the above consideration, we introduce, following [P2], a family of closed subsets of E to which the support of a cycle must belong. For any $c < 0$ we put

$$S_c^+ = \{v \in \mathbb{C} \mid Re(v) \geq c\}, \quad S_c^- = \{v \in \mathbb{C} \mid Re(v) \leq c\}.$$

Let Φ be the family of closed sets $A \subset E$ such that

$$A \cap f^{-1}(S_c^+) \quad \text{is compact for any } c < 0. \tag{2.11}$$

The following Lemma is easy to check.

Lemma 2.4. *The family Φ is a family of supports in the homology theory, namely it satisfies the conditions:*

(1) *If $A, A' \in \Phi$, then $A \cup A' \in \Phi$.*
(2) *If $A \in \Phi$ and $A' \subset A$ is closed, then $A' \in \Phi$.*
(3) *For any $A \in \Phi$ there is $A' \in \Phi$ such that A' is a neighbourhood of A.*

Restricting each element of Φ to a fiber E_w, we get a family Φ_w of closed subsets of E_w satisfying the conditions (1), (2) and (3) of Lemma 2.4. Therefore, Φ_w is a family of supports in E_w. If there is no fear of confusion we write Φ in place of Φ_w. For $w \in \Omega$ we consider a locally finite homology group with coefficient in $\mathbb{Z}$ in which any homology class is defined by a cycle whose support belongs to Φ_w. The k-th homology group is denoted by $H_k^\Phi(E_w)$ and is called the *homology group with supports in the family* Φ_w. The main theorem of this paper is the following.

Theorem 2.5.

(1) *For any $w \in \Omega$, we have*

$$H_k^\Phi(E_w) \simeq \begin{cases} 0 & \text{if } k \neq r, \\ \mathbb{Z}^\mu & \text{if } k = r, \end{cases}$$

where $\mu = \binom{n-1}{r}$.

(2) *$\mathcal{H} = \cup_{w \in \Omega} H_r^\Phi(E_w)$ forms a local system on Ω of free $\mathbb{Z}$-module of rank μ.*

Let $\Delta : w \mapsto \Delta(w)$ be a local horizontal section of the local system $\mathcal{H}$ on Ω. Then the integral (2.7) gives an analytic function of w which can be continued analytically along any path in Ω by virtue of (2) of the theorem.

3. Polynomial map depending on parameters

3.1. Polynomial map.

To prove Theorem 2.5, we need some results on a geometry of polynomial maps depending on several parameters.

Let $f : \mathbb{C}^r \to \mathbb{C}$ be a polynomial map. We equip $\mathbb{C}^r$ with the standard hermitian metric: $\|u\|^2 = \sum_i |u_i|^2$. It is well known ([B], [P2], [V-L]) that there exists a finite subset $\Sigma_f \subset \mathbb{C}$, called the *bifurcation set* of f, such that the restriction

$$f : \mathbb{C}^r \setminus f^{-1}(\Sigma_f) \to \mathbb{C} \setminus \Sigma_f$$

is a locally trivial C^∞ fibration. Assume that the bifurcation set Σ_f is chosen as small as possible. The set Σ_f contains all the critical values of f. It may contains points other than critical values. Take $c \in \Sigma_f$ which is not a critical value of f. Then, for any neighbourhood V of c and for any large open ball $B \subset \mathbb{C}^r$ centered at the origin, the restriction

$$f : f^{-1}(V) \cap (\mathbb{C}^r \setminus B) \to V$$

cannot be locally trivial fibration. In this sense, a point $c \in \Sigma_f$ is called *a critical value coming from (a critical point at) infinity.*

Let $a \in \mathbb{C}^r$ and let $\mathcal{O}_a$ be the ring of convergent power series at a. The Milnor number of f at a is defined by

$$\mu_a(f) := \dim_{\mathbb{C}} \mathcal{O}_a / < \partial_1 f, \dots, \partial_r f >, \quad \partial_i f = \partial f / \partial u_i,$$

where $< \partial_1 f, \dots, \partial_r f >$ is the ideal of $\mathcal{O}_a$ generated by $\partial_1 f, \dots, \partial_r f$. The number $\mu_a(f)$ is finite if and only if a is either a regular point or an isolated critical point of f. In particular, $\mu_a(f) = 0$ if and only if a is a regular point and $\mu_a(f) = 1$ if a a nondegenerated critical point of f.

The total Milnor number of f is defined by

$$\mu(f) := \dim_{\mathbb{C}} \mathbb{C}[u_1, \dots, u_r] / < \partial_1 f, \dots, \partial_r f >$$

where $< \partial_1 f, \dots, \partial_r f >$ denotes the ideal of $\mathbb{C}[u]$ generated by the entries; it is known to be equal to $\sum_{a \in \mathbb{C}^r} \mu_a(f)$. Let $\operatorname{grad} f = (\partial_1 f, \dots, \partial_r f)$ be the gradient vector of f. The norm of

$\operatorname{grad} f$ is defined to be $\|\operatorname{grad} f\|^2 := \sum_i |\partial_i f|^2 = \langle \operatorname{grad} f, \overline{\operatorname{grad} f}\rangle$. When $\mu(f)$ is finite, it is interpreted geometrically as follows ([M]). Take a sufficiently large closed ball B in u-space centered at the origin so that all the critical points of f are contained in the interior of B. Then $\mu(f)$ is equal to the mapping degree of the map

$$\frac{\operatorname{grad} f}{\|\operatorname{grad} f\|} : \partial B \to S^{2r-1},$$

where S^{2r-1} is the unit sphere in $\mathbb{C}^r$.

A sufficient condition for the nonexistence of critical values coming from infinity is known.

Definition 3.1 ([B]). A polynomial $f \in \mathbb{C}[u]$ is said to be *tame* if there is a constant $\delta > 0$ such that the set $\{u \in \mathbb{C}^r \mid \|\operatorname{grad} f\| \leq \delta\}$ is compact.

Remark 3.2. (i) A polynomial f is tame if and only if there exists a constant $\delta > 0$ and a sufficiently large $R > 0$ such that

$$\|\operatorname{grad} f\| \geq \delta \quad \text{for any } \|u\| \geq R.$$

(ii) If f is tame, the set of critical point of f is finite, therefore any critical point is isolated. In fact the set of critical points of f is an algebraic set defined by $\partial f/\partial u_1 = \cdots = \partial f/\partial u_r = 0$ which is compact by the tameness of f. This implies that the set of critical points is finite.

Proposition 3.3 ([B]). *If a polynomial map $f : \mathbb{C}^r \to \mathbb{C}$ is tame, the bifurcation set Σ_f consists only of the critical values of f.*

For a tame polynomial map f, the reduced homology group $\tilde{H}_*(f^{-1}(v), \mathbb{Z})$, $v \in \mathbb{C} \setminus \Sigma_f$, is determined from local informations at critical points of f. Let a be a critical point of f and $f(a) = c$ is its critical value. Take a small ball B in $\mathbb{C}^r$ centered at a so that $f^{-1}(c)$ and ∂B intersect each other transversally. If we take a small disk D centered at c so that any fiber $f^{-1}(v), v \in D$, intersect with ∂B transversally, then $f : f^{-1}(D^\circ) \cap B \to D^\circ, D^\circ = D \setminus \{a\}$, is a locally trivial fibration so called *Milnor fibration.* We know ([M]) that Milnor fiber is homotopic to a bouquet of $(r-1)$-spheres:

$$B \cap f^{-1}(v) \simeq \underbrace{S^{r-1} \vee \cdots \vee S^{r-1}}_{\mu_a(f)}.$$

It follows that the reduced homology group is

$$\tilde{H}_k(B \cap f^{-1}(v), \mathbb{Z}) \simeq \begin{cases} 0 & k \neq r-1, \\ \mathbb{Z}^{\mu_a(f)} & k = r-1. \end{cases}$$

The inclusion $B \cap f^{-1}(v) \to f^{-1}(v)$ induces an injective homomorphism $\tilde{H}_*(B \cap f^{-1}(v), \mathbb{Z}) \to \tilde{H}_*(f^{-1}(v), \mathbb{Z})$ and the image is generated by vanishing cycles attached to the critical point a. Gathering the local contribution from each critical point, we have the following.

Proposition 3.4 ([B], [P2]). *If f is tame, for any $v \in \mathbb{C} \setminus \Sigma_f$ we have*

$$\tilde{H}_k(f^{-1}(v), \mathbb{Z}) \simeq \begin{cases} 0 & k \neq r-1, \\ \mathbb{Z}^{\mu(f)} & k = r-1. \end{cases}$$

3.2. Tame polynomials with parameters.

For the application to Airy integral, we need similar results of §3.1 for polynomials depending on several parameters. Let Ω be a C^∞ manifold and let f be a polynomial of $u = (u_1, \dots, u_r)$ whose coefficients are C^∞ functions on Ω. For $w \in \Omega$, we put $f_w = f(\cdot, w)$.

Definition 3.5. A polynomials $f(u, w)$ is said to be *locally uniformly tame* if for any $w \in \Omega$ there exist a neighbourhood W of w, a positive constant δ and sufficiently large $R > 0$ such that

$$\|(\operatorname{grad}_u f_{w'})(u)\| \geq \delta \quad \text{for any } \|u\| \geq R \text{ and } w' \in W. \tag{3.1}$$

where $\operatorname{grad}_u f_{w'}$ is the gradient of $f_{w'}$ with respect to u. We sometimes write $\operatorname{grad} f$ in place of $\operatorname{grad}_u f$ if there is no fear of confusion.

Example 3.6. (i) Any polynomial of one variable

$$f = a_0 u^n + a_1 u^{n-1} + \cdots + a_n$$

with parameters $(a_0, a_1, \dots, a_n) \in \mathbb{C}^\times \times \mathbb{C}^n$ is locally uniformly tame.

(ii) $f = u_1^2 + u_2^2 + a_1 u_1 + a_2 u_2 + a_3$ is locally uniformly tame with $\Omega = \{(a_1, a_2, a_3) \in \mathbb{C}^3\}$, see also Proposition 3.9.

Proposition 3.7. *Suppose that $f(u, w)$ is a locally uniformly tame polynomial in u with coefficients depending smoothly on the paramenter $w \in \Omega$. Then the total Milnor number of f_w is constant with respect to w in any connected component of Ω.*

Proof. Let w^0 be any point of Ω. Since f is locally uniformly tame, we can take a connected neighbourhood W of w^0, a constants $\delta > 0$ and a closed ball B of a large radius such that the condition (3.1) holds. For any $w \in W$ we define the map

$$\frac{\operatorname{grad}_u f_w}{\|\operatorname{grad}_u f_w\|} : \partial B \to S^{2r-1}.$$

For any $w \in W$, we take a continuous curve $w(t)$ in Ω which join w^0 to w. By virtue of the condition (3.1), the map $\operatorname{grad} f_{w(t)}/\|\operatorname{grad} f_{w(t)}\|$ gives a homotopy between $\operatorname{grad} f_{w^0}/\|\operatorname{grad} f_{w^0}\|$ and $\operatorname{grad} f_w/\|\operatorname{grad} f_w\|$. Thus $\mu(f_w) = \mu(f_{w^0})$. □

With $f(u, w)$, we associate a map $F : \mathbb{C}^r \times \Omega \to \mathbb{C} \times \Omega$ defined by

$$F(u, w) = (f(u, w), w).$$

Let Σ be the bifurcation set of F, namely, the image of the set of critical points of F.

Proposition 3.8. *Suppose that $f(u, w)$ is a locally uniformly tame polynomial on $\mathbb{C}^r \times \Omega$. Then the restriction*

$$F : (\mathbb{C}^r \times \Omega) \setminus F^{-1}(\Sigma) \to (\mathbb{C} \times \Omega) \setminus \Sigma$$

is a locally trivial C^∞ fibration.

Proof. Take any point $b^0 = (v^0, w^0) \in (\mathbb{C} \times \Omega) \setminus \Sigma$. Let $V \subset \mathbb{C}$ be an open disk centered at v^0 and $W \subset \Omega$ be an open ball in a coordinate neighbourhood of w^0 in Ω with the system of coordinates $(w_1, \ldots, w_N)$ such that $U := V \times W$ is relatively compact in $(\mathbb{C} \times \Omega) \setminus \Sigma$. We may assume that the condition (3.1) holds for W. Define a vector field X_1 on $F^{-1}(U)$ by

$$X_1 = \|\operatorname{grad} f_w\|^{-2} \sum_i \overline{\frac{\partial f_w}{\partial u_i}} \frac{\partial}{\partial u_i}.$$

Notice that $\|X_1\| = 1/\|\operatorname{grad} f_w\|$ is bounded in $F^{-1}(U)$ by virtue of the choice of U and of the condition (3.1) of the locally uniform tameness of f. Take any point $b^1 = (v^1, w^1) \in U$ and define a vector field X_2 on $F^{-1}(U)$ by

$$X_2 = (v^1 - v^0)X_1 + \sum_j (w_j^1 - w_j^0) \frac{\partial}{\partial w_j}.$$

Note that X_2 is also bounded on $F^{-1}(U)$. Let $s \mapsto \phi(s, u, b^1)$ denote the integral curve of X_2 which pass $(u, w^0) \in F^{-1}(b^0)$ at $s = 0$. Fix b^1 for the moment. The boundedness of X_1 assure that ϕ is defined on some interval $s \in [0, \epsilon)$ for any $(u, w^0) \in F^{-1}(b^0)$. Moreover, since

$$\begin{aligned} \frac{df(\phi(s, u, b^1))}{ds} &= \left\langle (v^1 - v^0) \frac{\overline{\operatorname{grad} f}}{\|\operatorname{grad} f\|^2}, \operatorname{grad} f \right\rangle \\ &= v^1 - v^0, \end{aligned}$$

we see that $F(\phi(s, u, b^1)) = (v^0 + s(v^1 - v^0), s(w^1 - w^0))$. This implies that the integral curve $s \mapsto \phi(s, u, b^1)$ can be prolonged at least to be defined on $[0, 1]$ for any point $(u, w^0) \in F^{-1}(b^0)$. Now by the theorem on ordinary differential equations stating the dependence of solutions on initial data and parameters, we see that the map $F^{-1}(b^0) \times U \to F^{-1}(U)$ defined by the correspondence $(u, w^0, b^1) \mapsto \phi(1, u, b^1)$ is a diffeomorphism and gives a local trivialization of the fibration. $\square$

3.3 Example.

We consider here an example of locally uniformly tame polynomials of several variables. The example presented here may be well known, but we decided to include its proof for the completeness of presentation of this paper. Let $\rho = (\rho_1, \ldots, \rho_r)$ be a sequence of positive integers called a *system of weights*. A polynomial $g(u)$ is said to be quasihomogeneous with respect to the system of weights ρ when $g(u)$ is written as

$$g(u) = \sum_{i_1\rho_1 + \cdots + i_r\rho_r = N} g_{i_1 \ldots i_r} u_1^{i_1} \cdots u_r^{i_r}$$

for some positive integer N and with some nonzero coefficient. In this case we write $\deg_\rho g = N$ and say that g is of ρ-degree N. In general, if g is a polynomial with a quasihomogeneous part of highest ρ-degree $= N$, we also denote $\deg_\rho g = N$.

Proposition 3.9. *Let $f(u, w) = f_1(u) + f_2(u, w)$, where $f_1(u)$ is a quasihomogeneous polynomial with the isolated critical point 0 and $f_2(u, w)$ is a polynomial in u with coefficients depending smoothly on $w \in \Omega$ such that $\deg_\rho f_2 < \deg_\rho f_1$. Then $f(u, w)$ is locally uniformly tame.*

Proof. First we show that $f_1(u)$ is tame. It is sufficient to prove that there exist a positive number $\delta > 0$ and a sufficiently large number $R > 0$, such that $\|\operatorname{grad} f_1(u)\| \geq \delta$ holds for all $\|u\| \geq R$. Let $S^{2r-1} = \{u \in \mathbb{C}^r \mid \|u\| = 1\}$ be the unit sphere in u-space centered at the origin. For any $a = (a_1, \ldots, a_r) \in S^{2r-1}$ we consider in $\mathbb{C}^r$ a curve

$$\gamma_a(t) := (t^{\rho_1} a_1, \ldots, t^{\rho_r} a_r), \quad t \geq 0.$$

Since $f_1(u)$ has only one isolated critical point $u = 0$, the gradient vector $\operatorname{grad} f_1$ does not vanish on S^{2r-1}. It follows that for some positive constant δ_1 we have

$$\max_j |\partial_j f_1| \geq \delta_1 \quad \text{on } S^{2r-1}.$$

On the other hand, since $\partial_j f_1$ is quasihomogenous of ρ-degree $N - \rho_j$, we have

$$(\partial_j f_1)(\gamma_a(t)) = t^{N-\rho_j} (\partial_j f_1)(a).$$

Therefore, if i is chosen for $a \in S^{2r-1}$ so that $\max_j |(\partial_j f_1)(a)| = |(\partial_i f_1)(a)|$, we have

$$\begin{aligned}\|(\operatorname{grad} f_1)(\gamma_a(t))\| &= \|(t^{N-\rho_1}(\partial_1 f_1)(a), \dots, t^{N-\rho_r}(\partial_r f_1)(a))\| \\ &\geq t^{N-\rho_i}|(\partial_i f_1)(a)| \\ &\geq t^{N-\rho_i}\delta_1.\end{aligned}$$

This implies that $f_1(u)$ is tame. Next we show that $f(u, w)$ is locally uniformly tame. Let W be any compact subset of Ω and restrict $f(u, w)$ on $\mathbb{C}^r \times W$. Take $a \in S^{2r-1}$. Since $\partial_j f_2(u, w), j = 1, \dots, r$, is a linear combination of monomials of ρ-degree less than $N - \rho_j - 1$ and W is compact, there is a constant $C_j > 0$ such that we have

$$|\partial_j f_2(\gamma_a(t), w)| \leq C_j t^{N-\rho_j-1} \quad \text{for } \forall a \in S^{2r-1}, \forall w \in W,\ t \geq R > 0.$$

For $a \in S^{2r-1}$, let i be as above. Then we have

$$\begin{aligned}\|(\operatorname{grad} f)(\gamma_a(t), w)\| &\geq |(\partial_i f_1)(\gamma_a(t)) + (\partial_i f_2)(\gamma_a(t), w)| \\ &\geq |(\partial_i f_1)(\gamma_a(t))| - |(\partial_i f_2)(\gamma_a(t), w)| \\ &\geq t^{N-\rho_i}\left(\delta_1 - \frac{C_i}{t}\right) \\ &\geq \frac{1}{2}\delta_1 t^{N-\rho_i}\end{aligned}$$

for $t \geq R$, with $R > 0$ sufficiently large. This proves that $f(u, w)$ is locally uniformly tame. □

4 Proof of Theorem

4.1 Homology group with a family of supports.

Consider an integral

$$\int_\Delta e^{P(u_1,\dots,u_r)} a(u_1, \dots, u_r) du_1 \wedge \cdots \wedge du_r, \tag{4.1}$$

where $P(u)$ and $a(u)$ are polynomials in $(u_1, \dots, u_r)$ and are called *phase* and *amplitude*, respectively. In [P2], F. Pham constructed the homology group whose cycle Δ gives a meaning to the integral (1.1). In this subsection we review his results and in the following subsections we apply them to prove our theorem.

Hereafter we assume that P is tame and Σ_P is the set of its critical values. For any $c < 0$ let S_c^+ and S_c^- be as in §2.2 and Ψ be the family of supports in $\mathbb{C}^r$ consisting of closed sets C satisfying

$$\forall c < 0, C \cap P^{-1}(S_c^+) \quad \text{is compact.} \tag{4.2}$$

Denote by H_*^Ψ the locally finite homology group with coefficients in $\mathbb{Z}$ and with a family of supports Ψ.

By virtue of Proposition 3.3, the inclusion

$$(\mathbb{C}^r, P^{-1}(S_c^-)) \to (\mathbb{C}^r, P^{-1}(S_{c'}^-)) \tag{4.3}$$

is a homotopy equivalence for $c < c' < 0$ such that $S_{c'}^-$ contains no point of Σ_P. The family $\{(\mathbb{C}^r, P^{-1}(S_c^-))\}_{c<0}$ forms a projective system. Taking into consideration of the definition of locally finite chains and the definition of the family Ψ, the group of locally finite chains C_*^Ψ with a family of supports Ψ is identified with a projective limit of the system of groups of relative chains

$$C_*^\Psi(\mathbb{C}^r) = \varprojlim C_*(\mathbb{C}^r, P^{-1}(S_c^-)) \quad (c \to -\infty).$$

Using the fact that the inclusion (4.3) is a homotopy equivalence for sufficiently small $c < c' < 0$, the homology group for $C_*^\Psi(\mathbb{C}^r)$ can be identified with a relative homology group:

$$H_*^\Psi(\mathbb{C}^r) \simeq H_*(\mathbb{C}^r, P^{-1}(S_c^-)) \qquad (c \ll 0). \tag{4.4}$$

Once the identification (4.4) is established we follow the standard argument of singularity theory using deformation retraction and excision ([B], [P2]). Since $\mathbb{C}^r$ is contractible, the homology exact sequence for the pair $(\mathbb{C}^r, P^{-1}(S_c^-))$ leads to

$$H_k(\mathbb{C}^r, P^{-1}(S_c^-)) \simeq \tilde{H}_{k-1}(P^{-1}(S_c^-), \mathbb{Z}) \simeq \tilde{H}_{k-1}(P^{-1}(v), \mathbb{Z}), \tag{4.5}$$

where $\tilde{H}_{k-1}$ denotes the reduced homology group and v is an arbitrary point in S_c^-. The second isomorphism is obtained by deformation retraction by using the triviality of the fibration $P : P^{-1}(S_c^-) \to S_c^-$. Now by virtue of Proposition 3.4, we have the following proposition.

Proposition 4.1 ([P2]). *For a tame polynomial, we have for any $v \in \mathbb{C} \setminus \Sigma_P$*

$$H_k^\Psi(\mathbb{C}^r) \simeq \tilde{H}_{k-1}(P^{-1}(v), \mathbb{Z}) \simeq \begin{cases} 0 & k \neq r, \\ \mathbb{Z}^{\mu(P)} & k = r. \end{cases}$$

More concretely, cycles for $H_r^\Psi(\mathbb{C}^r)$ are described as follows. Note that the isomorphism $H_r(\mathbb{C}^r, P^{-1}(S_c^-)) \simeq H_{r-1}(P^{-1}(S_c^-), \mathbb{Z})$ is established by taking a boundary of a relative cycle. Take $c \in \Sigma_P$ and let L_c be the half line in $\mathbb{C}$ which starts from c and goes to infinity in the direction $arg(v - c) = \theta$, $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. We choose θ so that L_c contains no other point of Σ_P than c. On each point $v \in L_c \setminus c$, we take vanishing cycles $e_{c1}(v), \ldots, e_{cm_c}(v)$ which vanish at critical points over c, depend continuously on v and are linearly independent in $H_{r-1}(P^{-1}(v), \mathbb{Z})$, where $m_c = \sum_{a \in P^{-1}(c)} \mu_a(P)$. Then $\Delta_{ci} := \cup_{v \in L_c} e_{ci}(v)$ gives an element of $H_r^{\Psi}(\mathbb{C}^r)$. We have a generator $\{\Delta_{ci} \mid c \in \Sigma_P, i = 1, \ldots, m_c\}$ of $H_r^{\Psi}(\mathbb{C}^r)$.

4.2 Application to the Airy integral.

Let f be the rational function on $\mathbb{P}^r \times \Omega$ defined in §2.1, namely, $f(t,z,\alpha) = \sum_{i=1}^n \alpha_i \theta_i(tz)$ with a pole divisor $H = \{(t,z,\alpha) \in \mathbb{P}^r \times \Omega \mid tz_0 = 0\}$. We have set $E := (\mathbb{P}^r \times \Omega) \setminus H$. We extend it to a holomorphic map $F : E \to B := \mathbb{C} \times \Omega$ by

$$F(t,z,\alpha) = (f(t,z,\alpha), z, \alpha).$$

Let C be the set of critical points of F and $\Sigma := F(C) \subset B$ be the set of its critical values. For $w \in \Omega$ we denote the restriction of F to $E_w := \mathbb{P}^r \setminus H_w \simeq \mathbb{C}^r$ by F_w.

We want to show that the restriction $F : E \setminus F^{-1}(\Sigma) \to B \setminus \Sigma$ gives a locally trivial C^∞-fibration by the help of Propositions 3.8 and 3.9. In order to make emerge the principal part of the holomorphic map F_w for each $w \in \Omega$, let us consider the action of $G = GL(r+1)$ on $\mathbb{P}^r \times \Omega$ defined by

$$g \cdot (t,z,\alpha) = (tg^{-1}, gz, \alpha). \tag{4.6}$$

Since the defining equation $tz_0 = 0$ of the divisor H is invariant by the action of G, the action of G on $\mathbb{P}^r \times \Omega$ induces that on E. We also define the actions of G on Ω and on B by

$$g \cdot (z,\alpha) = (gz, \alpha) \quad \text{for } (z,\alpha) \in \Omega \tag{4.7}$$

$$g \cdot (v,z,\alpha) = (v, gz, \alpha) \quad \text{for } (v,z,\alpha) \in B \tag{4.8}$$

Remark that the following two fibration structures are preserved by the action of G. Firstly, we see that the action of G is compatible with the fibration structure (E, π, Ω), namely,

$$\pi(g \cdot e) = g \cdot \pi(e), \quad g \in G,\ e \in E.$$

Secondly, the action of G preserves the fibration structure of the map $F : E \to B$, namely, we have the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{g\cdot} & E \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ B & \xrightarrow{g\cdot} & B \end{array}$$

This fact follows from the invariance of f by the action of G on E :

$$(g^* f)(t,z,\alpha) := f(tg^{-1}, gz, \alpha) = f(t,z,\alpha).$$

The following lemma is an immediate consequence of the G-invariance of f.

Lemma 4.2. *Both the set of critical points C and the set of critical values Σ are invariant by the action G.*

Take any point $w = (z, \alpha) \in \Omega$. By the definition of Z, we have $\det z' \neq 0$, where $z = (z', z'')$. So we put $g = (z')^{-1}$ and consider the action of g on Ω. Then $g \cdot w = ((1_{r+1}, (z')^{-1} z''), \alpha)$ and $H_{g \cdot w} = \{t \in \mathbb{P}^r \mid t_0 = 0\}$. Taking this fact into account, put

$$\Omega' = \{(z, \alpha) \in \Omega \mid z' = 1_{r+1}\}$$

and consider the inclusion map $i : \Omega' \to \Omega$. Let (i^*E, π, Ω') be the restriction of (E, π, Ω) to the fibers over Ω'. Then we have the commutative diagram

$$\begin{array}{ccc} G \times i^*E & \longrightarrow & E \\ {\scriptstyle 1\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times \Omega' & \longrightarrow & \Omega, \end{array}$$

where the maps denoted by the horizontal arrows are those induced from the actions of G on E and Ω. These maps are biholomorphic. Note that the fibration (i^*E, π, Ω') is trivial and the total space i^*E is identified with $\mathbb{C}^r \times \Omega'$ by taking the affine coordinates of the fiber E_w as

$$u = (1, u_1, \ldots, u_r) = (t_0/t_0, t_1/t_0, \ldots, t_r/t_0).$$

We first restrict our consideration to the fibration $F : i^*E \to B' := \mathbb{C} \times \Omega'$. Put $\Sigma' := \Sigma \cap B'$.

Proposition 4.3. *The restriction $F : i^*E \backslash F^{-1}(\Sigma') \to B' \backslash \Sigma'$ is a locally trivial C^∞ fibration with*

$$\tilde{H}_{k-1}(F^{-1}(b), \mathbb{Z}) \simeq \begin{cases} 0 & k \neq r, \\ \mathbb{Z}^\mu & k = r, \end{cases}$$

where $\mu = \binom{n-1}{r}$.

To prove Proposition 4.3, we put

$$P(u, w) = \sum_{i=1}^{n} \alpha_i \theta_i(uz), \quad w = (z, \alpha) \in \Omega',$$

where $u = (1, u_1, \ldots, u_r)$, and we apply Propositions 3.7 and 3.8 to P. We see that P is a polynomial in $u_1, \ldots, u_r$ whose coefficients depend holomorphically on $w \in \Omega'$. Let $\rho = (1, 2, \ldots, r)$ be a system of weights of $u_1, \ldots, u_r$. It induces a structure of graded ring on the polynomial ring $\mathbb{C}[u]$. We need the following two lemmas.

Lemma 4.4. *For $w = (z, \alpha) \in \Omega'$, the polynomial $P(u, w)$ in u has ρ-degree n and the quasihomogeneous part of ρ-degree n is given by*

$$P_n(u) = \alpha_n \theta_n(1, u_1, \ldots, u_r, 0, \ldots, 0). \tag{4.9}$$

Proof. Put $l(u) = (1, l_1(u), \ldots, l_n(u)) = uz$. Since z has the form $(1_{r+1}, z'')$, we have $l_1(u) = u_1, \ldots, l_r(u) = u_r$. By the definition (2.2) we have

$$\theta_m(1, x_1, \ldots, x_r) = \sum_{1 \le k \le m} \frac{(-1)^{k+1}}{k} \sum_{i_1 + \cdots + i_k = m, i_j \ge 1} x_{i_1} \cdots x_{i_k}.$$

First we show that $\deg_\rho \theta_m(l(u)) \le m$. In fact, for an index $(i_1, \ldots, i_k)$ such that $i_1 + \cdots + i_k = m$, we have

$$\deg_\rho l_{i_1}(u) \cdots l_{i_k}(u) \le i_1 + \cdots + i_k = m$$

because $l_i(u) = u_i$ for $1 \le i \le r$ and $\deg_\rho l_i(u) < r$ for $i \ge r+1$. This proves the first half of the assertion. To show the latter half, note that $\theta_n(1, u_1, \ldots, u_r, 0, \ldots, 0)$ is a quasihomogeneous polynomial in $u_1, \ldots, u_r$ which is a sum of terms $l_{i_1}(u) \cdots l_{i_k}(u)$ satisfying $i_1 + \cdots + i_k = n$ and $1 \le i_1, \ldots, i_k \le r$. Since $\deg_\rho \theta_m(l(u)) < n$ for $m < n$, it is sufficient to show

$$\deg_\rho[\theta_n(l(u)) - \theta_n(1, u_1, \ldots, u_r, 0, \ldots, 0)] < n. \tag{4.10}$$

Since $\theta_n(l(u)) - \theta_n(1, u_1, \ldots, u_r, 0, \ldots, 0)$ is a linear combination of $l_{i_1}(u) \cdots l_{i_k}(u)$ with an index $(i_1, \ldots, i_k)$ satisfying $i_1 + \cdots + i_k = n$, with some $i_p \ge r+1$, we get

$$\begin{aligned} \deg_\rho[\theta_n(l(u)) - \theta_n(1, u_1, \ldots, u_r, 0, \ldots, 0)] &\le \max_{i_1 + \cdots + i_k = n} \deg_\rho l_{i_1}(u) \cdots l_{i_k}(u) \\ &< i_1 + \cdots + i_k = n. \end{aligned}$$

This proves (4.10) and the second assertion of the lemma. □

Lemma 4.5. *The family of polynomials $P(u, w)$ ($w \in \Omega'$) is a locally uniformly tame and, for any w, the total Milnor number of $P(u, w)$ is $\mu = \binom{n-1}{r}$.*

Proof. Taking account of Proposition 3.9, it is sufficent to show that the quasihomogeneous polynomial $P_n(u)$ defined by (4.9) has only one isolated critical point $u = 0$. A proof of this fact is already given in [K1]. However, for the sake of completeness of presentation, we give its another simple proof. Consider the polynomial map $\phi : \mathbb{C}^r \to \mathbb{C}^r$ from $(x_1, \ldots, x_r)$-space to $(u_1, \ldots, u_r)$-space defined by

$$\phi^* u_i = (-1)^i \times (\text{ } i\text{-th elementary symmetric function of } x \text{ }).$$

Then using the relation

$$\log(1+u_1T+\cdots+u_rT^r)=\log\prod_{i=1}^{r}(1-x_iT)=-\sum_{k=1}^{\infty}\sum_{i=1}^{r}\frac{x_i^k}{k}T^k,$$

and noting that the left hand side equals to $\sum_{k=1}^{\infty}\theta_k(u)T^k$, we see that

$$\phi^*P_n=-\frac{\alpha_n}{n}(x_1^n+\cdots+x_r^n).$$

Differentiating both sides with respect to x_i, we get

$$-\alpha_n x_i^{n-1}=\frac{\partial\phi^*P_n}{\partial x_i}=\sum_j\frac{\partial P_n}{\partial u_j}\frac{\partial u_j}{\partial x_i}.$$

If $P_n(u)$ has a critical point other than 0, the power sum ϕ^*P_n must have a critical point other than 0, which is absurd from the form of the derivative of power sum. Thus P_n has only one isolated singularity $u=0$. Now we can apply Proposition 3.9 to conclude that $P(u,w)$ is locally uniformly tame. The total Milnor number of $P(u,w)$ is equal to that of $P_n(u)$ by Proposition 3.7. Since $P_n(u)$ is quasihomogeneous, by the well known formula ([AVG] §12.3, Corollary 3), the Milnor number μ is computed by

$$\mu=\prod_{i=1}^{r}\left(\frac{n}{i}-1\right)=\binom{n-1}{r}.$$

This proves the lemma. □

Proof of Proposition 4.3. Apply Proposition 3.8 to the polynomial $P(u,w)$. □

Combining Proposition 4.3 with the action of G on E, we can obtain the following:

Proposition 4.6. *The restriction $F: E\setminus F^{-1}(\Sigma)\to B\setminus\Sigma$ is a locally trivial C^∞ fibration. For any $b\in B\setminus\Sigma$, we have*

$$\tilde H_{k-1}(F^{-1}(b),\mathbb{Z})\simeq\begin{cases}0 & k\neq r,\\ \mathbb{Z}^\mu & k=r,\end{cases}$$

where $\mu=\binom{n-1}{r}$.

Proof. The actions of G on E and B defined by (4.6) and (4.8) induces the biholomorphic maps

$$G\times i^*E\to E,\quad G\times B'\to B$$

which are compatible with the map F:

$$\begin{tikzcd} G \times i^*E \arrow[r] \arrow[d, "1\times F|_{i^*E}"'] & E \arrow[d, "F"] \\ G \times B' \arrow[r] & B, \end{tikzcd}$$

Notice that $g \cdot \Sigma = \Sigma$ by Lemma 4.2. Take any point $b_0 \in B \setminus \Sigma$. There exist $g_0 \in G$ and $b_0' \in B' \setminus \Sigma'$ such that $g_0 \cdot b_0' = b_0$ by virtue of the bijection $G \times B' \to B$. Take an open neighbourhood $b_0' \in V \subset B' \backslash \Sigma'$ such that the restriction $F : F^{-1}(V \backslash (\Sigma' \cap V)) \to V \backslash (\Sigma' \cap V)$ is a trivial C^∞-fibration and let U be any neigbourhood of g_0. Then the maps in the commutative diagram

$$\begin{tikzcd} U \times F^{-1}(V \setminus (\Sigma' \cap V)) \arrow[r] \arrow[d, "1\times F"'] & E \arrow[d, "F"] \\ U \times (V \setminus (\Sigma' \cap V)) \arrow[r] & B \end{tikzcd}$$

give a local trivialization. The second assertion is a consequence of Proposition 4.3. □

Combining the result in §4.1 with Proposition 4.6, we can show Theorem 2.5.

Proof of Theorem 2.5. As is shown in §4.1, for each $w \in \Omega$, we have $H_k^\Phi(E_w) \simeq \tilde{H}_{k-1}(F^{-1}(b), \mathbb{Z})$ for any $b \in (\mathbb{C} \times \{w\}) \setminus \Sigma_w$. On the otherhand we know that $\tilde{H}_{k-1}(F^{-1}(b), \mathbb{Z}) = 0$ if $k \neq r$ by Proposition 4.6. This proves the first part of the theorem. We show the latter half. Take any point $w_0 \in \Omega$ and choose v_0 such that $(v_0, w_0) \in B_{w_0} \setminus \Sigma_{w_0}$. We take a small simply connected neighbourhood $w_0 \in W \subset \Omega$ so that $b(w) = (v_0, w) \in B \setminus \Sigma$ for any $w \in W$. Then combining $H_r^\Phi(E_w) \simeq \tilde{H}_{r-1}(F^{-1}(b(w)), \mathbb{Z})$ and the local triviality of the fibration $F : E \backslash F^{-1}(\Sigma) \to B \backslash \Sigma$, we see that $\bigcup_{w \in W} H_r^\Phi(E_w)$ is trivial and is identified with $V \times H_r^\Phi(E_{w_0})$. This proves the theorem. □

4.3 Action of G on the homology groups.

We can relate the homology group $H_r^\Phi(E_w), w \in \Omega \setminus \Omega'$, directly to the homology group $H_r^\Phi(E_{w'})$ for some $w \in \Omega'$ by using the action of G on the fibration (E, π, Ω). To state this, we recall some definition. Let X and Y be topological spaces equipped with families of supports Φ and Ψ respectively. A continuous map $\phi : X \to Y$ is said to be *admissible* for Φ and Ψ if $\phi(A) \in \Psi$ for any $A \in \Phi$ and if, for any compact set $K \subset Y$ and for any $A \in \Phi$, $A \cap \phi^{-1}(K)$ is compact. In this case the map ϕ induces a homomorphism of homologies:

$$\phi_* : H_k^\Phi(X) \to H_k^\Psi(Y).$$

Lemma 4.7. *Let the action of G on E be that defined by (4.6). Then each $g \in G$ defines an automorphism of E admissible for Φ.*

Proof. Let $A \in \Phi$. By the definition of Φ, $A \cap f^{-1}(S_c^+)$ is compact for any $c < 0$. We want to show that, for any $g \in G$, $g \cdot A \cap f^{-1}(S_c^+)$ is also compact. Since f is invariant by the action of G, we have $g \cdot f^{-1}(S_c^+) = f^{-1}(S_c^+)$; it follows that

$$g \cdot A \cap f^{-1}(S_c^+) = g \cdot (A \cap f^{-1}(S_c^+))$$

and that it is compact. Next we check the latter half of the conditions of admissibility. Let K be any compact subset of E. Since the action of G is holomorphic, $g^{-1} \cdot K$ is compact. Therefore, $A \cap g^{-1} \cdot K$ is also compact. $\Box$

Notice that the action of $g \in G$ on E sends E_w to $E_{g \cdot w}$ and that this map is admissible with respect to Φ_w and $\Phi_{g \cdot w}$. This remark leads to the following:

Proposition 4.8. *For any $g \in G$ and $w \in \Omega$, we have the isomorphism*

$$g_* : H_r^{\Phi}(E_b) \to H_r^{\Psi}(E_{g \cdot b}).$$

Acknowlegement. The author would like to express his thanks to Professor Frédéric Pham and to the staffs of Université de Nice for the encouragement and the hospitality. This work is done during his stay in Nice.

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