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CERTAIN SUBCLASSES OF PRESTARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract

The object of the present paper is to investigate coefficient estimates and closure theorems for the subclasses $R_\gamma[A, B, \alpha]$ and $C_\gamma[A, B, \alpha]$ of prestarlike functions with negative coefficients, which are introduced here. Various distortion theorems for fractional calculus of functions $f(z)$ belonging to these subclasses are also proven.

1. Introduction and Definitions

Let A denote the class of (*normalized*) functions of the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$U:=\{z:z\in\mathbb{C} \quad \text{and} \quad |z|<1\},$$

and let S denote the subclass of A consisting of functions which are also *univalent* in U. Then a function $f(z)$ in S is said to be *starlike of order* γ in U if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\gamma \quad (z\in U;\quad 0\leq\gamma<1). \tag{1.2}$$

We denote by $S^*(\gamma)$ the class of all functions in S which are starlike of order γ in U. It is well-known that

$$S^*(\gamma)\subseteq S^*(0)\equiv S^*.$$

The class $S^*(\gamma)$, introduced by Robertson [9], was studied subsequently by Schild [10], MacGregor [3], and Pinchuk [8]. Moreover, the function:

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$$s_\gamma(z):=\frac{z}{(1-z)^{2(1-\gamma)}} \tag{1.3}$$

is the familiar extremal function for the class $S^*(\gamma)$. Setting

$$c(\gamma, n)=\frac{\prod_{k=2}^{n}(k-2\gamma)}{(n-1)!} \quad (n\in\mathbb{N}\setminus\{1\};\quad \mathbb{N}:=\{1,2,3,\cdots\}), \tag{1.4}$$

$s_\gamma(z)$ can be written in the form:

$$s_\gamma(z)=z+\sum_{n=2}^{\infty}c(\gamma, n)z^n. \tag{1.5}$$

We note that $c(\gamma, n)$ is a decreasing function in γ and that

$$\lim_{n\to\infty} c(\gamma, n)=\begin{cases}\infty & \left(\gamma<\frac{1}{2}\right)\\ 1 & \left(\gamma=\frac{1}{2}\right)\\ 0 & \left(\gamma>\frac{1}{2}\right).\end{cases} \tag{1.6}$$

For A and B fixed,

$$-1\leq A<B\leq 1,\quad 0<B\leq 1,\quad \text{and}\quad 0\leq\alpha<1,$$

we say that a function $f(z)\in S$ is in the class $S^*(A, B, \alpha)$ if and only if

$$\frac{zf'(z)}{f(z)}\prec\frac{1+[B+(B-A)(1-\alpha)]z}{1+Bz} \qquad (z\in U)$$

or, equivalently, $f(z)\in S^*(A, B, \alpha)$ if and only if

$$\left|\frac{\frac{zf'(z)}{f(z)}-1}{B\frac{zf'(z)}{f(z)}-[B+(A-B)(1-\alpha)]}\right|<1 \qquad (z\in U). \tag{1.7}$$

Furthermore, a function $f(z)$ is said to belong to the class $K(A, B, \alpha)$ if and only if

$$zf'(z)\in S^*(A, B, \alpha).$$

Let $(f*g)(z)$ denote the Hadamard product (or convolution) of two functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z)=z+\sum_{n=2}^{\infty} b_n z^n, \tag{1.8}$$

then

$$(f*g)(z):=z+\sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.9}$$

Let $\boldsymbol{R}_\gamma(A, B, \alpha)$ be the subclass of $\boldsymbol{A}$ consisting of functions $f(z)$ such that

$$\left|\frac{\dfrac{zh'(z)}{h(z)}-1}{B\dfrac{zh'(z)}{h(z)}-[B+(A-B)(1-\alpha)]}\right|<1 \quad (z\in \boldsymbol{U}) \tag{1.10}$$

$$(A \text{ and } B \text{ fixed};\quad -1\leq A<B\leq 1;\quad 0<B\leq 1;\quad 0\leq\alpha<1),$$

where

$$h(z)=(f*s_\gamma)(z) \quad (0\leq\gamma<1). \tag{1.11}$$

Also let $\boldsymbol{C}_\gamma(A, B, \alpha)$ be the subclass of $\boldsymbol{A}$ consisting of functions $f(z)$ satisfying the condition:

$$zf'(z)\in \boldsymbol{R}_\gamma(A, B, \alpha)$$

$$(-1\leq A<B\leq 1;\quad 0<B\leq 1;\quad 0\leq\alpha<1;\quad 0\leq\gamma<1).$$

We observe that $\boldsymbol{R}_\gamma(-1, 1, \alpha)=\boldsymbol{R}_\gamma(\alpha)$ is the class of γ-prestarlike functions of order α, which was introduced by Sheil-Small *et al.* [11]. Also

$$\boldsymbol{R}_\gamma(-\beta, \beta, \alpha)=\boldsymbol{R}_\gamma(\alpha, \beta) \quad (0<\beta\leq 1)$$

is the class of γ-prestarlike functions of order α and type β, which was studied by Ahuja and Silverman [1].

Finally, let $\boldsymbol{T}$ denote the subclass of $\boldsymbol{A}$ consisting of functions $f(z)$ whose nonzero coefficients, from the second on, are negative. Thus a function $f(z)$, analytic in $\boldsymbol{U}$, is in the class $\boldsymbol{T}$ if it can be expressed as

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n \quad (a_n\geq 0). \tag{1.12}$$

We denote by $\boldsymbol{T}^*(A, B, \alpha)$, $\boldsymbol{C}^*(A, B, \alpha)$, $\boldsymbol{R}_\gamma[A, B, \alpha]$, and $\boldsymbol{C}_\gamma[A, B, \alpha]$ the classes obtained by taking intersections, respectively, of the classes $\boldsymbol{S}^*(A, B, \alpha)$, $\boldsymbol{K}(A, B, \alpha)$, $\boldsymbol{R}_\gamma(A, B, \alpha)$, and $\boldsymbol{C}_\gamma(A, B, \alpha)$ with the class $\boldsymbol{T}$. The classes $\boldsymbol{T}^*(A, B, \alpha)$ and $\boldsymbol{C}^*(A, B, \alpha)$ were studied by Aouf

[2], the class $\boldsymbol{R}_\gamma[-1, 1, \alpha]=\boldsymbol{R}_\gamma[\alpha]$ was studied by Silverman and Silvia [12], the class $\boldsymbol{C}_\gamma[-1, 1, \alpha]=\boldsymbol{C}_\gamma[\alpha]$ was studied by Owa and Uralegaddi [7], and the class $\boldsymbol{R}_\gamma[-\beta, \beta, \alpha]=\boldsymbol{R}_\gamma[\alpha, \beta]$ $(0<\beta\leq 1)$ was studied by Ahuja and Silverman [1] and (subsequently) by Owa and Ahuja [5]. In the present paper we propose to investigate several important properties and characteristics of the general classes $\boldsymbol{R}_\gamma[A, B, \alpha]$ and $\boldsymbol{C}_\gamma[A, B, \alpha]$, which we have defined here.

2. Basic Characterizations Involving Coefficient Estimates

We first state and prove

Theorem 1. *A function $f(z)\in \boldsymbol{T}$ is in the class $\boldsymbol{R}_\gamma[A, B, \alpha]$ if and only if*

$$\sum_{n=2}^{\infty}[(1+B)(n-1)+(B-A)(1-\alpha)]c(\gamma, n)a_n \leq (B-A)(1-\alpha). \tag{2.1}$$

The result is sharp.

Proof. It is known from [2] that a necessary and sufficient condition for $f(z)\in \boldsymbol{T}$ to be in the class $\boldsymbol{T}^*(A, B, \alpha)$ is that

$$\sum_{n=2}^{\infty}[(1+B)(n-1)+(B-A)(1-\alpha)]a_n \leq (B-A)(1-\alpha).$$

Since

$$(f * s_\gamma)(z)=z-\sum_{n=2}^{\infty}c(\gamma, n)a_n z^n \quad (a_n\geq 0),$$

where $s_\gamma(z)$ is given by (1.5), the result (2.1) follows. Further, we can see that the function $f(z)$ given by

$$f(z)=z-\frac{(B-A)(1-\alpha)}{[(1+B)(n-1)+(B-A)(1-\alpha)]c(\gamma, n)}z^n \qquad (n\in \mathbb{N}\backslash\{1\}) \tag{2.2}$$

is an extremal function for Theorem 1.

Theorem 2. *Let $f(z)\in \boldsymbol{T}$. Then $f(z)$ is in the class $\boldsymbol{C}_\gamma[A, B, \alpha]$ if and only if*

$$\sum_{n=2}^{\infty}n[(1+B)(n-1)+(B-A)(1-\alpha)]c(\gamma, n)a_n \leq (B-A)(1-\alpha). \tag{2.3}$$

The result is sharp.

Proof. Since $f(z)\in C_\gamma[A, B, \alpha]$ if and only if $zf'(z)\in R_\gamma[A, B, \alpha]$, we have Theorem 2 by replacing a_n by na_n in Theorem 1. Further, we can see that the function $f(z)$ given by

$$f(z)=z-\frac{(B-A)(1-\alpha)}{n[(1+B)(n-1)+(B-A)(1-\alpha)]c(\gamma, n)}z^n \qquad (n\in \mathbb{N}\backslash\{1\}) \tag{2.4}$$

is an extremal function for Theorem 2.

Corollary 1. *Let $f(z)\in T$ be in the class $R_\gamma[A, B, \alpha]$. Then*

$$a_n\leq\frac{(B-A)(1-\alpha)}{[(1+B)(n-1)+(B-A)(1-\alpha)]c(\gamma, n)} \qquad (n\in \mathbb{N}\backslash\{1\}). \tag{2.5}$$

Equality in (2.5) *holds true for the function $f(z)$ given by* (2.2).

Corollary 2. *Let $f(z)\in T$ be in the class $C_\gamma[A, B, \alpha]$. Then*

$$a_n\leq\frac{(B-A)(1-\alpha)}{n[(1+B)(n-1)+(B-A)(1-\alpha)]c(\gamma, n)} \qquad (n\in \mathbb{N}\backslash\{1\}). \tag{2.6}$$

Equality in (2.6) *holds true for the function $f(z)$ given by* (2.4).

3. A Set of Closure Theorems

Theorem 3. *The class $R_\gamma[A, B, \alpha]$ is closed under convex linear combination.*

Proof. Let each of the functions $f_1(z)$ and $f_2(z)$ given by

$$f_j(z)=z-\sum_{n=2}^{\infty}a_{n,j}\,z^n \qquad (a_{n,j}\geq 0;\quad j=1, 2) \tag{3.1}$$

be in the class $R_\gamma[A, B, \alpha]$. Then it is sufficient to show that the function $h(z)$ defined by

$$h(z)=\lambda f_1(z)+(1-\lambda)f_2(z) \qquad (0\leq\lambda\leq 1) \tag{3.2}$$

is also in the class $R_\gamma[A, B, \alpha]$. Since, for $0\leq\lambda\leq 1$,

$$h(z)=z-\sum_{n=2}^{\infty}\{\lambda a_{n,1}+(1-\lambda)a_{n,2}\}z^n, \tag{3.3}$$

with the aid of Theorem 1, we have

$$\begin{aligned}\sum_{n=2}^{\infty}&[(1+B)(n-1)+(B-A)(1-\alpha)]c(\gamma, n)\{\lambda a_{n,1}+(1-\lambda)a_{n,2}\} \\ &\leq (B-A)(1-\alpha) \qquad (0\leq\lambda\leq 1),\end{aligned} \tag{3.4}$$

which implies that $h(z)\in \boldsymbol{R}_{\gamma}[A, B, \alpha]$.

Similarly, we can prove

Theorem 4. *The class $\boldsymbol{C}_{\gamma}[A, B, \alpha]$ is closed under convex linear combination.*

Theorem 5. *Let*

$$f_1(z)=z \tag{3.5}$$

and

$$f_n(z)=z-\frac{(B-A)(1-\alpha)}{[(1+B)(n-1)+(B-A)(1-\alpha)]c(\gamma, n)}z^n \qquad (n\in \mathbb{N}\setminus\{1\}). \tag{3.6}$$

Then $f(z)$ is in the class $\boldsymbol{R}_{\gamma}[A, B, \alpha]$ if and only if it can be expressed in the form:

$$f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z) \tag{3.7}$$

where

$$\lambda_n\geq 0 \ (n\in \mathbb{N}) \quad \text{and} \quad \sum_{n=1}^{\infty}\lambda_n=1.$$

Proof. Assume that

$$\begin{aligned}f(z)&=\sum_{n=1}^{\infty}\lambda_n f_n(z) \\ &=z-\sum_{n=2}^{\infty}\frac{(B-A)(1-\alpha)}{[(1+B)(n-1)+(B-A)(1-\alpha)]c(\gamma, n)}\lambda_n z^n.\end{aligned} \tag{3.8}$$

Then it follows that

$$\sum_{n=2}^{\infty} \frac{[(1+B)(n-1)+(B-A)(1-\alpha)]c(\gamma, n)}{(B-A)(1-\alpha)} \cdot \frac{(B-A)(1-\alpha)}{[(1+B)(n-1)+(B-A)(1-\alpha)]c(\gamma, n)} \lambda_n \tag{3.9}$$
$$= \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1.$$

Therefore, by Theorem 1, $f(z) \in \boldsymbol{R}_\gamma[A, B, \alpha]$.

Conversely, assume that the function $f(z)$ defined by (1.12) belongs to the class $\boldsymbol{R}_\gamma[A, B, \alpha]$. Then we have

$$a_n \leq \frac{(B-A)(1-\alpha)}{[(1+B)(n-1)+(B-A)(1-\alpha)]c(\gamma, n)} \qquad (n \in \mathbb{N}\backslash\{1\}). \tag{3.10}$$

Setting

$$\lambda_n = \frac{[(1+B)(n-1)+(B-A)(1-\alpha)]c(\gamma, n)}{(B-A)(1-\alpha)} a_n \qquad (n \in \mathbb{N}\backslash\{1\}) \tag{3.11}$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n, \tag{3.12}$$

we see that $f(z)$ can be expressed in the form (3.7). This completes the proof of Theorem 5.

In precisely the same manner, we can prove

Theorem 6. *Let*

$$f_1(z) = z \tag{3.13}$$

and

$$f_n(z) = z - \frac{(B-A)(1-\alpha)}{n[(1+B)(n-1)+(B-A)(1-\alpha)]c(\gamma, n)} z^n \qquad (n \in \mathbb{N}\backslash\{1\}). \tag{3.14}$$

Then $f(z)$ is in the class $C_\gamma[A, B, \alpha]$ if and only if it can be expressed in the form:

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) \tag{3.15}$$

where

$$\lambda_n \geq 0 \ (n \in \mathbb{N}) \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n = 1.$$

4. Operators of Fractional Calculus

Various operators of fractional calculus (that is, fractional integral and fractional derivative) have been studied in the literature rather extensively. We find it to be convenient to restrict ourselves to the following definitions used recently by Owa [4] and others (*cf.*, *e.g.*, Srivastave and Owa [13], [14]).

Definition 1 (Fractional Integral Operator). The *fractional integral of order* μ ($\mu > 0$) is defined, for a function $f(z)$, by

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\mu}} d\zeta, \tag{4.1}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\mu-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 2 (Fractional Derivative Operator). The *fractional derivative of order* μ ($0 \leq \mu < 1$) is defined, for a function $f(z)$, by

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\mu}} d\zeta, \tag{4.2}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\mu}$ is removed, as in Definition 1.

Theorem 7. *Let $f(z) \in \boldsymbol{T}$ be in the class $\boldsymbol{R}_\gamma[A, B, \alpha]$ with*

$$-1 \leq A < B \leq 1, \quad 0 < B \leq 1, \quad 0 \leq \alpha < 1, \quad \text{and} \quad 0 \leq \gamma \leq \frac{1}{2}. \tag{4.3}$$

Then

$$\begin{aligned} &\frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left\{ 1 - \frac{(B-A)(1-\alpha)}{(2+\mu)[1+B+(B-A)(1-\alpha)](1-\gamma)} |z| \right\} \\ &\leq |D_z^{-\mu} f(z)| \leq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left\{ 1 + \frac{(B-A)(1-\alpha)}{(2+\mu)[1+B+(B-A)(1-\alpha)](1-\gamma)} |z| \right\} \\ &\qquad (\mu > 0; \quad z \in \boldsymbol{U}). \end{aligned} \tag{4.4}$$

The result is sharp.

Proof. Let

$$\begin{aligned} F(z) &= \Gamma(2+\mu) z^{-\mu} D_z^{-\mu} f(z) \\ &= z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\mu)}{\Gamma(n+1+\mu)} a_n z^n \\ &= z - \sum_{n=2}^{\infty} \Theta(n) a_n z^n, \end{aligned} \tag{4.5}$$

where

$$\Theta(n) = \frac{\Gamma(n+1)\Gamma(2+\mu)}{\Gamma(n+1+\mu)} \qquad (n \in \mathbb{N}\setminus\{1\}). \tag{4.6}$$

Then it is easily verified that

$$0 < \Theta(n) \leq \Theta(2) = \frac{2}{2+\mu} \qquad (\mu > 0; \quad n \in \mathbb{N}\setminus\{1\}). \tag{4.7}$$

We note also that

$$c(\gamma, n+1) \geq c(\gamma, n) \qquad (0 \leq \gamma \leq \frac{1}{2}; \quad n \in \mathbb{N}\setminus\{1\}).$$

Since $f(z) \in \boldsymbol{R}_\gamma[A, B, \alpha]$, Theorem 1 implies that

$$\begin{aligned} &2[1+B+(B-A)(1-\alpha)](1-\gamma)\sum_{n=2}^{\infty} a_n \\ &\quad \leq \sum_{n=2}^{\infty} [(1+B)(n-1)+(B-A)(1-\alpha)]c(\gamma, n)a_n \leq (B-A)(1-\alpha), \end{aligned} \tag{4.8}$$

so that

$$\sum_{n=2}^{\infty} a_n \leq \frac{(B-A)(1-\alpha)}{2[1+B+(B-A)(1-\alpha)](1-\gamma)}. \tag{4.9}$$

Thus we find from (4.7) and (4.9) that

$$\begin{aligned}|F(z)| &\geq |z|-\Theta(2)|z|^2\sum_{n=2}^{\infty}a_n \\ &\geq |z|-\frac{(B-A)(1-\alpha)}{(2+\mu)[1+B+(B-A)(1-\alpha)](1-\gamma)}|z|^2\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}|F(z)| &\leq |z|+\Theta(2)|z|^2\sum_{n=2}^{\infty}a_n \\ &\leq |z|+\frac{(B-A)(1-\alpha)}{(2+\mu)[1+B+(B-A)(1-\alpha)](1-\gamma)}|z|^2,\end{aligned} \tag{4.11}$$

which prove the inequalities (4.3) of Theorem 7. Further, equalities in (4.3) are attained for the function $f(z)$ given by

$$D_z^{-\mu}f(z)=\frac{z^{1+\mu}}{\Gamma(2+\mu)}\left\{1-\frac{(B-A)(1-\alpha)}{(2+\mu)[1+B+(B-A)(1-\alpha)](1-\gamma)}z\right\} \tag{4.12}$$

or, equivalently, by

$$f(z)=z-\frac{(B-A)(1-\alpha)}{2[1+B+(B-A)(1-\alpha)](1-\gamma)}z^2. \tag{4.13}$$

Thus we complete the proof of Theorem 7.

Corollary 3. *Under the conditions of Theorem 7, $D_z^{-\mu}f(z)$ is included in the disk with center at the origin and radius r_1 given by*

$$r_1=\frac{1}{\Gamma(2+\mu)}\left\{1+\frac{(B-A)(1-\alpha)}{(2+\mu)[1+B+(B-A)(1-\alpha)](1-\gamma)}\right\}. \tag{4.14}$$

Using Theorem 2, we have

Theorem 8. *Let $f(z)\in T$ be in the class $C_\gamma[A, B, \alpha]$ with the parametric constraints given by* (4.3). *Then*

$$\begin{aligned}&\frac{|z|^{1+\mu}}{\Gamma(2+\mu)}\left\{1-\frac{(B-A)(1-\alpha)}{2(2+\mu)[1+B+(B-A)(1-\alpha)](1-\gamma)}|z|\right\} \\ &\leq |D_z^{-\mu}f(z)| \leq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)}\left\{1+\frac{(B-A)(1-\alpha)}{2(2+\mu)[1+B+(B-A)(1-\alpha)](1-\gamma)}|z|\right\} \\ &(\mu>0;\quad z\in U).\end{aligned} \tag{4.15}$$

The result is sharp for the function

$$f(z)=z-\frac{(B-A)(1-\alpha)}{4[1+B+(B-A)(1-\alpha)](1-\gamma)}z^2. \tag{4.16}$$

Corollary 4. *Under the conditions of Theorem* 8, $D_z^{-\mu}f(z)$ *is included in the disk with center at the origin and radius* r_2 *given by*

$$r_2=\frac{1}{\Gamma(2+\mu)}\left\{1+\frac{(B-A)(1-\alpha)}{2(2+\mu)[1+B+(B-A)(1-\alpha)](1-\gamma)}\right\}. \tag{4.17}$$

Theorem 9. *Let* $f(z)\in \boldsymbol{T}$ *be in the class* $\boldsymbol{R}_\gamma[A, B, \alpha]$ *with the parametric constraints given by* (4.3). *Then*

$$\begin{aligned}\frac{|z|^{1-\mu}}{\Gamma(2-\mu)}&\left\{1-\frac{(B-A)(1-\alpha)}{(2-\mu)[1+B+(B-A)(1-\alpha)](1-\gamma)}|z|\right\}\\ \leq |D_z^\mu f(z)| \leq &\frac{|z|^{1-\mu}}{\Gamma(2-\mu)}\left\{1+\frac{(B-A)(1-\alpha)}{(2-\mu)[1+B+(B-A)(1-\alpha)](1-\gamma)}|z|\right\}\\ &(0\leq\mu<1;\quad z\in \boldsymbol{U}).\end{aligned} \tag{4.18}$$

The result is sharp.

Proof. Let

$$\begin{aligned}G(z)&=\Gamma(2-\mu)z^\mu D_z^\mu f(z)\\ &=z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\mu)}{\Gamma(n+1-\mu)}a_n z^n\\ &=z-\sum_{n=2}^{\infty}n\Phi(n)a_n z^n,\end{aligned} \tag{4.19}$$

where

$$\Phi(n)=\frac{\Gamma(n)\Gamma(2-\mu)}{\Gamma(n+1-\mu)} \qquad (n\in \mathbb{N}\backslash\{1\}). \tag{4.20}$$

Noting that

$$0<\Phi(n) \leq \Phi(2) = \frac{1}{2-\mu} \qquad (0\leq\mu<1;\quad n\in \mathbb{N}\backslash\{1\}), \tag{4.21}$$

and since

$$\begin{aligned}&\frac{1}{2}[1+B+(B-A)(1-\alpha)]c(\gamma,2)\sum_{n=2}^{\infty}n\,a_n\\&\leq\sum_{n=2}^{\infty}[(1+B)(n-1)+(B-A)(1-\alpha)]c(\gamma,n)a_n\\&\leq(B-A)(1-\alpha),\end{aligned}\tag{4.22}$$

which follows readily from Theorem 1, we have

$$\sum_{n=2}^{\infty}n\,a_n\leq\frac{(B-A)(1-\alpha)}{[1+B+(B-A)(1-\alpha)](1-\gamma)}.\tag{4.23}$$

Thus we find that

$$\begin{aligned}|G(z)|&\geq|z|-\Phi(2)|z|^2\sum_{n=2}^{\infty}n\,a_n\\&\geq|z|-\frac{(B-A)(1-\alpha)}{(2-\mu)[1+B+(B-A)(1-\alpha)](1-\gamma)}|z|^2\end{aligned}\tag{4.24}$$

and

$$\begin{aligned}|G(z)|&\leq|z|+\Phi(2)|z|^2\sum_{n=2}^{\infty}n\,a_n\\&\leq|z|+\frac{(B-A)(1-\alpha)}{(2-\mu)[1+B+(B-A)(1-\alpha)](1-\gamma)}|z|^2,\end{aligned}\tag{4.25}$$

which yield the inequalities (4.18) of Theorem 9 under the parametric constraints given by (4.3). Since equalities in (4.18) are attained for the function $f(z)$ given by

$$D_z^\mu f(z)=\frac{z^{1-\mu}}{\Gamma(2-\mu)}\left\{1-\frac{(B-A)(1-\alpha)}{(2-\mu)[1+B+(B-A)(1-\alpha)](1-\gamma)}z\right\},\tag{4.26}$$

or, equivalently, by (4.11), we complete the proof of Theorem 9.

Corollary 5. *Under the conditions of Theorem* 9, $D_z^\mu f(z)$ *is included in a disk with its center at the origin and radius* r_3 *given by*

$$r_3=\frac{1}{\Gamma(2-\mu)}\left\{1+\frac{(B-A)(1-\alpha)}{(2-\mu)[1+B+(B-A)(1-\alpha)](1-\gamma)}\right\}.\tag{4.27}$$

Finally, using Theorem 2, we obtain

Theorem 10. *Let $f(z)\in T$ be in the class $C_\gamma[A, B, \alpha]$ with the parametric constraints given by* (4.3). *Then*

$$\frac{|z|^{1-\mu}}{\Gamma(2-\mu)}\left\{1-\frac{(B-A)(1-\alpha)}{2(2-\mu)[1+B+(B-A)(1-\alpha)](1-\gamma)}|z|\right\}$$
$$\leq |D_z^\mu f(z)| \leq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)}\left\{1+\frac{(B-A)(1-\alpha)}{2(2-\mu)[1+B+(B-A)(1-\alpha)](1-\gamma)}|z|\right\} \tag{4.28}$$
$$(0\leq\mu<1;\quad z\in U).$$

The result is sharp for the function $f(z)$ given by

$$D_z^\mu f(z) = \frac{z^{1-\mu}}{\Gamma(2-\mu)}\left\{1-\frac{(B-A)(1-\alpha)}{2(2-\mu)[1+B+(B-A)(1-\alpha)](1-\gamma)}z\right\}, \tag{4.29}$$

or, equivalently, by (4.16).

Corollary 6. *Under the conditions of Theorem* 10, *$D_z^\mu f(z)$ is included in a disk with its center at the origin and radius r_4 given by*

$$r_4 = \frac{1}{\Gamma(2-\mu)}\left\{1+\frac{(B-A)(1-\alpha)}{2(2-\mu)[1+B+(B-A)(1-\alpha)](1-\gamma)}\right\}. \tag{4.30}$$

5. A Generalized Fractional Integral Operator

In this section we shall make use of the generalized fractional integral operator $I_{0,z}^{\beta,\delta,\eta}$ given by (*cf*. Srivastava *et al.* [15]).

Definition 3. For real numbers $\beta>0$, δ, and η, the *generalized fractional integral operator* $I_{0,z}^{\beta,\delta,\eta}$ is defined by

$$I_{0,z}^{\beta,\delta,\eta}f(z) = \frac{z^{-\beta-\delta}}{\Gamma(\beta)}\int_0^z (z-\zeta)^{\beta-1}{}_2F_1\left(\beta+\delta, -\eta; \beta; 1-\frac{\zeta}{z}\right)f(\zeta)d\zeta, \tag{5.1}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, with the order

$$f(z)=O(|z|^\epsilon)\quad (z\to 0;\quad \epsilon>\max\{0, \delta-\eta]-1), \tag{5.2}$$

$$ {}_2F_1(a, b; c; z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \tag{5.3}$$

$(\nu)_n$ is the Pochhammer symbol defined by

$$(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 & (n=0) \\ \nu(\nu+1)\cdots(\nu+n-1) & (n\in\mathbb{N}), \end{cases} \tag{5.4}$$

and (just as in Definitions 1 and 2) the multiplicity of $(z-\zeta)^{\beta-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$.

Remark 1. For $\delta=-\beta$, we note that

$$I_{0,z}^{\beta,-\beta,\eta}f(z)=D_z^{-\beta}f(z). \tag{5.5}$$

In order to prove our results for the generalized fractional integral operator $I_{0,z}^{\beta,\delta,\eta}$, we recall here the following lemma due to Srivastava *et al.* [15].

Lemma (Srivastava *et al.* [15]). *If* $\beta>0$ *and* $\kappa>\delta-\eta-1$, *then*

$$I_{0,z}^{\beta,\delta,\eta}z^{\kappa} = \frac{\Gamma(\kappa+1)\Gamma(\kappa-\delta+\eta+1)}{\Gamma(\kappa-\delta+1)\Gamma(\kappa+\beta+\eta+1)}z^{\kappa-\delta}. \tag{5.6}$$

With the aid of this Lemma, we prove

Theorem 11. *Let* $\beta>0$, $\delta<2$, $\beta+\eta>-2$, $\delta-\eta<2$, *and* $\delta(\beta+\eta)\leq 3\beta$. *If* $f(z)\in \boldsymbol{T}$ *is in the class* $\boldsymbol{R}_\gamma[A,B,\alpha]$ *with the parametric constraints given by* (4.3), *then*

$$\begin{aligned} &\frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\beta+\eta)} \\ &\quad\cdot\left\{1-\frac{(B-A)(1-\alpha)(2-\delta+\eta)}{[1+B+(B-A)(1-\alpha)](1-\gamma)(2-\delta)(2+\beta+\eta)}|z|\right\} \\ &\leq |I_{0,z}^{\beta,\delta,\eta}f(z)| \leq \frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\beta+\eta)} \\ &\quad\cdot\left\{1+\frac{(B-A)(1-\alpha)(2-\delta+\eta)}{[1+B+(B-A)(1-\alpha)](1-\gamma)(2-\delta)(2+\beta+\eta)}|z|\right\} \quad (z\in U_0), \end{aligned} \tag{5.7}$$

where

$$U_0 = \begin{cases} U & (\delta\leq 1) \\ U\setminus\{0\} & (\delta>1). \end{cases} \tag{5.8}$$

Equalities in (5.7) *are attained for the function* $f(z)$ *given by* (4.13).

Proof. By using the Lemma, we have

$$\begin{aligned} I_{0,z}^{\beta,\delta,\eta}f(z) &= \frac{\Gamma(2-\delta+\eta)}{\Gamma(2-\delta)\Gamma(2+\beta+\eta)}z^{1-\delta} \\ &\quad -\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(n-\delta+\eta+1)}{\Gamma(n-\delta+1)\Gamma(n+\beta+\eta+1)}a_n z^{n-\delta}. \end{aligned} \tag{5.9}$$

Letting

$$\begin{aligned} H(z) &= \frac{\Gamma(2-\delta)\Gamma(2+\beta+\eta)}{\Gamma(2-\delta+\eta)}z^{\delta}\, I_{0,z}^{\beta,\delta,\eta}f(z) \\ &= z-\sum_{n=2}^{\infty}\Psi(n)a_n z^n, \end{aligned} \tag{5.10}$$

where

$$\Psi(n)=\frac{(2-\delta+\eta)_{n-1}(1)_n}{(2-\delta)_{n-1}(2+\beta+\eta)_{n-1}} \qquad (n\in \mathbb{N}\backslash\{1\}), \tag{5.11}$$

we can see that $\Psi(n)$ is non-increasing for integers n $(n\in \mathbb{N}\backslash\{1\})$, and we have

$$0<\Psi(n)\leq \Psi(2)=\frac{2(2-\delta+\eta)}{(2-\delta)(2+\beta+\eta)} \qquad (n\in \mathbb{N}\backslash\{1\}). \tag{5.12}$$

Therefore, by using (4.9) and (5.12), we have

$$\begin{aligned} |H(z)| &\geq |z|-\Psi(2)|z|^2\sum_{n=2}^{\infty}a_n \\ &\geq |z|-\frac{(B-A)(1-\alpha)(2-\delta+\eta)}{[1+B+(B-A)(1-\alpha)](1-\gamma)(2-\delta)(2+\beta+\eta)}|z|^2 \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} |H(z)| &\leq |z|+\Psi(2)|z|^2\sum_{n=2}^{\infty}a_n \\ &\leq |z|+\frac{(B-A)(1-\alpha)(2-\delta+\eta)}{[1+B+(B-A)(1-\alpha)](1-\gamma)(2-\delta)(2+\beta+\eta)}|z|^2. \end{aligned} \tag{5.14}$$

This completes the proof of Theorem 11.

Similarly, by applying Theorem 2 (instead of Theorem 1) to the function $f(z)$ belonging to the class $C_\gamma[A, B, \alpha]$, we can derive

Theorem 12. *Let* $\beta>0$, $\delta<2$, $\beta+\eta>-2$, $\delta-\eta<2$, *and* $\delta(\beta+\eta)\leq 3\beta$. *If* $f(z)\in \boldsymbol{T}$ *in the class* $C_\gamma[A, B, \alpha]$ *with the parametric constraints given by* (4.3), *then*

$$\begin{aligned}&\frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\beta+\eta)}\\ &\quad\cdot\left\{1-\frac{(B-A)(1-\alpha)(2-\delta+\eta)}{2[1+B+(B-A)(1-\alpha)](1-\gamma)(2-\delta)(2+\beta+\eta)}|z|\right\}\\ &\leq |I_{0,z}^{\beta,\delta,\eta}f(z)| \leq \frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\beta+\eta)}\\ &\quad\cdot\left\{1+\frac{(B-A)(1-\alpha)(2-\delta+\eta)}{2[1+B+(B-A)(1-\alpha)](1-\gamma)(2-\delta)(2+\beta+\eta)}|z|\right\} \quad (z\in U_0),\end{aligned} \tag{5.15}$$

where U_0 *is defined by* (5.8). *Equalities in* (5.15) *are attained for the function* $f(z)$ *given by* (4.16).

Remark 2. Taking $\beta=-\delta=\mu$ in Theorems 11 and 12, we get the results of Theorems 7 and 8, respectively.

Remark 3. Putting $A=-1$ and $B=1$ in Theorems 11 and 12, we get the corresponding results for the classes $\boldsymbol{R}_\gamma[\alpha]$ and $C_\gamma[\alpha]$, respectively.

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