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On Some Hamiltonian Structures of Painlevé Systems, III

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0. Introduction

This paper is the third part of our series of papers. In the second part([12]), we have given a description of the space E_J for the J-th Painlevé system (H_J), which gives a symplectic structure to E_J $(J=V, IV, III, II)$. The purpose of this paper is to prove that there exist no other Hamiltonian systems on the symplectic space E_J than the J-th Painlevé system (H_J) for $J=V, IV, III, II$ as we have shown the fact for $J=VI$ in [11].

The J-th Painlevé system which is equivalent to the J-th Painlevé equation is a Hamiltonian system

$$(H_J) \qquad dx/dt=\partial H_J/\partial y, \qquad dy/dt=-\partial H_J/\partial x,$$

where H_J for $J=V\sim II$ is a function given by

$$H_V(x,y,t)=\frac{1}{t}[x(x-1)^2y^2-\{\kappa_0(x-1)^2+\kappa_t x(x-1)-\eta tx\}y+\kappa(x-1)]$$
$$(\kappa:=\frac{1}{4}\{(\kappa_0+\kappa_t)^2-\kappa_\infty^2\}),$$
$$H_{IV}(x,y,t)=2xy^2-\{x^2+2tx+2\kappa_0\}y+\kappa_\infty x,$$
$$H_{III}(x,y,t)=\frac{1}{t}[2x^2y^2-\{2\eta_\infty tx^2+(2\kappa_0+1)x-2\eta_0 t\}y+\eta_\infty(\kappa_0+\kappa_\infty)tx],$$
$$H_{II}(x,y,t)=\frac{1}{2}y^2-(x^2+\frac{t}{2})y-(\alpha+\frac{1}{2})x,$$

x, y, and t are complex variables and $\kappa_0, \kappa_t, \kappa_\infty, \eta, \eta_0, \eta_\infty, \alpha$ are complex constants([4]).

In order to state our results, we recall the results of our paper [12]. The space $E_J(J=V, IV, III, II)$ is constructed as follows. We first take a minimal compactification $\overline{\Sigma}_\epsilon$ of $\mathbf{C}^2$ ([6]) obtained by glueing four $U_i=\mathbf{C}^2\ni(x_i,y_i), i=0,1,2,3$, via the following identifications:

$$\begin{aligned}
&x_0=x_1, && y_0=1/y_1,\\
&x_0=1/x_2, && y_0=x_2(\epsilon-x_2y_2),\\
&x_2=x_3, && y_2=1/y_3,
\end{aligned}$$

where ϵ is a complex constant depending on the parameters in H_J. Secondly, for every $t \in B_J$, we make a finite number of quadric transformations to $\overline{\Sigma}_\epsilon \times t$ and get $\overline{E_J(t)}$. Lastly we obtain $E_J(t)$ by removing some divisors which consist of *vertical* leaves and *inaccessible* singular points and we define E_J by

$$E_J = \bigcup_{t \in B_J} E_J(t) \times t.$$

Then a description of each $E_J (J = V, IV, III, II)$ is given as follows:

i) *The space E_V for the fifth Painlevé system in case of $\eta \neq 0$ is obtained by glueing five copies of $\mathbf{C}^2 \times B_V$:*

$$\begin{aligned}
V(00) \times B_V &= \mathbf{C}^2 \times B_V \ni (x, y, t) = (x(00), y(00), t),\\
V(0\infty) \times B_V &= \mathbf{C}^2 \times B_V \ni (x(0\infty), y(0\infty), t),\\
V(1\infty) \times B_V &= \mathbf{C}^2 \times B_V \ni (x(1\infty), y(1\infty), t),\\
V(\infty 0+) \times B_V &= \mathbf{C}^2 \times B_V \ni (x(\infty 0+), y(\infty 0+), t),\\
V(\infty 0-) \times B_V &= \mathbf{C}^2 \times B_V \ni (x(\infty 0-), y(\infty 0-), t),
\end{aligned}$$

via the following symplectic transformations

$$x(00) = y(0\infty)(\kappa_0 - x(0\infty)y(0\infty)), \quad y(00) = 1/y(0\infty), \tag{0.1}$$

$$x(00) = 1 + x(1\infty), \quad y(00) = -\frac{\eta t}{x(1\infty)^2} + \frac{\kappa_t + 1}{x(1\infty)} + y(1\infty), \tag{0.2}$$

$$x(00) = 1/x(\infty 0+), \quad y(00) = x(\infty 0+)(\epsilon(+) - x(\infty 0+)y(\infty 0+)), \tag{0.3}$$

$$x(\infty 0+) = y(\infty 0-)(\kappa_\infty - x(\infty 0-)y(\infty 0-)), \quad y(\infty 0+) = 1/y(\infty 0-), \tag{0.4}$$

where

$$B_V = \mathbf{C} - \{0\}, \tag{0.5}$$

$$\epsilon(+) = (\kappa_0 + \kappa_t + \kappa_\infty)/2. \tag{0.6}$$

ii) *The space E_{IV} for the fourth Painlevé system is obtained by glueing four copies of* $\mathbf{C}^2 \times B_{IV}$*:*

$$\begin{aligned} V(00) \times B_{IV} &= \mathbf{C}^2 \times B_{IV} \ni (x, y, t) = (x(00), y(00), t), \\ V(0\infty) \times B_{IV} &= \mathbf{C}^2 \times B_{IV} \ni (x(0\infty), y(0\infty), t), \\ V(\infty 0) \times B_{IV} &= \mathbf{C}^2 \times B_{IV} \ni (x(\infty 0), y(\infty 0), t), \\ V(\infty\infty) \times B_{IV} &= \mathbf{C}^2 \times B_{IV} \ni (x(\infty\infty), y(\infty\infty), t), \end{aligned}$$

via the following symplectic transformations

$$x(00) = y(0\infty)(\kappa_0 - x(0\infty)y(0\infty)), \quad y(00) = 1/y(0\infty), \tag{0.7}$$

$$x(00) = 1/x(\infty 0), \quad y(00) = x(\infty 0)(\kappa_\infty - x(\infty 0)y(\infty 0)), \tag{0.8}$$

$$\begin{aligned} &x(\infty 0) = x(\infty\infty), \\ &y(\infty 0) = -\frac{1/2}{x(\infty\infty)^3} - \frac{t}{x(\infty\infty)^2} + \frac{2\kappa_\infty - \kappa_0 + 1}{x(\infty\infty)} + y(\infty\infty), \end{aligned} \tag{0.9}$$

where

$$B_{IV} = \mathbf{C}. \tag{0.10}$$

iii) *The space E_{III} for the third Painlevé system in case of $\eta_0\eta_\infty \neq 0$ is obtained by glueing four copies of* $\mathbf{C}^2 \times B_{III}$*:*

$$\begin{aligned} V(00) \times B_{III} &= \mathbf{C}^2 \times B_{III} \ni (x, y, t) = (x(00), y(00), t), \\ V(0\infty) \times B_{III} &= \mathbf{C}^2 \times B_{III} \ni (x(0\infty), y(0\infty), t), \\ V(\infty 0) \times B_{III} &= \mathbf{C}^2 \times B_{III} \ni (x(\infty 0), y(\infty 0), t), \\ V(\infty\eta_\infty t) \times B_{III} &= \mathbf{C}^2 \times B_{III} \ni (x(\infty\eta_\infty t), y(\infty\eta_\infty t), t), \end{aligned}$$

via the following symplectic transformations

$$x(00) = x(0\infty), \quad y(00) = -\frac{\eta_0 t}{x(0\infty)^2} + \frac{\kappa_0 + 1}{x(0\infty)} + y(0\infty), \tag{0.11}$$

$$x(00) = 1/x(\infty 0), \quad y(00) = x(\infty 0)(\epsilon - x(\infty 0)y(\infty 0)), \tag{0.12}$$

$$x(\infty 0) = x(\infty\eta_\infty t), \quad y(\infty 0) = -\frac{\eta_\infty t}{x(\infty\eta_\infty t)^2} + \frac{\kappa_\infty}{x(\infty\eta_\infty t)} + y(\infty\eta_\infty t), \tag{0.13}$$

where

$$B_{III} = \mathbf{C} - \{0\}, \tag{0.14}$$

$$\epsilon = (\kappa_0 + \kappa_\infty)/2. \tag{0.15}$$

iv) *The space E_{II} for the second Painlevé system is obtained by glueing three copies of $\mathbf{C}^2 \times B_{II}$:*

$$\begin{aligned} V(00) \times B_{II} &= \mathbf{C}^2 \times B_{II} \ni (x, y, t) = (x(00), y(00), t), \\ V(\infty 0) \times B_{II} &= \mathbf{C}^2 \times B_{II} \ni (x(\infty 0), y(\infty 0), t), \\ V(\infty\infty) \times B_{II} &= \mathbf{C}^2 \times B_{II} \ni (x(\infty\infty), y(\infty\infty), t), \end{aligned}$$

via the following symplectic transformations

$$x(00) = 1/x(\infty 0), \quad y(00) = x(\infty 0)(\epsilon - x(\infty 0)y(\infty 0)), \tag{0.16}$$

$$\begin{aligned} &x(\infty 0) = x(\infty\infty), \\ &y(\infty 0) = -\frac{2}{x(\infty\infty)^4} - \frac{t}{x(\infty\infty)^2} - \frac{2\alpha}{x(\infty\infty)} + y(\infty\infty), \end{aligned} \tag{0.17}$$

where

$$B_{II} = \mathbf{C}, \tag{0.18}$$

$$\epsilon = -\alpha - \frac{1}{2}. \tag{0.19}$$

In Section 1, we state main results. In the following sections, Sections 2, 3, 4, and 5, we prove THEOREM 2 according to $J = V, IV, III, II$.

1. Main Theorems

By a Hamiltonian system holomorphic on E_J, we mean a family of Hamiltonian functions $\{K(*; x(*), y(*), t)\}_*$ such that each $K(*) = K(*; x(*), y(*), t)$ is a holomorphic function in the chart $V(*) \times B_J$ and every $K(*)$ is the transform of $K(00)$ by the symplectic transformation between $(x(*), y(*), t)$ and $(x(00), y(00), t)$. Let $\{K(*)\}_*$ be a holomorphic Hamiltonian system on E_J. We say that it is algebraic or meromorphically continued to the space $\overline{E_J}$ if each $K(*)$ is meromorphically continued to $\overline{V(*)} \times B_J$ which is the closure in $\overline{E_J}$. Then our main theorem is stated as follows.

THEOREM 1 *Any Hamiltonian system which is holomorphic and algebraic on E_J ($J = V \sim II$) must coincide to the J-th Painlevé system (H_J).*

It should be noticed that, although a Hamiltonian system $\{K(*)\}_*$ on E_J does not define a function on E_J, the difference $\{K(*) - K'(*)\}_*$ of any two Hamiltonian systems $\{K(*)\}_*$ and $\{K'(*)\}_*$ on E_J defines a function on E_J, by adding functions of t if it is necessary. Therefore we see that the above theorem is equivalent to the following theorem.

THEOREM 2 *Any function holomorphic and algebraic on E_J is a function which depends only on t.*

2. Proof of THEOREM 2 for $J = V$

In this section we prove THEOREM 2 in the case of $J = V$. Let f be a holomorphic and algebraic function on E_V, namely, holomorphic on E_V and meromorphically continued to $\overline{E_V}$. Since f is holomorphic on a chart $V(00) \times B_V \ni (x, y, t)$, f is developpable into a power series of x and y as

$$f = \sum_{i,j=0}^{\infty} a_{ij}(t)x^i y^j, \tag{2.1}$$

which is convergent for any $x, y \in \mathbf{C}$ and $t \in B_V$. We will prove that f is a function depending only on t, namely, $a_{ij}(t) = 0$ for all $(i, j) \neq (0, 0)$, by using the assumption that f is horomorphic on every chart of E_V and f is meromorphically continued to $\overline{E_V}$.

2.1 Linear equations for $a_{ij}(t)$

In this subsection, we derive linear equations for the coefficients $a_{ij}(t)$ from the assumption that f is holomorphic on every other chart.

For the sake of simplicity, we use the notation as follows: for functions $g(x, y, t)$ and $h(x, y, t)$, $g(x, y, t) \equiv h(x, y, t)$ means that $g(x, y, t) - h(x, y, t)$ is a function entire in x and y and the symbols $\partial_\alpha, \partial_\varepsilon, \partial_{\kappa_\infty}$, and ∂_{κ_0} denote the differential operators $\partial/\partial\alpha, \partial/\partial\varepsilon, \partial/\partial\kappa_\infty$, and $\partial/\partial\kappa_0$ respectively. And $(u_{ij})_{0 \le i \le m-1, 0 \le j \le n-1}$ denotes an $m \times n$ matrix with (i, j)-components u_{ij}, $(u_i)_{0 \le i \le m-1}$ denotes an $m \times 1$ matrix (m-column vector), and $(u_{ij})_{i \ge 0, 0 \le j \le n-1}$ denotes an $\infty \times n$ matrix.

First, we study the condition that f is holomorphic on $V(\infty0+) \times B_V$. Let $(X, Y, t) := (x(\infty0+), y(\infty0+), t)$ and $\varepsilon := \varepsilon(+)$, where $\varepsilon(+)$ is a constant given by (0.6). From (0.3), it follows that

$$\begin{aligned} f &= \sum_{i,j=0}^{\infty} a_{ij} X^{-(i-j)} (\varepsilon - XY)^j \equiv \sum_{\mu=1}^{\infty} \sum_{j=0}^{\infty} a_{\mu+j,j} \frac{1}{X^\mu} (\varepsilon - XY)^j \\ &= \sum_{\mu=1}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{j} a_{\mu+j,j} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial \varepsilon^k} (\varepsilon^j) \frac{Y^k}{X^{\mu-k}} \\ &\equiv \sum_{\mu=1}^{\infty} \sum_{k=0}^{\mu-1} \sum_{j=0}^{\infty} a_{\mu+j,j} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial \varepsilon^k} (\varepsilon^j) \frac{Y^k}{X^{\mu-k}}. \end{aligned}$$

As f is holomorphic on $X = 0$ by assumption, all the coefficients of $Y^k/X^{\mu-k}$ $(\mu \geq 1, 0 \leq k \leq \mu - 1)$ must vanish. Therefore we obtain

$$\sum_{j=0}^{\infty} a_{\mu+j,j} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial \varepsilon^k} (\varepsilon^j) = 0 \qquad (\mu \geq 1, \quad 0 \leq k \leq \mu - 1), \tag{2.2}$$

the left-hand side of which is the coefficient of $Y^k/X^{\mu-k}$ $(0 \leq k \leq \mu - 1)$. We write these equations in matrix form as

$$\begin{pmatrix} a_{\mu,0} & a_{\mu+1,1} & \cdots \end{pmatrix} \left(\frac{1}{j!} \partial_\varepsilon^j (\varepsilon^i) \right)_{i \geq 0, 0 \leq j \leq \mu-1} = 0 \qquad (\mu \geq 1). \tag{2.3}$$

Secondly, we study f in $V(\infty 0-) \times B_V \ni (X, Y, t)$. From (0.3) and (0.4), it follows $x = 1/Y(\kappa_\infty - XY)$, $y = Y(\kappa_\infty - XY)\{\varepsilon - (\kappa_\infty - XY)\}$. Therefore, by using (2.2), we have

$$\begin{aligned} f &\equiv \sum_{\mu=1}^{\infty} \sum_{k=\mu}^{\infty} \sum_{j=0}^{\infty} a_{\mu+j,j} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial \varepsilon^k} (\varepsilon^j) \frac{(\kappa_\infty - XY)^{k-\mu}}{Y^\mu} \\ &= \sum_{\mu=1}^{\infty} \sum_{k=\mu}^{\infty} \sum_{j=0}^{\infty} a_{\mu+j,j} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial \varepsilon^k} (\varepsilon^j) \frac{1}{Y^\mu} \sum_{l=0}^{k-\mu} \frac{(-1)^l}{l!} \frac{\partial^l}{\partial \kappa_\infty^l} (\kappa_\infty^{k-\mu}) (XY)^l \\ &\equiv \sum_{\mu=1}^{\infty} \sum_{l=0}^{\mu-1} \sum_{j=\mu}^{\infty} \sum_{k=\mu}^{\infty} a_{\mu+j,j} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial \varepsilon^k} (\varepsilon^j) \frac{(-1)^l}{l!} \frac{\partial^l}{\partial \kappa_\infty^l} (\kappa_\infty^{k-\mu}) \frac{X^l}{Y^{\mu-l}}. \end{aligned}$$

Here we used (2.2) in the first equation. Therefore we have

$$\sum_{j=\mu}^{\infty} \sum_{k=\mu}^{j} a_{\mu+j,j} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial \varepsilon^k} (\varepsilon^j) \frac{(-1)^l}{l!} \frac{\partial^l}{\partial \kappa_\infty^l} (\kappa_\infty^{k-\mu}) = 0, \tag{2.4}$$

$$(\mu \geq 1, \quad 0 \leq l \leq \mu - 1)$$

by observing the coefficients of $X^l/Y^{\mu-l} (0 \le l \le \mu - 1)$. We write these equations as

$$(2.5)\qquad \begin{pmatrix} a_{2\mu,\mu} & a_{2\mu+1,\mu+1} & \cdots \end{pmatrix} \left(\sum_{k=\mu}^{\mu+i} \frac{(-1)^k}{k!} \partial_\varepsilon^k (\varepsilon^{\mu+i}) \frac{1}{j!} \partial_{\kappa_\infty}^j (\kappa_\infty^{k-\mu}) \right)_{i\ge 0, 0\le j\le \mu-1} = 0 \quad (\mu \ge 1).$$

Thirdly we study f in $V(0\infty) \times B_V \ni (X, Y, t)$. By (0.1), we have

$$f \equiv \sum_{\nu=1}^{\infty} \sum_{k=0}^{\nu-1} \sum_{i=0}^{\infty} a_{i,\nu+i} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial \kappa_0^k} (\kappa_0^i) \frac{X^k}{Y^{\nu-k}}.$$

Therefore we have

$$(2.6)\qquad \sum_{i=0}^{\infty} a_{i,\nu+i} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial \kappa_0^k} (\kappa_0^i) = 0 \qquad (\nu \ge 1, \quad 0 \le k \le \nu - 1),$$

by observing the coefficients of $X^k/Y^{\nu-k}$ $(0 \le k \le \nu - 1)$. These equations are written as

$$(2.7)\qquad \begin{pmatrix} a_{0,\nu} & a_{1,\nu+1} & \cdots \end{pmatrix} \left(\frac{1}{j!} \partial_{\kappa_0}^j (\kappa_0^i) \right)_{i\ge 0, 0\le j\le \nu-1} = 0 \quad (\nu \ge 1).$$

Lastly, we study f in $V(1\infty) \times B_V \ni (X, Y, t)$. We notice that the transformation between (x, y, t) and (X, Y, t) is given by $x = 1+X$, $y = \alpha/X^2 + \beta/X + Y$, where $\alpha := -\eta t$, $\beta := \kappa_t + 1$. Here we notice $\alpha \ne 0$. In order to obtain simple expressions of the coefficients of Y^λ/X^τ $(\tau \ge 1, \lambda \ge 0)$, we introduce new variables v_j^i defined by

$$v_j^i := \sum_{k=0}^{\infty} a_{kj} \binom{k}{i} \quad \text{i.e.} \quad \begin{pmatrix} v_j^0 & v_j^1 & \cdots \end{pmatrix} = \begin{pmatrix} a_{0j} & a_{1j} & \cdots \end{pmatrix} \left(\binom{i}{j} \right)_{i,j\ge 0}.$$

We notice

$$\det \left(\binom{i}{j} \right)_{0\le i,j\le N} = 1,$$

for every positive integer N. By the use of v_j^i,

$$f = \sum_{i,j=0}^{\infty} a_{ij} (1+X)^i \left(\frac{\alpha}{X^2} + \frac{\beta}{X} + Y \right)^j = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} v_j^i X^i \left(\frac{\alpha}{X^2} + \frac{\beta}{X} + Y \right)^j.$$

Now we assume that $a_{ij} = 0$ $(j > J)$ for some positive integer J, which implies $v_{ij} =$

0 $(j > J)$. Under this assumption, we obtain

$$f = \sum_{i=0}^{\infty}\sum_{j=0}^{J} v_j^i X^i \left(\frac{\alpha}{X^2} + \frac{\beta}{X} + Y\right)^j \equiv \sum_{i=0}^{\infty}\sum_{j=0}^{J-1} v_{J-j}^i X^i \left(\frac{\alpha}{X^2} + \frac{\beta}{X} + Y\right)^{J-j}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{J-1} v_{J-j}^i X^i \sum_{0\le p,q,r\le J-j,p+q+r=J-j} \frac{(J-j)!}{p!q!r!}\left(\frac{\alpha}{X^2}\right)^p\left(\frac{\beta}{X}\right)^q Y^r$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{J-1} v_{J-j}^i \sum_{0\le q,r,q+r\le J-j} \frac{1}{q!r!}\beta^q\partial_\alpha^{q+r}(\alpha^{(J-j)})\frac{Y^r}{X^{2(J-j)-q-i-2r}}.$$

By decomposing the set of $2(J-j)-(q+i)$, which appear in powers of $1/X$, into even integers or odd integers, namely, by putting $2(J-j)-(q+i) = 2J-2k$ or $2J-2k-1$, we obtain

$$f \equiv \sum_{k=0}^{J-1}\sum_{r=0}^{J-k-1}\left\{\sum_{2j+(q+i)=2k} v_{J-j}^i \frac{1}{q!r!}\beta^q\partial_\alpha^{q+r}(\alpha^{(J-j)})\frac{Y^r}{X^{2J-2k-2r}}\right.$$

$$\left.+ \sum_{2j+(q+i)=2k+1} v_{J-j}^i \frac{1}{q!r!}\beta^q\partial_\alpha^{q+r}(\alpha^{(J-j)})\frac{Y^r}{X^{2J-(2k+1)-2r}}\right\}$$

$$= \sum_{k=0}^{J-1}\sum_{r=0}^{J-k-1}\left\{\sum_{q=0}^{2k}\sum_{j=0}^{[k-q/2]} v_{J-j}^{2k-q-2j}\frac{1}{q!r!}\beta^q\partial_\alpha^{q+r}(\alpha^{(J-j)})\frac{Y^r}{X^{2J-2k-2r}}\right.$$

$$\left.+\sum_{q=0}^{2k+1}\sum_{j=0}^{[k+1/2-q/2]} v_{J-j}^{(2k+1)-q-2j}\frac{1}{q!r!}\beta^q\partial_\alpha^{q+r}(\alpha^{(J-j)})\frac{Y^r}{X^{2J-(2k+1)-2r}}\right\}.$$

Here, for a real number a, $[a]$ denotes the maximal integer not greater than a. Since f is holomorphic on $X=0$, we have

$$\sum_{q=0}^{2k}\sum_{j=0}^{[k-q/2]} v_{J-j}^{2k-q-2j}\frac{1}{q!r!}\beta^q\partial_\alpha^{q+r}(\alpha^{J-j}) = 0,$$

$$\sum_{q=0}^{2k+1}\sum_{j=0}^{[k+1/2-q/2]} v_{J-j}^{(2k+1)-q-2j}\frac{1}{q!r!}\beta^q\partial_\alpha^{q+r}(\alpha^{J-i}) = 0,$$

for $0\le k\le J-1, 0\le r\le J-k-1$, which are written as

$$(2.8)\qquad \begin{aligned}&(v_J^{2k}\ \ v_{J-1}^{2k-2}\ \ \cdots\ \ v_{J-k}^0)\left(\frac{1}{j!}\partial_\alpha^j(\alpha^{J-i})\right)_{0\le i\le k, 0\le j\le J-k-1}\\ &\qquad + P_0(\alpha,\beta,v) = 0,\end{aligned}$$

$$(2.9)\qquad \begin{aligned} &(v_J^{2k+1} \quad v_{J-1}^{2k-1} \quad \cdots \quad v_{J-k}^{1}) \left(\frac{1}{j!}\partial_\alpha^j(\alpha^{J-i})\right)_{0\le i\le k, 0\le j\le J-k-1} \\ &+(v_J^{2k} \quad v_{J-1}^{2k-2} \quad \cdots \quad v_{J-k}^{0}) \left(\frac{1}{j!}\beta\partial_\alpha^{j+1}(\alpha^{J-i})\right)_{0\le i\le k, 0\le j\le J-k-1} \\ &+P_1(\alpha,\beta,v)=0, \end{aligned}$$

for $0 \le k \le J-1$. Here $P_0(\alpha,\beta,v)$ and $P_1(\alpha,\beta,v)$ are row vectors which vanish in the case of $v_{J-j}^i = 0 \quad (0 \le j \le k-1, 0 \le i \le 2k-1-2j)$.

2.2 Reduction of f to a polynomial

In this subsection, we prove that $f = \sum_{i,j\ge 0} a_{ij}(t)x^i y^j$ is a polynomial of x and y. By recalling the construction of $\overline{E_V}$, we see that $\overline{V(00)} \times B_V$ contains a divisor $\{(x_1, y_1, t) \in \mathbf{C}^2 \times B_V \mid y_1 = 0, x_1 \neq 0, 1\}$ where $x = x_1, y = 1/y_1$. Therefore, by our assumption, $f(x_1, 1/y_1, t)$ must be meromorphic on $y_1 = 0, x_1 \neq 0, 1$, which implies that

$$a_{ij} = 0 \qquad (j > N)$$

for some nonnegative integer N. Then, from (2.3), we have

$$(a_{\mu,0} \quad a_{\mu+1,1} \quad \cdots) \left(\frac{1}{j!}\partial_\epsilon^j(\epsilon^i)\right)_{0\le i,j\le N} = 0,$$

for every $\mu > N$. Therefore by noting

$$\det\left(\frac{1}{j!}\partial_\epsilon^j(\epsilon^i)\right)_{0\le i,j\le N} = 1,$$

we obtain

$$a_{\mu+j,j} = 0 \quad (\mu > N, j \ge 0),$$

which shows that f is a polynomial of x and y .

2.3 Completion of the Proof of THEOREM 2 for $J = V$

In order to prove the theorem for $J = V$, it is sufficient to show

Proposition 2.1 *For every positive integer m, $a_{ij} = 0 \quad (i \quad or \quad j > 3m)$ implies*

$$a_{ij} = 0 \quad (i \quad or \quad j > 3m-3).$$

From now on, we assume

$$(2.10)\qquad a_{ij} = 0 \qquad (i \quad \text{or} \quad j > 3m)$$

for an arbitrary fixed positive integer m. We first obtain

Proposition 2.2 $a_{ij} = 0 \quad (i - j > m)$.

Proof. Under the assumption (2.10), from (2.3) we have

$$\begin{pmatrix} a_{\mu,0} & a_{\mu+1,1} & \cdots & a_{3m,3m-\mu} \end{pmatrix} \left(\frac{1}{j!} \partial_\epsilon^j (\epsilon^i) \right)_{0 \le i \le 3m-\mu, 0 \le j \le \mu-1} = 0.$$

Therefore, for every μ such that $3m - \mu \le \mu - 1$ *i.e.* $2\mu > 3m$, we have

$$\begin{pmatrix} a_{\mu,0} & a_{\mu+1,1} & \cdots & a_{3m,3m-\mu} \end{pmatrix} \left(\frac{1}{j!} \partial_\epsilon^j (\epsilon^i) \right)_{0 \le i,j \le 3m-\mu} = 0,$$

which yields

$$a_{\mu+j,j} = 0 \quad (2\mu > 3m, \quad 0 \le j \le 3m - \mu),$$

since

$$\det \left(\frac{1}{j!} \partial_\epsilon^j (\epsilon^i) \right)_{0 \le i,j \le 3m-\mu} = 1.$$

In the same way, from (2.5) we obtain

$$\begin{pmatrix} a_{2\mu,\mu} & a_{2\mu+1,\mu+1} & \cdots & a_{3m,3m-\mu} \end{pmatrix} \left(\sum_{k=\mu+j}^{\mu+i} \frac{(-1)^k}{k!} \partial_\varepsilon^k (\varepsilon^{\mu+i}) \frac{1}{j!} \partial_{\kappa_\infty}^j (\kappa_\infty^{k-\mu}) \right)_{0 \le i,j \le 3m-2\mu} = 0,$$

for every μ such that $2\mu \le 3m < 3\mu$, which shows

$$a_{\mu+j,j} = 0 \quad (2\mu \le 3m < 3\mu, \quad \mu \le j \le 3m - \mu),$$

because

$$\det \left(\sum_{k=\mu+j}^{\mu+i} \frac{(-1)^k}{k!} \partial_\varepsilon^k (\varepsilon^{\mu+i}) \frac{1}{j!} \partial_{\kappa_\infty}^j (\kappa_\infty^{k-\mu}) \right)_{0 \le i,j \le 3m-2\mu} = (-1)^{\frac{3}{2}m(3m-2\mu+1)} \neq 0.$$

Now, by using again (2.3) for $\begin{pmatrix} a_{\mu,0} & a_{\mu+1,1} & \cdots & a_{2\mu-1,\mu-1} \end{pmatrix}$, we obtain

$$a_{\mu+j,j} = 0 \quad (2\mu \le 3m < 3\mu, \quad 0 \le j \le \mu - 1).$$

Thus we have shown Proposition 2.2. □

We next show

Proposition 2.3 $a_{i,j} = 0 \quad (j - i > m \quad or \quad j > 2m)$.

To prove Proposition 2.3, we introduce a notion of a state. By a *state* $S(k,l)$ of a polynomial $f = \sum a_{ij}x^i y^j$, we mean a state

$$a_{ij} = 0, \qquad j > l \quad \text{or} \quad j - i > l - k.$$

Assume that f is in a state $S(k,l)$. Then $a_{i,i+(l-k)} = 0$ for $i > k$, $a_{ij} = 0$ for $j > l$, and $a_{il} = 0$ for $0 \le i < k$ or $i > 3m$. Therefore, if $l - k \ge k+1$, which means the number of equations is greater than or equal to that of unkowns, it follows from (2.7) that $a_{i,i+(l-k)} = 0$ for $0 \le i \le k$. In short, if $l \ge 2k+1$, then we can reduce $S(k,l)$ to $S(k+1,l)$ by using the linear system (2.7). We call this process Reduction A. On the other hand, if $2[(l+1)/2] \ge 3m - k + 1$, then we can reduce $S(k,l)$ to $S((k-1)^+,(l-1)^+)$ $(a^+ = \max\{a,0\})$ by the following Proposition 2.4. We call this process Reduction B.

Proposition 2.4 *If $a_{ij} = 0$ $(j > J)$, namely, $v_j^i = 0$ $(j > J)$, for a positive integer J, then*

$$v_{J-j}^i = 0 \quad (0 \le j \le n, \quad 0 \le i \le 2n+1-2j),$$

where $n := [(J-1)/2]$.

Proof. We prove the proposition by the following induction. If $a_{ij} = 0$ $(j > J)$ i.e. $v_j^i = 0$ $(j > J)$, then by putting $k = 0$ in (2.8) and (2.9), we have

$$v_J^0 = v_J^1 = 0,$$

because $\alpha \ne 0$.

Suppose $v_{J-j}^i = 0$ $(0 \le j \le k-1, 0 \le i \le 2k-1-2j)$ for k such that $1 \le k \le n = [(J-1)/2]$. Then from (2.8) and (2.9), we obtain

$$(2.11) \qquad \begin{pmatrix} v_J^{2k} & v_{J-1}^{2k-2} & \cdots & v_{J-k}^0 \end{pmatrix} \left(\frac{1}{j!}\partial_\alpha^j(\alpha^{J-i}) \right)_{0 \le i \le k, 0 \le j \le J-k-1} = 0,$$

$$(2.12) \qquad \begin{aligned} &\begin{pmatrix} v_J^{2k+1} & v_{J-1}^{2k-1} & \cdots & v_{J-k}^1 \end{pmatrix} \left(\frac{1}{j!}\partial_\alpha^j(\alpha^{J-i}) \right)_{0 \le i \le k, 0 \le j \le J-k-1} \\ &+ \begin{pmatrix} v_J^{2k} & v_{J-1}^{2k-2} & \cdots & v_{J-k}^0 \end{pmatrix} \left(\frac{1}{j!}\beta\partial_\alpha^{j+1}(\alpha^{J-i}) \right)_{0 \le i \le k, 0 \le j \le J-k-1} = 0. \end{aligned}$$

Now from $k \le n = [(J-1)/2]$, we get $J - k - 1 \ge k$. So, from (2.11), we have

$$\begin{pmatrix} v_J^{2k} & v_{J-1}^{2k-2} & \cdots & v_{J-k}^0 \end{pmatrix} \left(\frac{1}{j!}\partial_\alpha^j(\alpha^{J-i}) \right)_{0 \le i,j \le k} = 0.$$

Since

$$\det\left(\frac{1}{j!}\partial_\alpha^j(\alpha^{J-i})\right)_{0\le i,j\le k} = \alpha^{(k+1)(J-k)} \neq 0,$$

we obtain

$$v_{J-j}^{2k-2j} = 0 \quad (0 \le j \le k). \tag{2.13}$$

In the same way, from (2.12) and (2.13), we obtain

$$v_{J-j}^{(2k+1)-2j} = 0 \quad (0 \le j \le k),$$

therefore

$$v_{J-j}^{i} = 0 \quad (0 \le j \le k, 0 \le i \le 2k+1-2j).$$

Then we can prove the proposition by induction with respect to k.

Proof of Proposition 2.3 We want to show that we can reduce a polynomial f satisfying (2.10) to the state $S(m,2m)$ by a successive use of Reductions A and B. We say that a state $S(k,l)$ is *reducible* if Reduction A or B is possible and it is *irreducible* if neither Reduction A nor B is possible. Then, a necessary and sufficient condition for a state $S(k,l)$ to be reducible is $l \ge 2k+1$ or $2[(l+1)/2] \ge 3m-k+1$, and hence $S(0,3m)$ is reducible and $S(m,2m)$ is irreducible.

Let us consider a set Σ of all states $S(k,l)$ such that

$$0 \le k \le 3m, \quad 2m \le l \le 3m, \quad l \ge 2k-1, \quad l \ge 3m-k-2.$$

We see that every state in Σ except $S(m,2m)$ is reducible and Σ is *stable* under Reductions A and B, which means that every state in $\Sigma - \{S(m,2m)\}$ is reduced to a state or states in Σ by Reductions A and B, by noting that Reduction A is impossible for $S(k,l)$ with $l = 2k-1$ or $l = 2k$ and Reduction B is impossible for it with $l = 3m-k-2$ or $l = 3m-k-1$.

We introduce a linear order $\succ$ in the set Σ by: $S(k,l) \succ S(k',l')$ if and only if $l > l'$, or $l = l'$ and $l-k > l'-k'$. Then we see that $S(0,3m)$ is the highest state and $S(m,2m)$ is the lowest one with respect to the order, and moreover, Reductions A and B reduce a state in $\Sigma - \{S(m,2m)\}$ to strictly lower ones in Σ. By virtue of these properties, we can verify that there exists a chain of Reductions A and B which reduces $S(0,3m)$ to $S(m,2m)$. Thus we have proved that if f satisfies (2.10) then it must be in the state $S(m,2m)$. Thus we have proved Proposition 2.3. □

Proof of Proposition 2.1 From Propositions 2.2 and 2.3, we have already obtained $a_{ij} = 0 \quad (i-j \quad \text{or} \quad j-i > m)$ and $a_{ij} = 0 \quad (j > 2m)$ *i.e.* $v_j^i = 0 \quad (j > 2m)$. So from

Proposition 2.4, we have

$$v^i_{2m-j} = 0 \quad (0 \le j \le m-1, \quad 0 \le i \le 2m-1-2j). \tag{2.14}$$

By letting $J = 2m, k = m$ in (2.8) and (2.9), from (2.14) we have

$$\begin{pmatrix} v^{2m}_{2m} & v^{2m-2}_{2m-1} & \cdots & v^0_m \end{pmatrix} \left(\frac{1}{j!} \partial^j_\alpha (\alpha^{2m-i}) \right)_{0 \le i \le m, 0 \le j \le m-1} = 0, \tag{2.15}$$

$$\begin{aligned} &\begin{pmatrix} v^{2m+1}_{2m} & v^{2m-1}_{2m-1} & \cdots & v^1_m \end{pmatrix} \left(\frac{1}{j!} \partial^j_\alpha (\alpha^{2m-i}) \right)_{0 \le i \le m, 0 \le j \le m-1} \\ &+ \begin{pmatrix} v^{2m}_{2m} & v^{2m-2}_{2m-1} & \cdots & v^0_m \end{pmatrix} \left(\frac{1}{j!} \beta \partial^{j+1}_\alpha (\alpha^{2m-i}) \right)_{0 \le i \le m, 0 \le j \le m-1} = 0. \end{aligned} \tag{2.16}$$

Now we set

$$M_k := \operatorname{diag}(\alpha^{-m}, \alpha^{-m+1}, \cdots, \alpha^{-1}) \begin{pmatrix} 1 & (-1) & \cdots & (-1)^{m-1} \\ & 1 & \ddots & \vdots \\ & & \ddots & (-1) \\ & & & 1 \end{pmatrix}^k .$$

Multiplying both sides of (2.15) on the right by the matrix M_m, we obtain

$$\begin{pmatrix} v^{2m}_{2m} & v^{2m-2}_{2m-1} & \cdots & v^0_m \end{pmatrix} \left(\binom{m-i}{j} \alpha^{m-i} \right)_{0 \le i \le m, 0 \le j \le m-1} = 0.$$

Therefore, by observing the $(m-1)$-th component, we obtain

$$m\alpha v^{2m}_{2m} + v^{2m-2}_{2m-1} = 0, \tag{2.17}$$

because $\alpha \neq 0$. Here we note

$$\begin{pmatrix} v^{2m}_{2m} & v^{2m-2}_{2m-1} & \cdots & v^0_m \end{pmatrix} \left(\binom{2m-k-i}{j} \alpha^{m-i} \right)_{0 \le i \le m, 0 \le j \le m-1} = 0, \tag{2.18}$$

which is obtained by multiplying (2.15) on the right by M_k $(0 \le k \le m)$.

Next we observe the second row vector on the left-hand side of (2.16):

$$
\begin{aligned}
&(v_{2m}^{2m} \quad v_{2m-1}^{2m-2} \quad \cdots \quad v_m^0) \left(\frac{1}{j!}\beta\partial_\alpha^{j+1}(\alpha^{2m-i}) \right)_{0\le i\le m, 0\le j\le m-1} \\
&= \beta (v_{2m}^{2m} \quad v_{2m-1}^{2m-2} \quad \cdots \quad v_m^0) \left(0 \ \cdots \ 0 \ \left(m \tbinom{2m-i}{m} \alpha^{m-i} \right)_{0\le i\le m} \right) \\
&= \beta (v_{2m}^{2m} \quad v_{2m-1}^{2m-2} \quad \cdots \quad v_m^0) \\
&\qquad \left(0 \ \cdots \ 0 \ \left(m\left\{ \tbinom{2m-i}{m} + \sum_{l=1}^{m}(-1)^l \sum_{k=0}^{m-1} \tbinom{2m-k-i}{m-l} \right\}\alpha^{m-i} \right)_{0\le i\le m} \right) \\
&= \beta (v_{2m}^{2m} \quad v_{2m-1}^{2m-2} \quad \cdots \quad v_m^0) \left(0 \ \cdots \ 0 \ \left(m \tbinom{m-i}{m} \alpha^{m-i} \right)_{0\le i\le m} \right).
\end{aligned}
$$

Here we used (2.15) and (2.18) in the first and the second equalities respectively. Therefore, (2.16) is written as

$$
\begin{aligned}
&(v_{2m}^{2m+1} \quad v_{2m-1}^{2m-1} \quad \cdots \quad v_m^1) \left(\frac{1}{j!}\partial_\alpha^{j}(\alpha^{2m-i}) \right)_{0\le i\le m, 0\le j\le m-1} \\
&+ \beta (v_{2m}^{2m} \quad v_{2m-1}^{2m-2} \quad \cdots \quad v_m^0) \left(0 \ \cdots \ 0 \ \left(m \tbinom{m-i}{m} \alpha^{m-i} \right)_{0\le i\le m} \right) = 0.
\end{aligned}
\tag{2.19}
$$

Multiplying (2.19) on the right by M_m, we obtain

$$
\begin{aligned}
&(v_{2m}^{2m+1} \quad v_{2m-1}^{2m-1} \quad \cdots \quad v_m^1) \left(\binom{m-i}{j} \alpha^{m-i} \right)_{0\le i\le m, 0\le j\le m-1} \\
&+ \beta (v_{2m}^{2m} \quad v_{2m-1}^{2m-2} \quad \cdots \quad v_m^0) \left(0 \ \cdots \ 0 \ \left(m \tbinom{m-i}{m} \alpha^{m-i-1} \right)_{0\le i\le m} \right) = 0.
\end{aligned}
$$

By comparing the $(m-1)$-th component of the system, we have

$$
m\alpha v_{2m}^{2m+1} + v_{2m-1}^{2m-1} + m\beta v_{2m}^{2m} = 0, \tag{2.20}
$$

since $\alpha \neq 0$.

And by putting $\mu = m, l = m-1$ in (2.4), we have

$$
a_{3m-1,2m-1} + \left(2m\,\varepsilon - m\kappa_\infty\right) a_{3m,2m} = 0. \tag{2.21}
$$

By Proposition 2.3 and the equation (2.6) for $\nu = m, k = m-1$, we have

$$
a_{m-1,2m-1} + m\kappa_0 a_{m,2m} = 0. \tag{2.22}
$$

Thus we have obtained (2.14), (2.17), (2.20), (2.21), and (2.22), from which we show the vanishing of the remaining a_{ij}'s. These equations is written as

$$(u_1 \quad u_2)\,\Phi_V = 0,$$

where $u_1 := (a_{m,2m} \;\; a_{m+1,2m} \;\; \cdots \;\; a_{3m,2m})$, $u_2 := (a_{m-1,2m-1} \;\; a_{m,2m-1} \;\; \cdots \;\; a_{3m-1,2m-1})$, and

$$\Phi_V := \begin{pmatrix} \varphi_1 & 0 & \varphi_3 & \varphi_5 & \varphi_7 & \varphi_9 \\ 0 & \varphi_2 & \varphi_4 & \varphi_6 & \varphi_8 & \varphi_{10} \end{pmatrix},$$

$$\varphi_1 := \left(\binom{m+i}{j}\right)_{0\le i\le 2m, 0\le j\le 2m-1}, \quad \varphi_2 := \left(\binom{m-1+i}{j}\right)_{0\le i\le 2m, 0\le j\le 2m-3},$$

$$\varphi_3 := \left(m\alpha\binom{m+i}{2m}\right)_{0\le i\le 2m}, \quad \varphi_4 := \left(\binom{m-1+i}{2m-2}\right)_{0\le i\le 2m},$$

$$\varphi_5 := \left(m\beta\binom{m+i}{2m} + m\alpha\binom{m+i}{2m+1}\right)_{0\le i\le 2m}, \quad \varphi_6 := \left(\binom{m-1+i}{2m-1}\right)_{0\le i\le 2m},$$

$$\varphi_7 := {}^t(m\kappa_0 \;\; 0 \;\; \cdots \;\; 0), \quad \varphi_8 := {}^t(1 \;\; 0 \;\; \cdots \;\; 0),$$

$$\varphi_9 := {}^t(0 \;\; \cdots \;\; 0 \;\; 2m\varepsilon - m\kappa_\infty), \quad \varphi_{10} := {}^t(0 \;\; \cdots \;\; 0 \;\; 1).$$

Since

$$\det\Phi_V = m \neq 0,$$

we obtain $u_1 = 0$ and $u_2 = 0$, which is equivalent to

$$a_{m+i,2m} = a_{(m-1)+i,2m-1} = 0 \quad (0 \le i \le 2m). \tag{2.23}$$

Now, from the equation (2.4) for $\mu = m, l = m-2$ and (2.23), we obtain

$$a_{3m-2,2m-2} = 0,$$

which proves Proposition 2.1. □

Thus we have completed the proof of THEOREM 2 in the case of $J = V$.

3. Proof of THEOREM 2 for $J = IV$

In this section we prove THEOREM 2 in the case of $J = IV$ by the same way as that in the previous section. Let f be a function holomorphic and algebraic on E_{IV}. We expand f into power series of x and y as (2.1) convergent for any $x, y \in \mathbf{C}$ and $t \in B_{IV}$, where $(x, y, t) = (x(00), y(00), t)$ is the coordinate system of $V(00) \times B_{IV}$.

3.1 Linear equations for $a_{ij}(t)$

We first study the function f in $V(\infty 0)\times B_{IV}\ni(X,Y,t)$. From (0.8), we have

$$f\equiv\sum_{\mu=1}^{\infty}\sum_{k=0}^{\mu-1}\sum_{j=0}^{\infty}a_{\mu+j,j}\frac{(-1)^k}{k!}\frac{\partial^k}{\partial\kappa_\infty{}^k}(\kappa_\infty^j)\frac{Y^k}{X^{\mu-k}}.$$

Because f is holomorphic on $X=0$ by assumption, we obtain

$$(3.1)\qquad \begin{pmatrix}a_{\mu,0} & a_{\mu+1,1} & \cdots\end{pmatrix}\left(\frac{1}{j!}\partial^j_{\kappa_\infty}(\kappa_\infty^i)\right)_{i\geq 0,0\leq j\leq\mu-1}=0\quad(\mu\geq 1).$$

Secondly we study f in $V(0\infty)\times B_{IV}\ni(X,Y,t)$. From (0.7), we have

$$f\equiv\sum_{\nu=1}^{\infty}\sum_{k=0}^{\nu-1}\sum_{i=0}^{\infty}a_{i,\nu+i}\frac{(-1)^k}{k!}\frac{\partial^k}{\partial\kappa_0^k}(\kappa_0^i)\frac{X^k}{Y^{\nu-k}}.$$

Therefore we have

$$(3.2)\qquad \begin{pmatrix}a_{0,\nu} & a_{1,\nu+1} & \cdots\end{pmatrix}\left(\frac{1}{j!}\partial^j_{\kappa_0}(\kappa_0^i)\right)_{i\geq 0,0\leq j\leq\nu-1}=0\quad(\nu\geq 1).$$

Lastly we study f in $V(\infty\infty)\times B_{IV}\ni(X,Y,t)$. We notice that the transformation between (x,y,t) and (X,Y,t) is: $x=1/X, y=\alpha/X+t+\beta X-X^2Y$, where $\alpha:=1/2,\beta:=\kappa_0-\kappa_\infty-1$. Here we assume that $a_{ij}=0\quad(i+j>M)$, for some positive integer M greater than 2. Then we have

$$f=\sum_{\xi=0}^{M}\sum_{j=0}^{\xi}a_{\xi-j,j}\sum_{0\leq q+r+s,q,r,s\leq j}\frac{t^q\beta^r(-1)^s}{q!r!s!}\frac{\partial^{q+r+s}}{\partial\alpha^{q+r+s}}(\alpha^j)\frac{Y^s}{X^{\xi-q-2r-3s}},$$

where for a function $g(x)$ we denote $\partial g(x)/\partial x|_{x=\alpha}$ by $\partial g(\alpha)/\partial\alpha$. In the rest of this subsection, we obtain linear equations for $a_{ij}(t)$ from the condition that f is holomorphic on $X=0$. From the condition that all the coefficients of $Y^s/X^{M-3s}\quad(0\leq s\leq\rho_1)$ must be zero, where $\rho_1=[(M+2)/3]-1$, it follows that

$$(3.3)\qquad \begin{pmatrix}a_{M,0} & a_{M-1,1} & \cdots & a_{0,M}\end{pmatrix}\left(\frac{1}{j!}\partial^j_\alpha(\alpha^i)\right)_{0\leq i\leq M,0\leq j\leq\rho_1}=0.$$

By observing the coefficients of $Y^s/X^{M-1-3s}\quad(0\leq s\leq\rho_2)$ with $\rho_2:=[(M+1)/3]-1$, we have

$$(3.4)\qquad \begin{aligned}&\begin{pmatrix}a_{M,0} & a_{M-1,1} & \cdots & a_{0,M}\end{pmatrix}\left(t\frac{1}{j!}\partial^{j+1}_\alpha(\alpha^i)\right)_{0\leq i\leq M,0\leq j\leq\rho_2}\\&+\begin{pmatrix}a_{M-1,0} & a_{M-2,1} & \cdots & a_{0,M-1}\end{pmatrix}\left(\frac{1}{j!}\partial^j_\alpha(\alpha^i)\right)_{0\leq i\leq M-1,0\leq j\leq\rho_2}=0.\end{aligned}$$

By observing the coefficients of Y^s/X^{M-2-3s} $(0 \le s \le \rho_3)$ with $\rho_3 := [M/3]-1$, we have

$$\begin{aligned}
&(a_{M,0} \quad a_{M-1,1} \quad \cdots \quad a_{0,M})\left((\beta + \frac{t^2}{2!}\partial_\alpha)\frac{1}{j!}\partial_\alpha^{j+1}(\alpha^i)\right)_{0\le i\le M, 0\le j\le \rho_3} \\
(3.5) \qquad &+ (a_{M-1,0} \quad a_{M-2,1} \quad \cdots \quad a_{0,M-1})\left(t\frac{1}{j!}\partial_\alpha^{j+1}(\alpha^i)\right)_{0\le i\le M-1, 0\le j\le \rho_3} \\
&+ (a_{M-2,0} \quad a_{M-3,1} \quad \cdots \quad a_{0,M-2})\left(\frac{1}{j!}\partial_\alpha^{j}(\alpha^i)\right)_{0\le i\le M-2, 0\le j\le \rho_3} = 0.
\end{aligned}$$

3.2 Reduction of f to a polynomial

Since $\overline{V(00)} \times B_{IV}$ contains a divisor $\{(x_1, y_1, t) \in \mathbf{C}^2 \times B_{IV} \mid y_1 = 0, x_1 \ne 0\}$ where $x = x_1, y = 1/y_1$, by our assumption, $f(x_1, 1/y_1, t)$ must be meromorphic on $y_1 = 0, x_1 \ne 0$, which implies that $a_{ij} = 0 \quad (j > N)$ for some positive integer N. Therefore, by (3.1) for every $\mu > N$, we have $a_{\mu+j,j} = 0 \quad (0 \le j \le N, \mu > N)$, which shows that f is a polynomial of x and y.

3.3 Completion of the Proof of THEOREM 2 for $J = IV$

In order to prove the theorem for $J = IV$, it is sufficient to show

Proposition 3.1 *For every positive integer m, $a_{ij} = 0 \quad (i \text{ or } j > 2m)$ implies*

$$a_{ij} = 0 \quad (i \text{ or } j > 2m-2)$$

From now on, we assume

$$a_{ij} = 0 \quad (i \text{ or } j > 2m) \tag{3.6}$$

for an arbitrary fixed positive integer m. We first obtain

Proposition 3.2 $\quad a_{i,j} = 0 \quad (i - j > m)$.

Proof. Under the assumption (3.6), we have

$$(a_{\mu,0} \quad a_{\mu+1,1} \quad \cdots \quad a_{2m,2m-\mu})\left(\frac{1}{j!}\partial_{\kappa_\infty}^j(\kappa_\infty^i)\right)_{0\le i\le 2m-\mu, 0\le j\le \mu-1} = 0$$

from (3.1). Therefore, for every $\mu > m$, we obtain

$$a_{\mu+j,j} = 0 \quad (0 \le j \le 2m-\mu). \qquad \square$$

Secondly we obtain

Proposition 3.3 $a_{i,j} = 0 \quad (j - i > m)$.

Proof. Under the assumption (3.6), we have

$$\begin{pmatrix} a_{0,\nu} & a_{1,\nu+1} & \cdots & a_{2m-\nu,2m} \end{pmatrix} \left(\frac{1}{j!} \partial^j_{\kappa_0}(\kappa^i_0) \right)_{0 \le i \le 2m-\nu, 0 \le j \le \nu-1} = 0$$

from (3.2). Therefore, for every $\nu > m$, we obtain

$$a_{i,\nu+i} = 0 \quad (0 \le i \le 2m - \nu). \qquad \square$$

Thirdly we obtain

Proposition 3.4 $a_{ij} = 0 \quad (i + j > 3m)$.

Proof. We prove this proposition by induction. For an arbitrary fixed integer k such that $3m < k \le 4m$, we assume $a_{ij} = 0 \quad (i + j > k)$. By this assumption and (3.6), from (3.3) we have

$$\begin{pmatrix} a_{2m,k-2m} & \cdots & a_{k-2m,2m} \end{pmatrix} \left(\frac{1}{j!} \partial^j_\alpha (\alpha^{i+k-2m}) \right)_{0 \le i,j \le 4m-k} = 0,$$

because $4m - k < [(k + 2)/3] - 1$. Hence we obtain

$$a_{k-j,j} = 0 \quad (k - 2m \le j \le 2m) \quad \text{i.e.} \quad a_{ij} = 0 \quad (i + j = k).$$

Then we can prove the proposition by induction with respect to k. $\square$

Proof of Proposition 3.1 From Propositions 3.2, 3.3, and 3.4,

$$a_{ij} = 0 \quad (i - j > m \quad \text{or} \quad j - i > m \quad \text{or} \quad i + j > 3m). \tag{3.7}$$

Therefore, from (3.3) and (3.7), we first have

$$u_1 \varphi_1 := u_1 \left(\frac{1}{j!} \partial^j_\alpha (\alpha^{i+m}) \right)_{0 \le i \le m, 0 \le j \le m-1} = 0, \tag{3.8}$$

where $u_1 := \begin{pmatrix} a_{2m,m} & a_{2m-1,m+1} & \cdots & a_{m,2m} \end{pmatrix}$. Secondly from (3.4) and (3.7), we have

$$u_1 \varphi_2 + u_2 \varphi_3 = 0, \tag{3.9}$$

where $u_2 := \begin{pmatrix} a_{2m-1,m} & a_{2m-2,m+1} & \cdots & a_{m,2m-1} \end{pmatrix}$, and

$$\varphi_2 := \left(\frac{t}{j!} \partial^{j+1}_\alpha (\alpha^{i+m}) \right)_{0 \le i \le m, 0 \le j \le m-1}, \quad \varphi_3 := \left(\frac{1}{j!} \partial^j_\alpha (\alpha^{i+m}) \right)_{0 \le i,j \le m-1}.$$

Thirdly, from (3.5) and (3.7), we obtain

$$u_1\varphi_4 + u_2\varphi_5 + u_3\varphi_6 = 0. \tag{3.10}$$

where $u_3 := (\,a_{2m-1,m-1} \quad a_{2m-2,m} \quad \cdots \quad a_{m-1,2m-1}\,)$ and

$$\varphi_4 := \left((\beta + \frac{t^2}{2!}\partial_\alpha)\frac{1}{j!}\partial_\alpha^{j+1}(\alpha^{i+m})\right)_{0\le i\le m, 0\le j\le m-1},$$
$$\varphi_5 := \left(\frac{t}{j!}\partial_\alpha^{j+1}(\alpha^{i+m})\right)_{0\le i,j\le m-1}, \quad \varphi_6 := \left(\frac{1}{j!}\partial_\alpha^{j}(\alpha^{i+m-1})\right)_{0\le i\le m, 0\le j\le m-1}.$$

Lastly, we have

$$a_{m-1,2m-1} + m\kappa_\infty a_{m,2m} = 0 \tag{3.11}$$

from (3.1) for $\mu = m$ and (3.7). And we have

$$a_{m-1,2m-1} + m\kappa_0 a_{m,2m} = 0 \tag{3.12}$$

from (3.2) for $\nu = m$ and (3.7).

Now we have obtained a $(3m+2)$ -system for $(3m+2)$ unknowns. By using the equations (3.8), (3.9), (3.10), (3.11), and (3.12), we have

$$(\,u_1 \quad u_2 \quad u_3\,)\,\Phi_{IV} = 0,$$

where $$\Phi_{IV} := \begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_4 & \varphi_7 & \varphi_9 \\ 0 & \varphi_3 & \varphi_5 & 0 & 0 \\ 0 & 0 & \varphi_6 & \varphi_8 & \varphi_{10} \end{pmatrix},$$

$$\varphi_7 := {}^t(\,m\kappa_0 \quad 0 \quad \cdots \quad 0\,), \quad \varphi_8 := {}^t(\,1 \quad 0 \quad \cdots \quad 0\,),$$
$$\varphi_9 := {}^t(\,0 \quad \cdots \quad 0 \quad m\kappa_0\,), \quad \varphi_{10} := {}^t(\,0 \quad \cdots \quad 0 \quad 1\,).$$

Since

$$\det\Phi_{IV} = m(-1)^{m-1}\alpha^{3m^2} \neq 0,$$

we obtain

$$a_{ij} = 0 \quad (i + j \ge 3m - 2),$$

which yields Proposition 3.1. □

Thus we have completed the proof of THEOREM 2 in the case of $J = IV$.

4. Proof of the THEOREM 2 for $J = III$

In this section we prove THEOREM 2 in the case of $J = III$. Let f be a holomorphic and algebraic function on E_{III}. We expand f into power series of x and y as (2.1) convergent for any $x, y \in \mathbf{C}$ and $t \in B_{III}$, where $(x, y, t) = (x(00), y(00), t)$ is the coordinate system of $V(00) \times B_{III}$.

4.1 Linear equations for $a_{ij}(t)$

First, from (0.12), the function f in $V(\infty 0) \times B_{III} \ni (X, Y, t)$ is given by

$$f \equiv \sum_{\mu=1}^{\infty} \sum_{k=0}^{\mu-1} \sum_{j=0}^{\infty} a_{\mu+j,j} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial \varepsilon^k} (\varepsilon^j) \frac{Y^k}{X^{\mu-k}},$$

where ε is a constant given in (0.15). Since f is holomorphic on $X = 0$ by assumption, we obtain

$$(4.1) \qquad \begin{pmatrix} a_{\mu,0} & a_{\mu+1,1} & \cdots \end{pmatrix} \left(\frac{1}{j!} \partial_\varepsilon^j (\varepsilon^i) \right)_{i \geq 0, 0 \leq j \leq \mu-1} = 0 \quad (\mu \geq 1).$$

Secondly we study f in $V(0\infty) \times B_{III} \ni (X, Y, t)$. We notice that the transformation between (x, y, t) and (X, Y, t) is : $x = X, y = \alpha/X^2 + \beta/X + Y$, where $\alpha := -\eta_0 t, \beta := \kappa_0 + 1$. We assume that $a_{ij} = 0 \quad (\nu := 2j - i > M)$, M being an integer greater than 1. Then we obtain

$$f \equiv \sum_{\nu=1}^{M} \sum_{j=[(\nu+1)/2]}^{\infty} a_{2j-\nu,j} \sum_{0 \leq q,r,q+r \leq j} \frac{1}{q!r!} \beta^q \partial_\alpha^{q+r} (\alpha^j) \frac{Y^r}{X^{\nu-q-2r}}.$$

At first, from the condition that all the coefficients of $Y^r/X^{M-2r} \quad (0 \leq r \leq \rho_1)$ must be zero, where $\rho_1 := [(M+1)/2] - 1$, it follows that

$$(4.2) \qquad \begin{pmatrix} a_{2\rho_1+2-M,\rho_1+1} & a_{2\rho_1+4-M,\rho_1+2} & \cdots \end{pmatrix} \left(\frac{1}{j!} \partial_\alpha^j (\alpha^{i+\rho_1+1}) \right)_{i \geq 0, 0 \leq j \leq \rho_1} = 0.$$

And next from the condition that all the coefficients of $Y^r/X^{M-2r-1} \quad (0 \leq r \leq \rho_2)$ must be zero, where $\rho_2 = [M/2] - 1$, we have

$$(4.3) \qquad \begin{aligned} &\begin{pmatrix} a_{2\rho_1+2-M,\rho_1+1} & a_{2\rho_1+4-M,\rho_1+2} & \cdots \end{pmatrix} \left(\frac{\beta}{j!} \partial_\alpha^{j+1} (\alpha^{i+\rho_1+1}) \right)_{i \geq 0, 0 \leq j \leq \rho_2} \\ &+ \begin{pmatrix} a_{2\rho_2+3-M,\rho_2+1} & a_{2\rho_2+5-M,\rho_2+2} & \cdots \end{pmatrix} \left(\frac{1}{j!} \partial_\alpha^j (\alpha^{i+\rho_2+1}) \right)_{i \geq 0, 0 \leq j \leq \rho_2} = 0. \end{aligned}$$

Lastly, we study f in $V(\infty\eta_\infty t) \times B_{III} \in (X, Y, t)$. The transformation between (x, y, t) and (X, Y, t) is: $x = 1/X, y = a + bX - X^2 Y$, where $a := \eta_\infty t, b := \varepsilon - \kappa_\infty$. We assume that

$a_{ij} = 0 \quad (i > I)$, where I is the integer greater than 1. Then we obtain

$$f = \sum_{j=0}^{\infty} \sum_{i=0}^{I} a_{ij} \sum_{0 \le q,r,q+r \le j} \frac{b^q (-1)^r}{q! r!} \partial_a^{q+r}(a^j) \frac{Y^r}{X^{i-q-2r}}.$$

By observing the coefficients of $Y^r/X^{I-2r} \quad (0 \le r \le \sigma_1)$, where $\sigma_1 := [(I+1)/2] - 1$, we obtain

$$(4.4) \qquad \begin{pmatrix} a_{I,0} & a_{I,1} & \cdots \end{pmatrix} \left(\frac{1}{j!} \partial_a^j a^i \right)_{i \ge 0, 0 \le j \le \sigma_1} = 0.$$

And by observing the coefficients of $Y^r/X^{I-2r-1} \quad (0 \le r \le \sigma_2)$, where $\sigma_2 := [I/2] - 1$, we obtain

$$(4.5) \qquad \begin{aligned} &\begin{pmatrix} a_{I,0} & a_{I,1} & \cdots \end{pmatrix} \left(\frac{b}{j!} \partial_a^{j+1} a^i \right)_{i \ge 0, 0 \le j \le \sigma_2} \\ &\quad + \begin{pmatrix} a_{I-1,0} & a_{I-1,1} & \cdots \end{pmatrix} \left(\frac{1}{j!} \partial_a^j a^i \right)_{i \ge 0, 0 \le j \le \sigma_2} = 0. \end{aligned}$$

4.2 Reduction of f to a polynomial

Since $\overline{V(00)} \times B_{III}$ contains a divisor $\{(x_1, y_1, t) \in \mathbf{C}^2 \times B_{III} \mid y_1 = 0, x_1 \neq 0\}$ where $(x = x_1, y = 1/y_1)$, by our assumption, $f(x_1, 1/y_1, t)$ must be meromorphic on $y_1 = 0, x_1 \neq 0$, which implies that $a_{ij} = 0 \quad (j > N)$, for some positive integer N. And by (4.1) for every $\mu > N$, we have $a_{\mu+j,j} = 0 \quad (0 \le j \le N, \quad \mu > N)$, which shows that f is a polynomial on x and y.

4.3 Completion of the Proof of THEOREM 2 for $J = III$

In order to prove the theorem for $J = III$, it is sufficient to show

Proposition 4.1 *For every positive integer m, $a_{ij} = 0 \quad (i \quad or \quad j > 2m)$ implies*

$$a_{ij} = 0 \quad (i \quad or \quad j > 2m-2)$$

From now on, we assume

$$a_{ij} = 0 \quad (i \quad \text{or} \quad j > 2m) \tag{4.6}$$

for an arbitrary fixed positive integer m. We first obtain

Proposition 4.2 $a_{i,j} = 0 \quad (i - j > m)$.

Proof. Under the assumption (4.6), we have

$$\begin{pmatrix} a_{\mu,0} & a_{\mu+1,1} & \cdots & a_{2m,2m-\mu} \end{pmatrix} \left(\frac{1}{j!} \partial_\varepsilon^j (\varepsilon^i) \right)_{0 \le i \le 2m-\mu, 0 \le j \le \mu-1} = 0$$

from (4.1). Therefore, for every $\mu > m$, we have

$$a_{\mu+j,j} = 0 \quad (0 \le j \le 2m - \mu). \qquad \square$$

Secondly we obtain

Proposition 4.3 $a_{i,j} = 0 \quad (2j - i > 2m)$.

Proof. We prove the proposition by the following induction. For arbitrary fixed integer k such that $m < k \le 2m$, we assume $a_{ij} = 0 \quad (\nu := 2j - i > 2k)$. By this assumption and (4.6), from (4.2) we have

$$\begin{pmatrix} a_{0,k} & a_{2,k+1} & \cdots & a_{4m-2k,2m} \end{pmatrix} \left(\frac{1}{j!} \partial_\alpha^j (\alpha^{i+k}) \right)_{0 \le i,j \le 2m-k} = 0,$$

because $2m - k \le k - 1$. Hence we obtain

$$a_{ij} = 0 \quad (2j - i = 2k).$$

And in the same way, from (4.3) we obtain

$$a_{ij} = 0 \quad (2j - i = 2k - 1).$$

Then we can prove the proposition by induction with respect to k. $\square$

Proof of Proposition 4.1 From Propositions 4.2 and 4.3, we have

$$a_{ij} = 0 \quad (i - j > m \quad \text{or} \quad 2j - i > 2m \quad \text{or} \quad i > 2m \quad \text{or} \quad j > 2m). \tag{4.7}$$

Then from (4.2) and (4.7) we first obtain

$$u_1\varphi_1 := u_1 \left(\frac{1}{j!}\partial_\alpha^j(\alpha^{i+m})\right)_{0\leq i\leq m, 0\leq j\leq m-1} = 0, \tag{4.8}$$

where $u_1 := (\, a_{0,m} \quad a_{2,m+1} \quad \cdots \quad a_{2m,2m}\,)$.

Secondly from (4.3) and (4.7) we obtain

$$u_1\varphi_2 + u_2\varphi_3 = 0, \tag{4.9}$$

where $u_2 := (\, a_{1,m} \quad a_{3,m+1} \quad \cdots \quad a_{2m-1,2m-1}\,)$, and

$$\varphi_2 := \left(\frac{\beta}{j!}\partial_\alpha^{j+1}(\alpha^{i+m})\right)_{0\leq i\leq m, 0\leq j\leq m-1}, \varphi_3 := \left(\frac{1}{j!}\partial_\alpha^j(\alpha^{i+m})\right)_{0\leq i,j\leq m-1}.$$

Thirdly from (4.4) and (4.7) we obtain

$$u_3\varphi_4 := u_3 \left(\frac{1}{j!}\partial_a^j(a^{i+m})\right)_{0\leq i\leq m, 0\leq j\leq m-1} = 0, \tag{4.10}$$

where $u_3 := (\, a_{2m,m} \quad a_{2m,m+1} \quad \cdots \quad a_{2m,2m}\,)$.

Fourthly from (4.5) and (4.7) we obtain

$$u_3\varphi_5 + u_4\varphi_6 = 0, \tag{4.11}$$

where $u_4 := (\, a_{2m-1,m-1} \quad a_{2m-1,m} \quad \cdots \quad a_{2m-1,2m-1}\,)$ and

$$\varphi_5 := \left(\frac{b}{j!}\partial_a^{j+1}(a^{i+m})\right)_{0\leq i\leq m, 0\leq j\leq m-1}, \varphi_6 := \left(\frac{1}{j!}\partial_a^j(a^{i+m-1})\right)_{0\leq i\leq m, 0\leq j\leq m-1}.$$

Lastly from (4.1) for $\mu = m$ and (4.7), we obtain

$$a_{2m-1,m-1} + m\varepsilon a_{2m,m} = 0. \tag{4.12}$$

Therefore, from (4.8), (4.9), (4.10), (4.11), and (4.12), we have

$$(\, u_1 \quad u_2 \quad u_3 \quad u_4\,)\,\Phi_{III} = 0,$$

$$\text{where} \quad \Phi_{III} := \begin{pmatrix} \varphi_1 & \varphi_2 & 0 & 0 & 0 & \varphi_9 & 0 \\ 0 & \varphi_3 & 0 & 0 & 0 & 0 & \varphi_{11} \\ 0 & 0 & \varphi_4 & \varphi_5 & \varphi_7 & \varphi_{10} & 0 \\ 0 & 0 & 0 & \varphi_6 & \varphi_8 & 0 & \varphi_{12} \end{pmatrix},$$

$$\varphi_7 := {}^t(m\varepsilon \quad 0 \quad \cdots \quad 0), \quad \varphi_8 := {}^t(1 \quad 0 \quad \cdots \quad 0), \quad \varphi_9 := {}^t(0 \quad \cdots \quad 0 \quad 1),$$
$$\varphi_{10} = \varphi_{12} := {}^t(0 \quad \cdots \quad 0 \quad -1), \quad \varphi_{11} := {}^t(0 \quad \cdots \quad 0 \quad 1).$$

Since

$$\det\Phi_{III} = (-1)^{m+1}\alpha^{2m^2}a^{2m^2}m \neq 0,$$

we obtain

$$(u_1 \quad u_2 \quad u_3 \quad u_4) = 0,$$

which proves Proposition 4.1. □

We have thus completed the proof of THEOREM 2 for $J = III$.

5. Proof of THEOREM 2 for $J = II$

Let f be a holomorphic and algebraic function on E_{II}. We expand f into power series of x and y as (2.1) convergent for any $x, y \in \mathbf{C}$ and $t \in B_{II}$, where $(x, y, t) = (x(00), y(00), t)$ is the coordinate system of $V(00) \times B_{II}$.

5.1 Linear equations for $a_{ij}(t)$

We first study the function f in $V(\infty 0) \times B_{II} \ni (X, Y, t)$. From (0.16), we have

$$f \equiv \sum_{\mu=1}^{\infty}\sum_{k=0}^{\mu-1}\sum_{j=0}^{\infty} a_{\mu+j,j}\frac{(-1)^k}{k!}\frac{\partial^k}{\partial\varepsilon^k}(\varepsilon^j)\frac{Y^k}{X^{\mu-k}},$$

where ε is a constant which was given in (0.19). Since f is holomorphic on $X = 0$ by assumption, we obtain

$$\text{(5.1)} \qquad (a_{\mu,0} \quad a_{\mu+1,1} \quad \cdots)\left(\frac{1}{j!}\partial_\varepsilon^j(\varepsilon^i)\right)_{i\geq 0, 0\leq j\leq \mu-1} = 0 \quad (\mu \geq 1).$$

Next we study f in $V(\infty\infty) \times B_{II} \ni (X, Y, t)$. The transformation between (x, y, t) and (X, Y, t) is: $x = 1/X$, $y = a/X^2 + t + bX - X^2Y$,where $a := 2, b := \varepsilon + 2\alpha$. Here if we assume that $a_{ij} = 0 \quad (\xi := 2j + i > M)$ for some positive integer M greater than 3, then we have

$$\text{(5.2)} \qquad f \equiv \sum_{\xi=1}^{M}\sum_{j=0}^{[\xi/2]} a_{\xi-2j,j}\sum_{0\leq q,r,s,q+r+s\leq j}\frac{t^q b^r(-1)^s}{q!r!s!}\partial_a^{q+r+s}(a^j)\frac{Y^s}{X^{\xi-2q-3r-4s}}.$$

In the rest of this subsection, from the condition that f is holomorphic on $X=0$, we obtain linear equations for $a_{ij}(t)$. By observing the coefficients of Y^s/X^{M-4s} $(0\le s\le \rho_1)$ in (5.2), we have

$$(5.3)\qquad \begin{pmatrix} a_{M,0} & a_{M-2,1} & \cdots & a_{M-2\nu_1,\nu_1}\end{pmatrix}\left(\frac{1}{j!}\partial_a^j(a^i)\right)_{0\le i\le\nu_1,0\le j\le\rho_1}=0,$$

where $\rho_1:=[(M+3)/4]-1, \nu_1:=[M/2]$. And by observing the coefficients of Y^s/X^{M-4s-1} $(0\le s\le\rho_2)$ in (5.2), we have

$$(5.4)\qquad \begin{pmatrix} a_{M-1,0} & a_{M-3,1} & \cdots & a_{M-1-2\nu_2,\nu_2}\end{pmatrix}\left(\frac{1}{j!}\partial_a^j(a^i)\right)_{0\le i\le\nu_2,0\le j\le\rho_2}=0,$$

where $\rho_2:=[(M+2)/4]-1$, $\nu_2:=[(M-1)/2]$. And by observing the coefficients of Y^s/X^{M-4s-2} $(0\le s\le\rho_3)$ in (5.2), we have

$$(5.5)\qquad \begin{aligned}&\begin{pmatrix} a_{M,0} & a_{M-2,1} & \cdots & a_{M-2\nu_1,\nu_1}\end{pmatrix}\left(t\frac{1}{j!}\partial_a^{j+1}(a^i)\right)_{0\le i\le\nu_1,0\le j\le\rho_3}\\ &+\begin{pmatrix} a_{M-2,0} & a_{M-4,1} & \cdots & a_{M-2-2\nu_3,\nu_3}\end{pmatrix}\left(\frac{1}{j!}\partial_a^j(a^i)\right)_{0\le i\le\nu_3,0\le j\le\rho_3}=0,\end{aligned}$$

where $\rho_3:=[(M+1)/4]-1$ and $\nu_3:=[(M-2)/2]$. And by observing the coefficients of Y^s/X^{M-4s-3} $(0\le s\le\rho_4)$ in (5.2), we have

$$(5.6)\qquad \begin{aligned}&\begin{pmatrix} a_{M,0} & a_{M-2,1} & \cdots & a_{M-2\nu_1,\nu_1}\end{pmatrix}\left(b\frac{1}{j!}\partial_a^{j+1}(a^i)\right)_{0\le i\le\nu_1,0\le j\le\rho_4}\\ &+\begin{pmatrix} a_{M-1,0} & a_{M-3,1} & \cdots & a_{M-1-2\nu_2,\nu_2}\end{pmatrix}\left(t\frac{1}{j!}\partial_a^{j+1}(a^i)\right)_{0\le i\le\nu_2,0\le j\le\rho_4}\\ &+\begin{pmatrix} a_{M-3,0} & a_{M-5,1} & \cdots & a_{M-3-2\nu_4,\nu_4}\end{pmatrix}\left(\frac{1}{j!}\partial_a^j(a^i)\right)_{0\le i\le\nu_4,0\le j\le\rho_4}=0,\end{aligned}$$

where $\rho_4:=[M/4]-1$ and $\nu_4:=[(M-3)/2]$.

5.2 Reduction of f to a polynomial

Since $\overline{V(00)}\times B_{II}$ contains a divisor $\{(x_1,y_1,t)\in \mathbf{C}^2\times B_{IV}\mid y_1=0\}$ where $x=x_1, y=1/y_1$, by our assumption such that $f(x_1,1/y_1,t)$ must be meromorphic on $y_1=0$, we have $a_{ij}=0$ $(j>N)$, N being some nonnegative integer. And by (5.1) for every $\mu>N$, we have $a_{\mu+j,j}=0$ $(0\le j\le N,\mu>N)$, which shows that f is the polynomial of x and y.

5.3 Completion of the Proof of THEOREM 2 for $J=II$

In order to prove the theorem for $J=II$, it is sufficient to show

Proposition 5.1 *For every positive integer m, $a_{ij} = 0 \quad (i \quad or \quad j > 2m)$, then*

$$a_{ij} = 0 \quad (i \quad or \quad j > 2m - 2).$$

From now on, we assume

$$a_{ij} = 0 \quad (i \quad \text{or} \quad j > 2m) \tag{5.7}$$

for arbitrary fixed positive integer m. We first obtain

Proposition 5.2 $a_{ij} = 0 \quad (i - j > m)$.

Proof. Under the assumption (5.7), we have

$$\begin{pmatrix} a_{\mu,0} & a_{\mu+1,1} & \cdots & a_{2m,2m-\mu} \end{pmatrix} \left(\frac{1}{j!} \partial_\varepsilon^j (\varepsilon^i) \right)_{0 \le i \le 2m-\mu, 0 \le j \le \mu-1} = 0 \quad (\mu \ge 1)$$

from (5.1). Therefore, for every $\mu > m$, we have

$$a_{\mu+j,j} = 0 \quad (0 \le j \le 2m - \mu). \qquad \square$$

Secondly we obtain

Proposition 5.3 $a_{ij} = 0 \quad (2j + i > 4m \quad \text{or} \quad 2j + i = 4m - 1)$.

Proof. We prove this proposition by induction. For arbitrary fixed integer k such that $2m < k \le 3m$, we assume $a_{ij} = 0 \cdot (2j + i > 2k)$. By this assumption and (5.3), (5.7), we have

$$\begin{pmatrix} a_{2m,k-m} & a_{2m-2,k-m+1} & \cdots & a_{2k-4m,2m} \end{pmatrix} \left(\frac{1}{j!} \partial_a^j (a^{i+k-m}) \right)_{0 \le i,j \le 3m-k} = 0,$$

since $3m - k < [(2k + 3)/4] - 1$. Hence we have

$$a_{ij} = 0 \quad (2j + i = 2k).$$

In the same way, from (5.4) and (5.7), we have

$$a_{ij} = 0 \quad (2j + i = 2k - 1).$$

Therefore, by induction with respect to k, we obtain

$$a_{ij} = 0 \quad (2j + i > 4m). \tag{5.8}$$

Next from (5.4) and (5.8), we have

$$\begin{pmatrix} a_{2m-1,m} & a_{2m-3,m+1} & \cdots & a_{1,2m-1} \end{pmatrix} \left(\frac{1}{j!}\partial_a^j(a^{i+m})\right)_{0\leq i,j\leq m-1} = 0.$$

Therefore we have

$$a_{ij} = 0 \quad (2j+i = 4m-1),$$

which completes the proof of the proposition. □

Proof of Proposition 5.1 From Propositions 5.2 and 5.3, we have

$$a_{ij} = 0 \quad (i-j > m \quad \text{or} \quad 2j+i > 4m \quad \text{or} \quad 2j+i = 4m-1). \tag{5.9}$$

From (5.3) and (5.9), we first obtain

$$u_1\varphi_1 := u_1 \left(\frac{1}{j!}\partial_a^j(a^{i+m})\right)_{0\leq i\leq m, 0\leq j\leq m-1} = 0, \tag{5.10}$$

where $u_1 := \begin{pmatrix} a_{2m,m} & a_{2m-2,m+1} & \cdots & a_{0,2m} \end{pmatrix}$.

Secondly, from (5.5) and (5.9), we obtain

$$u_1\varphi_2 + u_2\varphi_3 = 0, \tag{5.11}$$

where $u_2 := \begin{pmatrix} a_{2m-2,m} & a_{2m-4,m+1} & \cdots & a_{0,2m-1} \end{pmatrix}$ and

$$\varphi_2 := \left(t\frac{1}{j!}\partial_a^{j+1}(a^{i+m})\right)_{0\leq i\leq m, 0\leq j\leq m-1}, \quad \varphi_3 := \left(\frac{1}{j!}\partial_a^j(a^{i+m})\right)_{0\leq i,j\leq m-1}.$$

Thirdly, from (5.6) and (5.9), we obtain

$$u_1\varphi_4 + u_3\varphi_5 = 0, \tag{5.12}$$

where $u_3 := \begin{pmatrix} a_{2m-1,m-1} & a_{2m-3,m} & \cdots & a_{1,2m-2} \end{pmatrix}$ and

$$\varphi_4 := \left(b\frac{1}{j!}\partial_a^{j+1}(a^{i+m})\right)_{0\leq i\leq m, 0\leq j\leq m-1}, \quad \varphi_5 := \left(\frac{1}{j!}\partial_a^j(a^{i+m-1})\right)_{0\leq i,j\leq m-1}.$$

Lastly, from (5.1) for $\mu = m$ and (5.9), we have

$$m\varepsilon a_{2m,m} + a_{2m-1,m-1} = 0. \tag{5.13}$$

Therefore, from (5.10), (5.11), (5.12), and (5.13), we obtain

$$\begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \Phi_{II} = 0,$$

where $$\Phi_{II} := \begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_4 & \varphi_6 \\ 0 & \varphi_3 & 0 & 0 \\ 0 & 0 & \varphi_5 & \varphi_7 \end{pmatrix},$$

$$\varphi_6 := {}^t(m\varepsilon \quad 0 \quad \cdots \quad 0), \quad \varphi_7 := {}^t(1 \quad 0 \quad \cdots \quad 0).$$

Since

$$\det\Phi_{II} = (-1)^{m+3} m a^{3m^2} \neq 0,$$

we have

$$a_{ij} = 0 \quad (2j + i = 4m, 4m-2, 4m-3),$$

which proves Proposition 5.1. □

We have thus completed the proof of THEOREM 2 for $J = II$.

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