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# Polynomial Hamiltonian structure for the $A_3$ system

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**Abstract**

We deduce the polynomial Hamiltonian structure from the Hamiltonian system given by [4] by means of the holonomic deformation of linear ordinary differential equations of the $A_3$-type. The completely integrable Hamiltonian system obtained in the present article can be viewed as an extension of the second Painlevé equation to the case of several independent variables. We obtain also the family of particular solutions of the Hamiltonian system and give an integral representation of such solutions.

## Introduction.

Consider a linear ordinary differential equation of the form:

$$\frac{d^2y}{dx^2}+p_1(x)\frac{dy}{dx}+p_2(x)y=0, \tag{0.1}$$

with the coefficients

$$\begin{aligned} p_1(x) &= -P(x)-\sum_{k=1}^{g}\frac{1}{x-\lambda_k}, \\ p_2(x) &= -(2\alpha+1)x^g-2\sum_{j=1}^{g}H_jx^{g-j}+\sum_{k=1}^{g}\frac{\mu_k}{x-\lambda_k}, \end{aligned} \tag{0.2}$$

where

$$P(x)=2x^{g+1}+\sum_{j=1}^{g}jr_jx^{j-1}. \tag{0.3}$$

We make the following assumption: *none of the regular singular points $x=\lambda_k$ $(k=1,\cdots,g)$ is logarithmic singularity.* Under this assumption, the coefficients $H_j$ $(j=1,\cdots,g)$ of $p_2(x)$ can be written as rational functions of $\lambda=(\lambda_1,\cdots,\lambda_g)$, $\mu=(\mu_1,\cdots,\mu_g)$ and $r=(r_1,\cdots,r_g)$. In fact, since the exponents at $x=\lambda_k$ are 0 and 2, we obtain by the use of Frobenius method the following equations:

$$\mu_k^2+\alpha_0^{(k)}\mu_k+\beta_0^{(k)}=0 \quad (k=1,\cdots,g). \tag{0.4}$$

where

$$\alpha_0^{(k)} = -P(\lambda_k) - \sum_{l=1,(\neq k)}^{g} \frac{1}{\lambda_k - \lambda_l},$$

$$\beta_0^{(k)} = -(2\alpha+1)\lambda_k^g - 2\sum_{j=1}^{g} H_j \lambda_k^{g-j} + \sum_{l=1,(\neq k)}^{g} \frac{\mu_l}{\lambda_k - \lambda_l}.$$

By solving the linear equation (0.4) with respect to $H_j$, we can obtain the rational functions $H_j = H_j\,(r, \lambda, \mu)$. When considering the linear equation (0.1)-(0.2), we suppose that $H_j$ are the rational functions of this form.

Since $P(x)$ can be regarded as a polynomial representing the versal deformation of the singularity of the $\mathrm{A}_g$-type, we call the linear equation (0.1)-(0.2) that of the $\mathrm{A}_g$-type. The first author of the present article studied in [4] the holonomic deformation of the linear equation (0.1)-(0.2) in the case $g = 3$; he introduced the deformation parameters $s = (s_1, s_2, s_3)$ such that

$$r_1 = s_1 + \frac{3}{4}s_3^2, \quad r_2 = s_2, \quad r_3 = s_3, \tag{0.5}$$

and proved the following theorem:

**Theorem** ([4]) *The holonomic deformation of* (0.1)-(0.2) *is governed by the completely integrable Hamiltonian system of partial differential equations:*

$$\frac{\partial \lambda_k}{\partial s_j} = \frac{\partial H_j}{\partial \mu_k}, \quad \frac{\partial \mu_k}{\partial s_j} = -\frac{\partial H_j}{\partial \lambda_k} \qquad (j, k = 1, 2, 3). \tag{H}$$

We call the Hamiltonian system (H) as the *$A_3$-system.*

In the case $g = 1$, by regarding $s = r_1$ as deformation parameter, we obtain by the holonomic deformation the following Hamiltonian system:

$$\frac{d\lambda}{ds} = \mu - \lambda^2 - \frac{s}{2}, \quad \frac{d\mu}{ds} = 2\lambda\mu + \alpha + \frac{1}{2}, \tag{0.6}$$

see [5]. Here we put $\lambda = \lambda_1$ and $\mu = \mu_1$. This is called the second Painlevé system; in fact (0.6) is equivalent to the second Painlevé equation:

$$\frac{d^2\lambda}{ds^2} = 2\lambda^3 + s\lambda + \alpha.$$

When $g = 2$, the holonomic deformation was studied [3]. In fact we obtain the Hamiltonian system (H) with the Hamiltonians:

$$
\begin{aligned}
H_1 &= \frac{1}{2}\sum_{k=1}^{2} \frac{1}{\Lambda'(\lambda_k)} \left[\mu_k^2 - P(\lambda_k)\mu_k - (2\alpha+1)\lambda_k^2\right] , \\
H_2 &= -\sum_{k=1}^{2} \frac{Q(\lambda_k)}{\Lambda'(\lambda_k)} \left[\mu_k^2 - \{P(\lambda_k) + \frac{1}{Q(\lambda_k)}\}\mu_k - (2\alpha+1)\lambda_k^2\right] ,
\end{aligned} \tag{0.7}
$$

where $\Lambda(x) = (x-\lambda_1)(x-\lambda_2)$, $Q(x) = x - \lambda_1 - \lambda_2$, and $\Lambda'(x) = \frac{d}{dx}\Lambda(x)$. Here, as the deformation parameter $s = (s_1, s_2)$, we can take:

$$s_1 = r_1, \quad s_2 = r_2 .$$

The Hamiltonian system with (0.7) is called the $A_2$ *system*. When $g \leq 2$, the coefficients of the polynomial $P(x)$ can be regarded as the deformation parameter, while we have to find the change of variables such as (0.5) when considering the holonomic deformation in the case $g \geq 3$.

The second Painlevé system (0.6), i.e. the $A_1$-*system* at our terminology, possesses the polynomial Hamiltonian:

$$H(s,\lambda,\mu) = \frac{1}{2}\mu^2 - \left(\lambda^2 + \frac{s}{2}\right)\mu - \left(\alpha + \frac{1}{2}\right)\lambda.$$

On the other hand, in the cases $g = 2$ and $g = 3$, the Hamiltonians induced directly by the holonomic deformation are rational in the canonical variables. It is known ([3]) that, when $g = 2$, we can obtain from (0.7) polynomial Hamiltonians by means of certain canonical transformations. In fact, the Hamiltonian system with (0.7) is transformed into

$$\frac{\partial q_k}{\partial s_j} = \frac{\partial L_j}{\partial p_k} , \quad \frac{\partial p_k}{\partial s_j} = -\frac{\partial L_j}{\partial q_k} \qquad (j,k = 1,2) , \tag{0.8}$$

with the Hamiltonians:

$$
\begin{aligned}
L_1 \;&=\; -\frac{1}{2}q_1p_2^2 + p_1p_2 \\
&\quad - \left(q_1^2 + q_2 + \frac{1}{2}s_2\right)p_1 - \left(q_1q_2 - \frac{1}{2}s_2q_1 + \frac{1}{2}s_1\right)p_2 - \left(\alpha + \frac{1}{2}\right)q_1 , \\
L_2 \;&=\; \frac{1}{2}p_1^2 + \frac{1}{2}\left(q_1^2 + q_2 - \frac{1}{2}s_2\right)p_2^2 - q_1p_1p_2 \\
&\quad - \left(q_1q_2 - \frac{1}{2}s_2q_1 + \frac{1}{2}s_1\right)p_1 - \left(q_2^2 - \frac{1}{2}s_1q_1 - \frac{1}{4}s_2^2\right)p_2 - \left(\alpha + \frac{1}{2}\right)q_2 .
\end{aligned} \tag{0.9}
$$

We rewrite here the results obtained in [3], [6], where the notation is slightly different from ours. The data of the system (0.8)-(0.9):

$$(q, p, L, s)$$

is called the *polynomial Hamiltonian structure* of the $A_2$-system. Here we write $q = (q_1, q_2)$, $p = (p_1, p_2)$, $L = (L_1, L_2)$ and $s = (s_1, s_2)$. It is shown in [6] that, when $\alpha = -\frac{1}{2}$, the system (0.8)-(0.9) has particular solutions of the form:

$$(p_1, p_2) = (0, 0), \quad q_1 = \frac{\partial}{\partial s_1} \log u, \quad q_2 = \frac{\partial}{\partial s_2} \log u \ , \tag{0.10}$$

where $u = (u_1, u_2)$ is given by:

$$u = e^{-\frac{1}{4}s_2^2} \int_\gamma \exp\left( - \int^\xi P(\xi) \right) d\xi \ , \tag{0.11}$$

$\gamma$ being a certain path in the complex plane. Here $P(\xi) = 2\xi^3 + 2s_2\xi + s_1$, namely, the polynomial (0.3) with $g = 2$. Moreover, when $\alpha = -\frac{1}{2}$, the $A_1$-system (0.6) possesses particular solutions of the form:

$$\mu = 0, \qquad \lambda = \frac{d}{ds} \log u,$$
$$u = \int_\gamma \exp\left( -\frac{2}{3}\xi^3 - s\xi \right) d\xi \ ,$$

see [5], [6].

The purpose of the present article is double; we will determine the polynomial Hamiltonian structure for the $A_3$-system and then obtain particular solutions, having integral representations of the form:

$$\int_\gamma \exp\left[ -\frac{2}{5}\xi^5 - s_3\xi^3 - s_2\xi^2 - \left( s_1 + \frac{3}{4}s_3^2 \right) \xi - \frac{1}{2}s_2 s_3 \right] d\xi.$$

By taking into consideration (0.5), we obtain again the primitive function of the polynomial $-P(\xi)$.

In section 1, we will establish the polynomial Hamiltonian structure for the $A_3$-system. We associate with the system (H) the 2-form:

$$\Omega = \sum_{k=1}^{3} d\mu_k \wedge d\lambda_k - \sum_{j=1}^{3} dH_j \wedge ds_j \ ,$$

which can be written in the form:

$$\Omega = \sum_{i<j} \Omega_{ij}\, ds_i \wedge ds_j \ ,$$

$$\Omega_{ij} = \sum_{k=1}^{3} \left[ \frac{\partial H_j}{\partial \lambda_k} \frac{\partial H_i}{\partial \mu_k} - \frac{\partial H_i}{\partial \lambda_k} \frac{\partial H_j}{\partial \mu_k} \right] + \left( \frac{\partial}{\partial s_i} \right) H_j - \left( \frac{\partial}{\partial s_j} \right) H_i \ .$$

Here $\left(\frac{\partial}{\partial s_i}\right)$ denotes the differential with respect to $s_i$ such that $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and $\mu = (\mu_1, \mu_2, \mu_3)$ are viewed to be independent of $s_i$. It is known that the system (H) is completely integrable if and only if $\Omega_{ij}$ does not dependent on $\lambda$, $\mu$, see [2, Proposition 6.4, Chapter 1]. And the transformation of the Hamiltonians structure:

$$\rho : \quad (\lambda, \mu, H, s) \rightarrow (\overline{\lambda}, \overline{\mu}, \overline{H}, \overline{s})$$

is canonical, if and only if

$$\sum_{k=1}^{3} d\mu_k \wedge d\lambda_k - \sum_{j=1}^{3} dH_j \wedge ds_j = \sum_{k=1}^{3} d\overline{\mu}_k \wedge d\overline{\lambda}_k - \sum_{j=1}^{3} d\overline{H}_j \wedge d\overline{s}_j \ ,$$

where $\overline{\lambda} = (\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3)$, $\overline{\mu} = (\overline{\mu}_1, \overline{\mu}_2, \overline{\mu}_3)$ and $\overline{s} = (\overline{s}_1, \overline{s}_2, \overline{s}_3)$. Consider in particular the transformation of the Hamiltonians:

$$\overline{H}_i = H_i + f_i(s) \qquad (i = 1, 2, 3), \tag{0.12}$$

where $f_i(s)$ are arbitrary functions of $s$, and we put $\overline{\lambda} = \lambda$, $\overline{\mu} = \mu$, $\overline{s} = s$. Then the necessary and sufficient condition that (0.12) is canonical is given by:

$$\frac{\partial f_i}{\partial s_j} = \frac{\partial f_j}{\partial s_i} \qquad (i, j = 1, 2, 3) \ .$$

On the other hand, the Hamiltonian system (H) remains invariant under an arbitrary transformation of the form (0.12). We consider in what follows the transformation (0.12) which is not necessarily canonical.

Section 2 is devoted to the studies on particular solutions of the $A_3$-system. We will show that such solutions are defined by the completely integrable linear system.

## 1. Polynomial Hamiltonian structure.

Let $e_j$ $(j = 1, 2, 3)$ be the $j$-th elementary symmetric polynomial of $\lambda_1, \lambda_2, \lambda_3$, and $e_j^{(k)}$ be the $j$-th elementary symmetric polynomial of the two variables, $\lambda_i$ $(i = 1, 2, 3\ (\neq k))$. We set:

$$\Lambda(x) = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)\ ,$$

$$P(x) = 2x^4 + 3s_3x^2 + 2s_2x + s_1 + \frac{3}{4}s_3^2\ . \tag{1.1}$$

Then the Hamiltonian of the A3-system (H) is the following:

$$H_j = \frac{1}{2}\sum_{k=1}^{3}\left[N_kN^{jk}\mu_k^2 - U_{jk}\mu_k - N_kN^{jk}(2\alpha+1)\lambda_k^3\right] \qquad (j=1,2,3)\ , \tag{1.2}$$

where

$$N^{jk} = (-1)^{j-1}e_{j-1}^{(k)}, \qquad N_k = [\Lambda'(x)]^{-1}\ ,$$

and

$$U_{jk} = N_kN^{jk}P(\lambda_k) - \sum_{l=1,\,(\neq k)}^{3}\frac{N_kN_{jk}+N_lN^{jl}}{\lambda_l-\lambda_k}\ ,$$

$\Lambda'(x)$ being the derivative of $\Lambda(x)$, see [4].

Firstly we consider the variables $\sigma = (\sigma_1,\sigma_2,\sigma_3)$ defined by

$$\sigma_j = (-1)^{j-1}e_j \qquad (j=1,2,3). \tag{1.3}$$

It is easy to see:

**Proposition 1.1** *The transformation*

$$(\lambda,\mu,H,s)\to(\sigma,\rho,H,s)$$

*is canonical if and only if*

$$\begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3\end{pmatrix} = \begin{pmatrix}1 & -(\lambda_2+\lambda_3) & \lambda_2\lambda_3\\ 1 & -(\lambda_1+\lambda_3) & \lambda_1\lambda_3\\ 1 & -(\lambda_1+\lambda_2) & \lambda_1\lambda_2\end{pmatrix}\begin{pmatrix}\rho_1\\ \rho_2\\ \rho_3\end{pmatrix}. \tag{1.4}$$

*where* $\rho = (\rho_1,\rho_2,\rho_3)$ .

Here, for the sake of simplicity of presentation, we use the same notation; in the quartette $(\sigma,\rho,H,s)$, we regard $H_j$ as functions of $\sigma$, $\rho$ and $s$. By means of (1.3) and (1.4), the explicit form of $H$ is written as follows:

$$\begin{pmatrix}H_1\\ H_2\\ H_3\end{pmatrix} = \frac{1}{2}\begin{pmatrix}Q_1\\ Q_2\\ Q_3\end{pmatrix} - \frac{1}{2}A\begin{pmatrix}\rho_1\\ \rho_2\\ \rho_3\end{pmatrix} - \left(\alpha+\frac{1}{2}\right)\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix}\ , \tag{1.5}$$

where

$$Q_1 = \rho_2^2 - \sigma_2\rho_3^2 - 2\sigma_1\rho_2\rho_3 + 2\rho_1\rho_3 \ ,$$

$$Q_2 = -2\sigma_1\rho_2^2 + (\sigma_3 + \sigma_1\sigma_2)\rho_3^2 + 2\rho_1\rho_2 + 2\sigma_1^2\rho_2\rho_3 - 2\sigma_1\rho_1\rho_3 \ ,$$

$$Q_3 = \rho_1^2 + \sigma_1^2\rho_2^2 + (\sigma_2^2 - \sigma_1\sigma_3)\rho_3^2 - 2\sigma_1\rho_1\rho_2 + 2(\sigma_1\sigma_2 + \sigma_3)\rho_2\rho_3 - 2\sigma_2\rho_1\rho_3 \ ,$$

and $A = ((\alpha_{kl}))_{k,l=1,2,3}$ is the matrix such that

$$\begin{aligned}
\alpha_{11} &= 2{\sigma_1}^2 + 2\sigma_2 + 3s_3 \ , \\
\alpha_{22} &= 2{\sigma_2}^2 + 3s_3\sigma_2 - 2s_2\sigma_1 + s_1 + \frac{3}{4}s_3^2 \ , \\
\alpha_{33} &= 2{\sigma_3}^2 + 2s_2\sigma_3 - \left(s_1 + \frac{3}{4}s_3^2\right)\sigma_2 - \sigma_1 \ , \\
\alpha_{12} &= \alpha_{21} = 2\sigma_1\sigma_2 + 2\sigma_3 + 2s_2 \ , \\
\alpha_{13} &= \alpha_{31} = 2\sigma_1\sigma_3 + s_1 + \frac{3}{4}s_3^2 \ , \\
\alpha_{23} &= 2\sigma_2\sigma_3 + 3s_3\sigma_3 - \left(s_1 + \frac{3}{4}s_3^2\right)\sigma_1 + 1 \ . \\
\alpha_{32} &= 2\sigma_2\sigma_3 + 3s_3\sigma_3 - \left(s_1 + \frac{3}{4}s_3^2\right)\sigma_1 + 2.
\end{aligned}$$

Now the Hamiltonians $H_j$ are polynomials in the canonical variables. We will rewrite them and obtain the polynomial Hamiltonians, symmetric with respect to the independent variables. In fact, as to the A$_2$-system (0.8)-(0.9), we have the symmetry:

$$\left(\frac{\partial}{\partial s_1}\right)L_2 = \left(\frac{\partial}{\partial s_2}\right)L_1,$$

and then

$$\sum_{k=1}^{2} dp_k \wedge dq_k = \sum_{j=1}^{2} dL_j \wedge ds_j = 0 \ .$$

For the A$_3$-system, we make the transformation of the Hamiltonians:

$$\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = \begin{pmatrix} \overline{H}_1 \\ \overline{H}_2 \\ \overline{H}_3 \end{pmatrix} + (\alpha + \frac{1}{2}) \begin{pmatrix} 0 \\ s_3 \\ \frac{1}{2}s_2 \end{pmatrix} \tag{1.6}$$

and then consider the change of variables:

$$\begin{aligned}
&s_1 = t_1 - \frac{1}{4}t_3^2, \quad s_2 = t_2, \quad s_3 = t_3 \ ; \\
&\sigma_1 = q_1, \quad \sigma_2 = q_2 - t_3, \quad \sigma_3 = q_3 + \frac{1}{2}t_3q_1 - \frac{1}{2}t_2 \ .
\end{aligned} \tag{1.7}$$

**Proposition 1.2** *The change* (1.7) *of variables extends to the canonical transformation:*

$$(\sigma, \rho, \overline{H}, s) \to (q, p, L, t)$$

*where* $\overline{H} = (\overline{H}_1, \overline{H}_2, \overline{H}_3)$, $q = (q_1, q_2, q_3)$, $p = (p_1, p_2, p_3)$, $L = (L_1, L_2, L_3)$ *and* $t = (t_1, t_2, t_3)$ .

**Proof.** Besides (1.7), we consider the following change of variables:

$$\begin{aligned} &\rho_1 = p_1 - \frac{1}{2}t_3p_3, \quad \rho_2 = p_2, \quad \rho_3 = p_3 \ ; \\ &\overline{H}_1 = L_1, \quad \overline{H}_2 = L_2 - \frac{1}{2}p_3, \quad \overline{H}_3 = L_3 + \frac{1}{2}t_3L_1 - p_2 + \frac{1}{2}q_1p_3 \ . \end{aligned} \tag{1.8}$$

It is not difficult to verify the following equality:

$$\sum_{k=1}^{3} d\rho_k \wedge d\sigma_k - \sum_{j=1}^{3} d\overline{H}_j \wedge ds_j = \sum_{k=1}^{3} dp_k \wedge dq_k - \sum_{j=1}^{3} dL_j \wedge dt_j.$$

This proves the proposition; we do not enter into details of computation.

We can deduce from (1.5), (1.6), (1.7) and (1.8) the explicit form of the Hamiltonians $L_j$:

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} - B\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} - \left(\alpha + \frac{1}{2}\right)\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}, \tag{1.9}$$

where

$$\begin{aligned} R_1 &= p_2^2 - q_2p_3^2 - 2q_1p_2p_3 + 2p_1p_3 \ , \\ R_2 &= -2q_1p_2^2 + \left(q_1q_2 + \frac{1}{2}t_3q_1 + q_3 - \frac{1}{2}t_2\right)p_3^2 \\ &\qquad +2p_1p_2 + (2q_1^2 - t_3)p_2p_3 - 2q_1p_1p_3 \ , \\ R_3 &= p_1^2 + \left(q_1^2 - \frac{1}{2}t_3\right)p_2^2 + \left(q_2^2 - q_1q_3 - \frac{1}{2}t_3q_1^2 + \frac{1}{2}t_2q_1 - \frac{1}{2}t_3q_2 + \frac{1}{4}t_3^2\right)p_3^2 \\ &\qquad -2q_1p_1p_2 + (2q_1q_2 + t_3q_1 + 2q_3 - t_2)p_2p_3 - 2q_2p_1p_3 \ , \end{aligned}$$

and $B = ((\beta_{jk}))_{j,k=1,2,3}$ is the symmetric matrix such that

$$
\begin{aligned}
\beta_{11} &= q_1^2 + q_2 + \frac{1}{2}t_3 \,, \\
\beta_{22} &= q_2^2 - t_2 q_1 - \frac{1}{2}t_3 q_2 + \frac{1}{2}t_1 - \frac{1}{4}t_3^2 \,, \\
\beta_{33} &= q_3^2 + \frac{1}{2}t_2 t_3 q_1 - \frac{1}{2}t_1 q_2 - \frac{1}{4}t_2^2 + \frac{1}{8}t_3^3 \,, \\
\beta_{12} &= \beta_{21} = q_1 q_2 - \frac{1}{2}t_3 q_1 + q_3 + \frac{1}{2}t_2 \,, \\
\beta_{13} &= \beta_{31} = q_1 q_3 - \frac{1}{2}t_2 q_1 - \frac{1}{2}t_3 q_2 + \frac{1}{2}t_1 \,, \\
\beta_{23} &= \beta_{32} = q_2 q_3 + \left(\frac{1}{4}t_3^2 - \frac{1}{2}t_1\right) q_1 - \frac{1}{2}t_2 q_2 - \frac{1}{2}t_2 t_3 \,.
\end{aligned} \tag{1.10}
$$

Note that the Hamiltonians $L_i$ now possess the following symmetry:

$$
\left(\frac{\partial}{\partial t_j}\right) L_i = \left(\frac{\partial}{\partial t_i}\right) L_j \qquad (i \neq j) \,. \tag{1.11}
$$

Moreover we show the following proposition.

**Proposition 1.3** *For the polynomial Hamiltonian structure $(q, p, L, t)$, we have*

$$
\sum_{k=1}^{3} dp_k \wedge dq_k = \sum_{j=1}^{3} dL_j \wedge dt_j = 0 \,.
$$

**Proof.** If we regard $q$, $p$ and $L$ as functions of $t$, it can be written in the form

$$
\sum_{k=1}^{3} dp_k \wedge dq_k = \sum_{i<j} \sum_{l=1}^{3} \left( \frac{\partial L_i}{\partial p_l}\frac{\partial L_j}{\partial q_l} - \frac{\partial L_i}{\partial q_l}\frac{\partial L_j}{\partial p_l} \right) dt_i \wedge dt_j, \tag{1.12}
$$

and

$$
\begin{aligned}
&\sum_{j=1}^{3} dL_j \wedge dt_j = \\
&\sum_{i<j} \sum_{l=1}^{3} \left[ 2\left( \frac{\partial L_i}{\partial p_l}\frac{\partial L_j}{\partial q_l} - \frac{\partial L_i}{\partial q_l}\frac{\partial L_j}{\partial p_l} \right) + \left(\frac{\partial}{\partial t_i}\right) L_j - \left(\frac{\partial}{\partial t_j}\right) L_i \right] dt_i \wedge dt_j \,.
\end{aligned} \tag{1.13}
$$

By means of (1.11), for $1 \le i < j \le 3$, it suffices to show

$$
\sum_{l=1}^{3} \left( \frac{\partial L_i}{\partial p_l}\frac{\partial L_j}{\partial q_l} - \frac{\partial L_i}{\partial q_l}\frac{\partial L_j}{\partial p_l} \right) = 0 \,. \tag{1.14}
$$

Now, for $1 \leq i,j \leq 3$, we compute $\frac{\partial L_i}{\partial p_j}$ and $\frac{\partial L_i}{\partial q_j}$ , by using the explicit forms (0.9) of the Hamiltonians. Substituting these equalities into the left hand side of (1.14), we can show by computation that (1.14) holds. This proves the proposition.

## 2. Particular solutions.

In this section, we study particular solutions of the $A_3$-system when $\alpha = -\frac{1}{2}$. In this case, we obtain from (1.9):

$$\frac{\partial}{\partial q_k}\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \frac{1}{2}\frac{\partial}{\partial q_k}\begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} - \left(\frac{\partial}{\partial q_k}B\right)\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}.$$

Since $R_j$ $(j = 1,2,3)$ are quadratic in $p$, the system of equations:

$$\frac{\partial p_k}{\partial t_j} = -\frac{\partial L_j}{\partial q_k}$$

admits a particular solution of the form:

$$(p_1, p_2, p_3) = (0,0,0).$$

Then we deduce from (1.9) that $q = (q_1, q_2, q_3)$ satisfies equations of the form:

$$\text{(2.1)} \qquad \frac{\partial q_k}{\partial t_j} = -\beta_{jk}, \qquad (j,k = 1,2,3).$$

Since $B = (\beta_{jk})_{j,k=1,2,3}$ is a symmetric matrix, it is easy to see:

**Proposition 2.1** *For a solution $q = (q_1, q_2, q_3)$ of (2.1), the one-form:*

$$\omega = \sum_{j=1}^{3} q_j dt_j$$

*is closed.*

Now we introduce a new variable $u$ by:

$$\text{(2.2)} \qquad \omega = d\log u$$

or equivalently

$$q_j = \frac{\partial}{\partial t_j}\log u.$$

By substituting this into (2.1) and using (1.10), we can verify the following proposition.

**Proposition 2.2** *The function $u = (u_1, u_2, u_3)$ satisfies the following completely integrable linear system:*

(2.3)
$$\begin{aligned}
\frac{\partial^2 u}{\partial t_1^2} &= -\frac{\partial u}{\partial t_2} - \frac{1}{2}t_3 u\,, \\
\frac{\partial^2 u}{\partial t_1 \partial t_2} &= \frac{1}{2}t_3\frac{\partial u}{\partial t_1} - \frac{\partial u}{\partial t_3} - \frac{1}{2}t_2 u\,, \\
\frac{\partial^2 u}{\partial t_1 \partial t_3} &= \frac{1}{2}t_2\frac{\partial u}{\partial t_1} + \frac{1}{2}t_3\frac{\partial u}{\partial t_2} - \frac{1}{2}t_1 u\,, \\
\frac{\partial^2 u}{\partial t_2^2} &= t_2\frac{\partial u}{\partial t_1} + \frac{1}{2}t_3\frac{\partial u}{\partial t_2} - \left(\frac{1}{2}t_1 - \frac{1}{4}t_3^2\right)u\,, \\
\frac{\partial^2 u}{\partial t_2 \partial t_3} &= \left(\frac{1}{2}t_1 - \frac{1}{4}t_3^2\right)\frac{\partial u}{\partial t_1} + \frac{1}{2}t_2\frac{\partial u}{\partial t_2} + \frac{1}{2}t_2 t_3 u\,, \\
\frac{\partial^2 u}{\partial t_3^2} &= -\frac{1}{2}t_2 t_3\frac{\partial u}{\partial t_1} + \frac{1}{2}t_1\frac{\partial u}{\partial t_2} + \frac{1}{4}\left(t_2^2 - \frac{1}{2}t_3^3\right)u\,.
\end{aligned}$$

We show the

**Proposition 2.3.** *The system* (2.3) *has solutions of the form:*

(2.4)
$$u_l(t) = \int_{\gamma_l} \exp\left[-S(\xi,t) - \frac{1}{2}t_2 t_3\right] d\xi, \quad (l = 1,2,3,4),$$

*where $\gamma_l$ are paths in the complex plane described in Figure* 1, *and*

$$S(\xi,t) = \frac{2}{5}\xi^5 + t_3\xi^3 + t_2\xi^2 + \left(t_1 + \frac{1}{2}t_3^2\right)\xi.$$

**Proof.** Supposing that $u$ is written in the form:

$$u(t) = \int_\gamma e^{-t_1\xi}\phi(\xi, t_2, t_3)\,d\xi,$$

we obtain the system:

(2.5)
$$\begin{aligned}
\frac{\partial \phi}{\partial t_2} &= -\left(\lambda^2 + \frac{1}{2}t_3\right)\phi\,, \\
\frac{\partial \phi}{\partial t_3} &= -\left(\lambda^3 + t_3\lambda + \frac{1}{2}t_2\right)\phi\,, \\
\frac{\partial \phi}{\partial \lambda} &= -\left(2\lambda^4 + 3t_3\lambda^2 + 2t_2\lambda + \frac{1}{2}t_3^2\right)\phi\,.
\end{aligned}$$

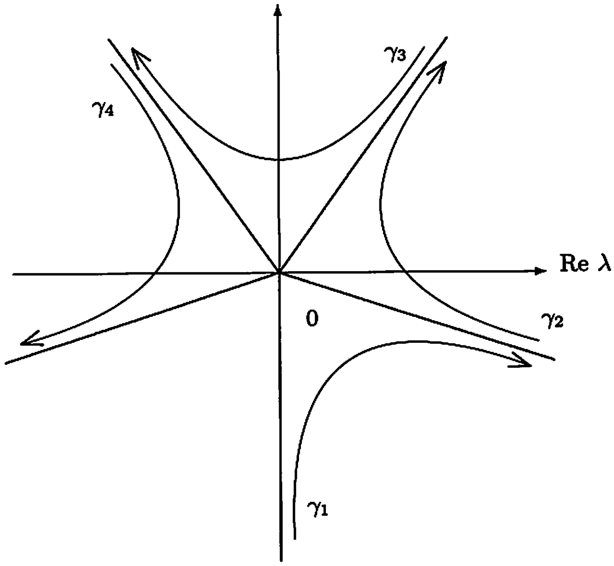


Figure 1:

It is easy to see that

$$\phi(\xi, t_2, t_3) = \exp\left[-\frac{2}{5}\xi^5 - t_3\xi^3 - t_2\xi^2 - \frac{1}{2}t_3^2\xi - \frac{1}{2}t_2t_3\right]$$

solves (2.5). This proves Proposition 2.3.

Let $J$ be the linear space of solutions of (2.3). It is known ([1]) that the linear map $\Phi$ from $J$ to $\mathbf{C}^4$ such that

$$\text{(2.6)} \qquad \Phi: \quad J \ni u(t) \longrightarrow \begin{pmatrix} u(0) \\ u_1(0) \\ u_2(0) \\ u_3(0) \end{pmatrix} \in \mathbf{C}^4$$

is an isomorphism, where $u_i = \frac{\partial u}{\partial t_i}$ for $i = 1, 2, 3$. Here we evaluate values of the functions at $(t_1, t_2, t_3) = (0, 0, 0)$, for the right hand side of (2.3) is polynomial in $t$. $J$ is of the dimension four. Finally, we prove the

**Proposition 2.4** *The solutions* (2.4) *of* (2.3) *form the basis of the linear space* $J$.

**Proof.** By means of (2.6), it suffices to show that, for $u_l$ $(l = 1, 2, 3, 4)$, $\Phi(u_l)$ are linear independent vectors in $\mathbf{C}^4$. In order to compute the vector $\Phi(u_l)$, we put $\theta = e^{\frac{2}{5}\pi i}$ and

$$a_l = \int_0^\infty \xi^{l-1} e^{-\frac{2}{5}\xi^5} d\xi \qquad (l = 1,2,3,4) .$$

Then we deduce from (2.4) that the matrix

$$(\Phi(u_1), \Phi(u_2), \Phi(u_3), \Phi(u_4))$$

is given by:

$$\begin{pmatrix} (\theta-1)a_1 & 0 & 0 & 0 \\ 0 & (1-\theta^2)a_2 & 0 & 0 \\ 0 & 0 & (1-\theta^3)a_3 & 0 \\ 0 & 0 & 0 & (1-\theta^4)a_4 \end{pmatrix} \begin{pmatrix} 1 & \theta & \theta^2 & \theta^3 \\ 1 & \theta^2 & \theta^4 & \theta^6 \\ 1 & \theta^3 & \theta^6 & \theta^9 \\ 1 & \theta^4 & \theta^8 & \theta^{12} \end{pmatrix} .$$

Since $(1-\theta^l)a_l \neq 0$, $\Phi(u_l)$ $(l = 1,2,3,4)$ are linearly independent. It follows that $u_l$ $(l = 1,2,3,4)$ form basis of $J$.

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