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Locally conformal Kähler manifolds with a family of constant curvature tensors

Yoshinobu KAMISHIMA

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Contents

Introduction

A *locally conformal Kähler structure* on a complex manifold (W, J) is a family of local forms $\{U_\alpha, \Omega_\alpha\}_{\alpha\in\Lambda}$ consisting of an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of W and a Kähler form Ω_α on each U_α satisfying that (1) each Ω_α is invariant under the complex structure J, (2) if $U_\alpha\cap U_\beta\neq\emptyset$, then $\Omega_\beta=\lambda_{\beta\alpha}\cdot\Omega_\alpha$, where $\lambda_{\beta\alpha}$ is a positive constant. A manifold equipped with this structure is said to be a *locally conformal Kähler* manifold. (Abbreviated to *l.c. Kähler manifold.*) As usual, $\{\lambda_{\alpha\beta}\}$ satisfies the cocycle condition so that there is a family of functions $\{f_\alpha\}$ on each U_α for which $f_\alpha\cdot f_\beta^{-1}=\lambda_{\beta\alpha}$. If $\Omega_\beta=\lambda_{\beta\alpha}\cdot\Omega_\alpha$, then $f_\beta\cdot\Omega_\beta=f_\alpha\cdot\Omega_\alpha$ on $U_\alpha\cap U_\beta$. Thus a locally conformal Kähler structure $\{U_\alpha, \Omega_\alpha\}_{\alpha\in\Lambda}$ defines a 2-form Ω on (W, J) such that $\Omega\,|\,U_\alpha=f_\alpha\cdot\Omega_\alpha$. If we set $g(X, Y)=\Omega(X, JY)$ and $g_\alpha(X, Y)=\Omega_\alpha(X, JY)$, then g is a Hermitian metric on the complex manifold W and each g_α is a Kähler metric on U_α with $g\,|\,U_\alpha=f_\alpha\cdot g_\alpha$. A $2n$-manifold M is said to be an *l. c. K-const* manifold if each g_α has constant holomorphic sectional curvature. On the other hand, by the property that $g\,|\,U_\alpha=f_\alpha\cdot g_\alpha$, if each g_α is of constant sectional curvature, then

g is (locally) conformal to a euclidean metric, *i.e.*, (M, g) is conformally flat $(n>1)$. In this note, we study the following:

(1) Which manifold occurs as l.c. K-const manifolds?

(2) Which manifold occurs as conformally flat l.c. Kähler manifolds?

Obviously Kähler manifolds are typical examples of l.c. Kähler manifolds. In this case, the question (1) is referred to the *complex space form problem.* Moreover, a Kähler manifold of constant holomorphic sectional curvature c is locally holomorphically isometric to the complex projective space $\mathbb{CP}^n$, a complex euclidean space $\mathbb{C}^n$, a complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^n$ up to a constant scalar multiple of the Kähler metric according as $c=+1, 0, -1$. As to (2), it is known that any conformally flat Kähler manifold of dimension $2n>4$ is flat (*i.e.*, locally isometric to a euclidean space). The scalar cuevature of a conformally flat Kähler 4-manifold is known to be zero. Moreover, in this case M is Bochner flat. So a conformally flat Kähler 4-manifold is locally holomorphically isometric to the complex euclidean space or the product of the complex hyperbolic line $\mathbb{H}_{\mathbb{C}}^1$ and the complex projective line $\mathbb{CP}^1$ (=2-sphere). In § 1, we answer Question 1 (cf. Theorem 4) and § 2, § 3 answer to Question 2. (Compare Corollary 6, Theorem 8). In particular we obtain the following classification in dimension 4. (See § 3 for the details.)

Theorem Let (M, g) be a compact conformally flat l.c. Kähler 4-manifold. Then there exists a function $\tau: M \to \mathbb{R}^+$ such that $(M, \tau\cdot g, J)$ is holomorphically isometric to one of the following l.c. Kähler manifolds. (In other words, M is *conformally equivalent* to one of the following l.c. Kähler manifolds *preserving the complex structures*:

1. A complex euclidean space form $T_{\mathbb{C}}^2/F$ $(F\subset \mathrm{U}(2))$.
2. A fiber space $\mathbb{H}_{\mathbb{C}}^1 \times_{\Gamma} \mathbb{CP}^1$ $(\Gamma\subset \mathrm{PU}(1,1)\times \mathrm{PU}(2))$.
3. An infra-Hopf manifold $S^3 \times_F S^1$ $(F\subset \mathrm{U}(2)\times S^1)$,

where Γ is a discrete uniform subgroup of $\mathrm{PU}(1,1)\times \mathrm{PU}(2)$ and F is a finite subgroup of $\mathrm{U}(2)$ $\times S^1$ acting freely.

Note that the group of l.c. Kähler transformations of M, $\mathrm{Aut}_{\,l.c.K}(M)$, is a subgroup of the group of holomorphic transformations of an l.c. Kähler manifold M. (See § 4 for the definition.) In § 4, using the Obata, Lelong-Ferrand's theorem in *Conformal geometry*, we shall prove that if M is a compact l.c. Kähler manifold of dimension $2n>2$, then $\mathrm{Aut}_{\,l.c.K}(M)$ is a compact Lie group. As an application, in § 5 we provide the structure of a compact Lie group $\mathrm{Aut}_{\,l.c.K}(M)$.

1. Bochner flat l.c. Kähler manifolds

In the sequel of [8] we have examined locally conformal Kähler manifolds since Tricerri and Vanheck [23] have defined a generalized Bochner curvature component and proved that it is a conformal invariant (*i.e.*, equal if the two Hermitian metrics are conformal under the complex structure fixed on it). Our technique will be generalized into locally conformal Kähler manifolds. (Compare [9].) The geometric properties of locally conformal Kähler manifolds have been extensively developed by Vaisman, Ornea, Kashiwada and others (cf. [25], [26], [27], [28], [19], [20]). Especially a Hopf manifold $S^1\times S^{2n-1}$ is known to be an l.c. Kähler manifold but not globally conformal Kähler.

When we examine compact l.c. Kähler manifolds with vanishing (generalized) Bochner tensor, the classification is different from that of the Kähler manifolds [8]. Indeed, we have shown that *among all compact l.c. Kähler manifolds, a Bochner flat l.c. Kähler manifold but not globally conformal Kähler is an infra-Hopf manifold* $S^{2n-1}\underset{F}{\times}S^1$ $(F\subset \mathrm{U}(n)\times S^1)$, *which is finitely covered by the Hopf manifold* $S^{2n-1}\times S^1$. (Compare [9], [12].)

In order to derive this classfication of comact locally conformal Kähler manifolds of dimension $2n$ $(n>1)$, we have proved the following uniformization in [9] as well as that of [8].

Theorem 1. *Let* (M, g, J) *be a locally conformal Kähler manifold of dimension* $2n$ $(n>1)$. *If the Bochner curvature tensor* B *with respect to the Hermitian metric* g *vanishes, then* M *is uniformized over* $Y_{\mathbb{C}}^n$ *with respect to* $\mathcal{G}$. *The model space* $(Y_{\mathbb{C}}^n, g_Y)$, *which is the connected simply connected Kähler manifold endowed with the transitive group of biholomorphic transformations* $\mathcal{G}$, *is one of the following*:

(1) A projective geometry $(\mathrm{PU}(n+1), \mathbb{CP}^n)$.

(2) A complex similarity geometry $(\mathbb{C}^n\rtimes(\mathrm{U}(n)\times\mathbb{R}^+), \mathbb{C}^n)$.

(3) A hyperbolic geometry $(\mathrm{PU}(n,1), \mathbb{H}_{\mathbb{C}}^n)$.

(4) A projective-hyperbolic geometry $(\mathrm{PU}(m,1)\times\mathrm{PU}(n-m+1), \mathbb{H}_{\mathbb{C}}^m\times\mathbb{CP}^{n-m})$; $(m=1, 2, \cdots, n-1)$.

Moreover, the uniformization $\{U_\alpha, \varphi_\alpha\}_{\alpha\in\Lambda}$ *of* (M, g, J) *has the following property that*

$$c_\alpha\cdot g_\alpha=\varphi_\alpha^* g_Y$$

where c_α *is a constant defined on* U_α, *and the canonical complex structrure* J_Y *on* $Y_{\mathbb{C}}^n$ *satisfies that*

$$\varphi_{\alpha*}\circ J=J_Y\circ\varphi_{\alpha*},$$

(In p.28 [9], we forgot to write g_α for g above.)

Concerning the above theorem we remark the following: First, in general, a *uniformization* on a manifold M with respect to $(\mathcal{G}, Y_{\mathbb{C}}^n)$ is a maximal collection of charts $\{U_\alpha, \varphi_\alpha\}_{\alpha\in\Lambda}$ of M satisfying that:

$$M = \bigcup_{\alpha\in\Lambda} U_\alpha,$$

$$\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)\ (\subset Y_{\mathbb{C}}^n) \text{ is a homeomorphism, and}$$

$$\text{if } U_\alpha \cap U_\beta \neq \emptyset, \text{ then the coordinate change } g_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} \text{ extends to an element of } \mathcal{G}.$$

Second, in view of the paper [8] we have not used any tensorial propeties of Kähler metric like Ricci curvature tensor, and scalar curvature tensors, but only using vanishing of the Bochner curvature tensor and its coincidence with the Chern-Moser-Webster curvature tensor. Given a Kähler manifold M, because of the classification (determination) of spherical CR manifolds with (positive dimensional) Lie group actions (they are CR-equivalent to S^{2n+1} with maximal free toral actions, or $S^{2n+1}-S^{2m-1}$ where S^{2m-1} is the boundary of complex m-dimensional hyperbolic space $\mathbb{H}_{\mathbb{C}}^m$, $m=1, 2, \cdots, n$, up to conjugacy), the universal covering space $\tilde{M}$ is uniformized over the complex Kähler manifold of the above four types, among which only $\mathbb{H}_{\mathbb{C}}^m \times \mathbb{CP}^{n-m}$ comes out as the product case, in addition, if $\tilde{M}$ is complete (for example M is compact), $\tilde{M}$ is conformally equivalent to the product $\mathbb{H}_{\mathbb{C}}^m \times \mathbb{CP}^{n-m}$ with the product Kähler metric. (In other words, its conformal class of (M, g) is holomorphically isometric to the product.) Hence we concluded that the constant holomorphic sectional curvatures c_1, c_2 of $\mathbb{H}_{\mathbb{C}}^m$ and $\mathbb{CP}^{n-m}$ in the product $\mathbb{H}_{\mathbb{C}}^m \times \mathbb{CP}^{n-m}$ satisfy $c_1 = -c_2$.

On the other hand, around 1970's, using the tensor calculus, there are several results concerning Bochner curvature flat metric under curvature assumptions (cf. [4], [5]). From this point, we look at what restraint paused on Kähler metrics on a Bochner flat Kähler manifold M^{2k}, and deduce that $c_1 = -c_2$ by calculation of the scalar curvature of $\mathbb{H}_{\mathbb{C}}^m \times \mathbb{CP}^{n-m}$. First, Theorem 3 of [17] says that if a Kähler space with vanishing Bochner tensor has constant scalar curvature, then either it is a space of constant holomorphic sectional curvature, or a locally product space of two spaces of constant holomorphic sectional cuevature $H>0$ and $-H$. Moreover, in [16] they study the curvature condition when the product metric supports a Bochner curvature flat metric. In fact, if the Kähler metric g of M^{2n} is of form g_1+g_2 and g supports a Bochner flat metric, then g_1 and g_2 are metrics of constant holomorphic sectional curvatures satisfying some condition on the scalar curvature. Especially, if M^{2n} is product $M = N_1 \times N_2$ with $\dim N_1 = 2m$, $\dim N_2 = 2(n-m)$, then the scalar curvature R_i of g_i $(i=1, 2)$ satisfies that

$$(*) \qquad \frac{R_1}{m(m+1)} + \frac{R_2}{(n-m)(n-m+1)} = 0.$$

It is also remarked that a non-flat metric of Bochner flat Kähler manifold cannot be a direct sum of more than two Kähler metrics.

Let $R_{\alpha\bar{\beta}\rho\bar{\sigma}}$ be the curvature tensor, and $R_{\rho\bar{\sigma}}=R^{\alpha}_{\alpha\rho\bar{\sigma}}$ and $\frac{1}{2}R=g^{\rho\bar{\sigma}}R_{\rho\bar{\sigma}}$ the Ricci tensor and the scalar curvature on M^{2k} respectively. Since the constant holomorphic sectional curvature satisfies that $R_{\alpha\bar{\beta}\rho\bar{\sigma}}=-\frac{1}{2}c(g_{\alpha\bar{\beta}}g_{\rho\bar{\sigma}}+g_{\alpha\bar{\sigma}}g_{\bar{\beta}\rho})$. In particular, letting $g_{\alpha\bar{\beta}}=\delta_{\alpha\bar{\beta}}$, we have

$$R_{\bar{\beta}\alpha\beta\bar{\alpha}}=\begin{cases}\frac{c}{2} & \text{if } \alpha\neq\beta\\ c & \text{if } \alpha=\beta.\end{cases}$$

Then the scalar curvature of M^{2k} satisfies that

$$\begin{aligned}\frac{1}{2}R&=g^{\rho\bar{\beta}}R_{\rho\bar{\beta}}=\sum_{\alpha}\delta^{\rho\bar{\beta}}R_{\alpha\bar{\alpha}\rho\bar{\beta}}\\ &=\sum_{\alpha,\beta}-R_{\bar{\beta}\beta\alpha\bar{\alpha}}=\sum_{\alpha,\beta}R_{\bar{\beta}\alpha\beta\bar{\alpha}}\\ &=\sum_{\alpha}(R_{\bar{1}\alpha1\bar{\alpha}}+R_{\bar{2}\alpha2\bar{\alpha}}+\cdots+R_{\bar{k}\alpha k\bar{\alpha}})=\frac{c}{2}k(k+1),\end{aligned}$$

where

$$\sum_{\alpha}R_{\bar{1}\alpha1\bar{\alpha}}=R_{\bar{1}11\bar{1}}+R_{\bar{1}21\bar{2}}+\cdots+R_{\bar{1}k1\bar{k}}=c+\frac{c}{2}+\cdots+\frac{c}{2}=\frac{c}{2}(k+1).$$

So, $R_1=c_1m(m+1)$ for $\mathbb{H}^m_{\mathbb{C}}$ and $R_2=c_2(n-m)(n-m+1)$ for $\mathbb{CP}^{n-m}$. Substituting these scalar curvature into (*), we obtain that $c_1+c_2=0$.

Combining these with our result yields that

Remark 2. *If the Kähler metric of a Bochner curvature flat manifold M^{2n} is the product as a Kähler metric, then it must be (locally) a product of only two manifolds M_1^m and M_2^{n-m}, with constant holomorphic sectional curvature but has opposite sign.*

Note that if each g_α has the contant holomorphic sectional curvature, then by the definition of Bochner curvature tensor, $B=0$ for g_α. Since the Bochner curvature tensor is conformal invariant (cf. [23]), the Bochner curvature tensor B of g also vanishes. By the definition, an l.c. K-const manifold M^{2n} is a Bochner flat l.c. Kähler manifold provided that $n>1$.

Let (M, g) be an l.c. Kähler manifold of dimension $2n$ $(n>1)$. Suppose that M is Bochner flat. (We retain the notation in [9].) First recall that the Hermitian metric g has the form; $g|U_\alpha=f_\alpha\cdot g_\alpha$ where f_α is a positive function and g_α is a Kähler metric defined on a neighborhood U_α of M.

On the intersection $U_\alpha \cap U_\beta \neq \emptyset$, Uniformization Theorem 1 shows that $\varphi_\beta = \hat{h} \circ \varphi_\alpha$ for $\hat{h} \in \mathcal{G}$. Then, except for (2) complex similarity geometry $(\mathbb{C}^n \rtimes (\mathrm{U}(n) \times \mathbb{R}^+), \mathbb{C}^n)$, we note that

$$\mathcal{G} = \mathrm{Iso}(Y_{\mathbb{C}}^n, g_Y).$$

So we have

$$(**) \qquad \varphi_\alpha^* g_Y = \varphi_\beta^* g_Y.$$

Since $c_\alpha \cdot g_\alpha = \varphi_\alpha^* g_Y$ and $g_\beta = \lambda_{\beta\alpha} \cdot g_\alpha = \dfrac{f_\alpha}{f_\beta} \cdot g_\alpha$ by the cocycle condition as before, we have

$$c_\beta \cdot f_\beta^{-1} = c_\alpha \cdot f_\alpha^{-1} \text{ on } U_\alpha \cap U_\beta.$$

This defines a global function $\tau : M \longrightarrow \mathbb{R}^+$ such that $\tau | U_\alpha = c_\alpha \cdot f_\alpha^{-1}$. (Compare [9].)
On the other hand, using the above uniformization we have a developing pair (locally conformal Kähler immersion and holonomy map) from the universal covering space of M:

$$(\rho, \mathrm{dev}) : (\pi_1(M), \tilde{M}) \longrightarrow (\mathcal{G}, Y_{\mathbb{C}}).$$

As above $\mathcal{G} = \mathrm{Iso}(Y_{\mathbb{C}}^n, g_Y)$, the pullback by dev defines a $\pi_1(M)$-invariant Kähler metric on $\tilde{M}$. So it induces a Kähler metric h on M such that $P^* h = \mathrm{dev}^* g_Y$, where $P : \tilde{M} \to M$ is the covering map. Let $\tau \cdot g$ be a conformal class of g on M. Then calculate

$$\begin{aligned} \tau \cdot g | U_\alpha &= \tau \cdot f_\alpha \cdot g_\alpha | U_\alpha \\ &= (c_\alpha f_\alpha^{-1}) \cdot (f_\alpha g_\alpha) \\ &= c_\alpha \cdot g_\alpha = \varphi_\alpha^* g_Y | U_\alpha. \end{aligned}$$

We can assume that $P : \tilde{U}_\alpha \to U_\alpha$ is homeomorphic. By the construction of developing map, $\varphi_\alpha \circ P = \mathrm{dev} | \tilde{U}_\alpha$. We have

$$\begin{aligned} P^*(\tau \cdot g) | \tilde{U}_\alpha &= P^*(\tau \cdot g | U_\alpha) \\ &= P^*(\varphi_\alpha^* g_Y) | \tilde{U}_\alpha \\ &= \mathrm{dev}^* g_Y | \tilde{U}_\alpha = P^* h | \tilde{U}_\alpha, \end{aligned}$$

i.e., $\tau \cdot g | U_\alpha = h | U_\alpha$ for each α. Hence $\tau \cdot g = h$.
As a consequence if we consider the conformal class $(M, \tau \cdot g)$ of (M, g), then M is globally conformal Kähler.

On the other hand, for (2) complex similarity geometry $\mathcal{G} = \mathbb{C}^n \rtimes (\mathrm{U}(n) \times \mathbb{R}^+) \subset \mathrm{Sim}(\mathbb{R}^{2n})$, we cannot expect the above $(**)$ because $\mathbb{R}^+$ acts as similarities on $\mathbb{C}^n$ with respect to the euclidean metric g_0. By Uniformization Theorem 1 again, we have

$$c_a \cdot g_a = \varphi_a^* g_Y$$

where $(Y_{\mathbb{C}}^n, g_Y) = (\mathbb{C}^n, g_0)$. Therefore, each g_a is flat, in other words, (M, g) is an l.c. Kähler-flat manifold. (See [25] for the definition.) In summary we have the following result noted by Vaisman.

Proposition 3. *A Bochner flat l.c. Kähler manifold* (M, g) *must be either l.c. Kähler-flat or globally conformal Kähler.*

I. Suppose that (M, g) is globally conformal Kähler, *i.e.*, $(M, \tau \cdot g)$ is Kähler. By Uniformization Theorem 1, $(M, \tau \cdot g)$ is uniformizable with respect to $(Y_{\mathbb{C}}^n, g_Y) = \mathbb{CP}^n$, $\mathbb{H}_{\mathbb{C}}^n$, $\mathbb{H}_{\mathbb{C}}^m \times \mathbb{CP}^{n-m}$. In addition if $(M, \tau \cdot g)$ is complete with respect to the Riemannian metric $\tau \cdot g$ (for example, when M is compact), then $\tilde{M}$ is conformally equivalent to $Y_{\mathbb{C}}$ preserving the complex structures. For this, if $(M, \tau \cdot g)$ is complete, then the developing map dev : $(\tilde{M}, \mathrm{dev}^* g_Y) \rightarrow (Y_{\mathbb{C}}, g_Y)$ is a covering map so a homeomorphism where $P^*(\tau \cdot g) = \mathrm{dev}^* g_Y$ as above. In particular, dev induces a (complex structure-preserving) conformal diffeomorphism of (M, g) onto the Kähler manifold $Y_{\mathbb{C}} / \rho(\pi_1(M))$ where $\rho(\pi_1(M)) \subset \mathcal{G}$.

II. Suppose that (M, g) is l.c. Kähler-flat. By Uniformization Theorem 1, there is the developing pair:

$$(\rho, \mathrm{dev}) : (\pi_1(M), \tilde{M}) \longrightarrow (\mathbb{C}^n \rtimes (\mathrm{U}(n) \times \mathbb{R}^+), \mathbb{C}^n).$$

Since the complex similarity group $\mathbb{C}^n \rtimes (\mathrm{U}(n) \times \mathbb{R}^+)$ is the subgroup of $\mathrm{Sim}(\mathbb{R}^{2n})$, the group of similarity transformations of M, M is said to be a similarity manifold. (Compare[6].)
Let g_0 be the euclidean metric invariant under the group of rigid motions $\mathrm{E}_{\mathbb{C}}(n) = \mathbb{C}^n \rtimes \mathrm{U}(n)$. If $\tilde{M}$ is complete with respect to the pullback metric $\mathrm{dev}^* g_0$, dev is a homeomorphism of $\tilde{M}$ onto $\mathbb{C}^n$ such that $\rho(\pi_1(M))$ is contained exactly in $\mathrm{E}_{\mathbb{C}}(n)$. Since $\mathrm{E}_{\mathbb{C}}(n) = \mathrm{Iso}(\mathbb{C}^n, g_0)$ in this case, $(**)$ holds. As in I, there exists a smooth function τ on M such that the conformal class $\tau \cdot g$ is a Kähler metric. $(M, \tau \cdot g)$ is holomorphically isometric to a complete complex euclidean space form $\mathbb{C}^n / \rho(\pi_1(M))$. (Equivalently, (M, g) is conformally equivalent to $\mathbb{C}^n / \rho(\pi_1(M))$ preserving complex structures. Note that all distinct affine isomorphism classes of $\mathbb{C}^n / \rho(\pi_1(M))$ consists of finitely many by the Bieberbach theorem on affine rigidity.) Otherwise, M is an incomplete similarity manifold. Just taking a domain from $\mathbb{C}^n - \{0\}$, there are infinitely many incomplete l.c. Kähler flat manifolds. Moreover, even if M is compact, dev : $\tilde{M} \longrightarrow \mathbb{C}^n$ is not necessarily a homeomorphism. Because the holonomy $\rho(\pi_1(M))$ dose not preserve g_0 in general. However when M is a compact incomplete similarity manifold, the theorem of Fried [6] shows that dev : $\tilde{M} \longrightarrow \mathbb{R}^{2n} - \{0\}$ is a homeomorphism such that $\rho(\pi_1(M))$

$\subset O(2n)\times\mathbb{R}^+$ (up to conjugacy). As a consequence, $\rho(\pi_1(M))$ stabilizes the origin, and since $\rho(\pi_1(M))\subset\mathbb{C}^n\rtimes(U(n)\times\mathbb{R}^+)$, we have that $\rho(\pi_1(M))\subset U(n)\times\mathbb{R}^+$ for which $\mathbb{R}^{2n}=\mathbb{C}^n$. Then dev induces a homeomorphism $\widehat{\mathrm{dev}}$ of M onto the infra-Hopf manifold $S^1\underset{F}{\times}S^{2n-1}$ for a finite subgroup $F\subset U(n)$. We show that $\widehat{\mathrm{dev}}$ is a conformal diffeomorphism preserving the complex structures. Let $P:\mathbb{C}^n-\{0\}\longrightarrow S^1\underset{F}{\times}S^{2n-1}$ be the covering map. Recall from [9] that a locally conformal Kähler structure $\{V_\alpha,\Theta_\alpha\}_{\alpha\in\Lambda}$ on $S^1\underset{F}{\times}S^{2n-1}$ defines a Hermitian metric h such that $h|V_\alpha=s_\alpha\cdot h_\alpha$ for some function s_α in which $h_\alpha(X,Y)=\Theta_\alpha(X,JY)$ is a Kähler metric satisfying that $P^*h_\alpha=g_{Y'}|\tilde{V}_\alpha$ for each $\alpha\in\Lambda$. Here $Y'=\mathbb{C}^n-\{0\}$. Calculate:

$$\begin{aligned}\widehat{\mathrm{dev}}^*h&=(P\circ\varphi_\alpha)^*h\\&=\varphi_\alpha^*P^*s_\alpha\cdot\varphi_\alpha^*g_{Y'}\\&=\widehat{\mathrm{dev}}^*s_\alpha\cdot c_\alpha\cdot g_\alpha\\&=\widehat{\mathrm{dev}}^*s_\alpha\cdot c_\alpha\cdot f_\alpha^{-1}\cdot g\quad\text{on } U_\alpha.\end{aligned}$$

Chasing over the intersections $U_\alpha\cap U_\beta$ as before, we obtain a global function $\mu:M\longrightarrow\mathbb{R}^+$ such that $\mu|U_\alpha=\widehat{\mathrm{dev}}^*s_\alpha\cdot c_\alpha\cdot f_\alpha^{-1}$. In particular, $\widehat{\mathrm{dev}}^*h=\mu\cdot g$ on M. It is easily seen that such diffeomorphism preserves the complex structure. (Compare [9].) Hence $(M,\mu\cdot g)$ is holomorphically isometric to the infra-Hopf manifold $(S^1\underset{F}{\times}S^{2n-1},h)$.

Finally we can state the following result as to the problem in Introduction, which is also a generalization and refinement of Theorem 3.4 of Vaisman [25].

Theorem 4. *Let (M,g) be an l.c. K-const manifold of dimenstion $2n$ $(n>1)$.*

(i) Suppose that the holomorphic sectional curvature of g_α, $k_\alpha=0$ for some α. Then M is a similarity manifold . According to whether (compact) complete or compact incomplete, M is conformally equivalent to a (compact) complete complex euclidean space form $\mathbb{C}^n/\rho(\pi_1(M))$, or an infra-Hopf manifold $S^1\underset{F}{\times}S^{2n-1}$.
Here the conformal diffeomorphism preserves the complex structures.

(ii) Suppose that the holomorphic sectional curvature of g_α, $k_\alpha\neq0$ for some α. Then (M,g) is globally conformal Kähler, i.e., there exists a smooth function τ on M such that the conformal class $\tau\cdot g$ is a Kähler metric. Moreover, $(M,\tau\cdot g)$ is locally holomorphically isometric to $\mathbb{CP}^n$ or $\mathbb{H}_{\mathbb{C}}^n$ according as $k_\alpha>0$, $k_\alpha<0$. In paticular, if M is compact, then $(M,\tau\cdot g)$ is holomorphically isometric to the complex projective space $\mathbb{CP}^n$ or the Kähler hyperbolic space form $\mathbb{H}_{\mathbb{C}}^n/\rho(\pi_1(M))$ where $\rho(\pi_1(M))\subset PU(n,1)$. It is unique up to the scalar multiple of the Kähler metric.

Note that by the definition of l.c. K-const manifold, the product $\mathbb{H}_{\mathbb{C}}^m\times\mathbb{CP}^{n-m}$ does not show up.

In *(i)*, it is known that all Chern classes vanish (especially in the case that M is incomplete but not compact). I was taught by Kashiwada that a more general version of Theorem 4 can be obtained by using the Hermitian Einstein-Weyl connection. (See [18] for the related work.)

2. Conformally flat l.c. Kähler manifolds

As is noted in Introduction, the Riemannian metric of constant sectional curvature is locally conformal to the euclidean metric. Let (M, g) be an l.c. Kähler manifold of dimension $2n$ ($n>1$) where g is the canonical Hermitian metric obtained in Introduction. Then (M, g) is conformally flat if and only if each g_α is of zero sectional curvature provided that $2n>4$. We treat the case that $2n$ is bigger than 4 in this section.

Proposition 5. *Let (M, g) be an l.c. Kähler $2n$-manifold ($n>2$). If (M, g) is conformally flat, then M is uniformizable over $\mathbb{R}^{2n}$ with respect to* $\mathrm{Sim}(\mathbb{R}^{2n})$. *In particular, M is a similarity manifold.*

In contrast to conformally flat Kähler manifolds, we shall prove the following.

Corollary 6. *Let (M, g) be a compact conformally flat l.c. Kähler manifold of dimension $2n$ ($n>2$). Then M is conformally equivalent to either a complex euclidean space form $\mathbb{C}^n/\Gamma$ ($\Gamma\subset \mathrm{E_C}(n)$), or an infra-Hopf manifold $S^{2n-1}\times_F S^1$ ($F\subset \mathrm{U}(n)\times S^1$) preserving the complex structures.*

Proof of Proposition 5. As $n>2$, the Weyl curvature tensor $W(g)=0$. Since g is locally conformally equivalent to g_α on each U_α, $W(g_\alpha)=0$ on U_α. Thus each U_α is a conformally flat Kähler manifold. It is known (see p. 69 in [29], cf. also [1], [22]) that whenever $n>2$, any conformally flat Kähler $2n$-manifold is flat. Let g_0 be the standard euclidean metric on $\mathbb{R}^{2n}$ as before. As we can assume that each U_α is simply connected passing to a small open subset of U_α if necessary, there is an isometric embedding (developing map) $\varphi:(U_\alpha, g_\alpha)\rightarrow(\mathbb{R}^{2n}, g_0)$. Applying to each neighborhood U_α in M, we obtain a family of charts $\{U_\alpha, \varphi_\alpha\}_{\alpha\in\Lambda}$ on M.
Suppose that $U_\alpha\cap U_\beta\neq\emptyset$. Since $\varphi_\alpha: U_\alpha\rightarrow\mathbb{R}^{2n}$ is a local isometry as well as $\varphi_\beta: U_\beta\rightarrow\mathbb{R}^{2n}$, we have

$$g_\alpha(X, Y)=g_0((\varphi_\alpha)_*X, (\varphi_\alpha)_*Y), \quad g_\beta(X, Y)=g_0((\varphi_\beta)_*X, (\varphi_\beta)_*Y)$$

for $X, Y\in T_x(U_\alpha\cap U_\beta)$. Put $g_{\beta\alpha}=\varphi_\beta\circ\varphi_\alpha^{-1}: \varphi_\alpha(U_\alpha\cap U_\beta)\rightarrow\varphi_\beta(U_\alpha\cap U_\beta)$. Recall that $g_\beta=\lambda_{\beta\alpha}\cdot g_\alpha$ for a constant $\lambda_{\beta\alpha}$ on $U_\alpha\cap U_\beta$. If $v, w\in T_{\varphi_\alpha(x)}\mathbb{R}^{2n}$, then

$$\begin{aligned} g_0((g_{\beta\alpha})_*v, (g_{\beta\alpha})_*w) &= g_0((\varphi_\beta)_*(\varphi_\alpha)_*^{-1}v, (\varphi_\beta)_*(\varphi_\alpha)_*^{-1}w) \\ &= g_\beta((\varphi_\alpha)_*^{-1}v, (\varphi_\alpha)_*^{-1}w) \\ &= \lambda_{\beta\alpha}\cdot g_\alpha((\varphi_\alpha)_*^{-1}v, (\varphi_\alpha)_*^{-1}w) \\ &= \lambda_{\beta\alpha}\cdot g_0(v, w). \end{aligned}$$

As $\lambda_{\beta\alpha}$ is a constant, $g_{\beta\alpha}$ is a locally homothetic transformation defined on a neighborhood of $\mathbb{R}^{2n}$. Then the local change $g_{\beta\alpha}$ is a restriction of an element of $\mathrm{Sim}(\mathbb{R}^{2n})=\mathbb{R}^{2n}\rtimes(\mathrm{O}(2n)\times\mathbb{R}^+)$. Hence $\{U_\alpha, \varphi_\alpha\}_{\alpha\in\Lambda}$ gives a uniformization on (M, g) with respect to $(\mathrm{Sim}(\mathbb{R}^{2n}), \mathbb{R}^{2n})$. □

Proof of Corollary 6. Now, M is a compact similarity manifold. By the result of Fried [6], the developing map $\mathrm{dev}:\tilde{M}\to\mathbb{R}^{2n}$ is homeomorphic or $\mathrm{dev}:\tilde{M}\to\mathbb{R}^{2n}-\{0\}$ is homeomorphic according as whether M is complete or incomplete. Put $\Gamma=\rho(\pi_1(M))$ and $\mathrm{E}(2n)=\mathbb{R}^{2n}\rtimes\mathrm{O}(2n)$ which is the group of rigid motions of $\mathbb{R}^{2n}$. Then dev induces a homeomorphism $\widehat{\mathrm{dev}}:M\longrightarrow X/\Gamma$ where $X=\mathbb{R}^{2n}$ with $\Gamma\subset\mathrm{E}(2n)$ or $X=\mathbb{R}^{2n}-\{0\}=S^{2n-1}\times\mathbb{R}^+$ with $\Gamma\subset\mathrm{O}(2n)\times\mathbb{R}^+$ respectively.

Case 1. $(\mathrm{E}(2n), \mathbb{R}^{2n})$. Let $\hat{g}_0$ be the induced Riemannian metric on $\mathbb{R}^{2n}/\Gamma$. Since $g_{\beta\alpha}$ lies in $\mathrm{E}(2n)=\mathrm{Iso}(\mathbb{R}^{2n}, g_0)^+$, the above equation implies that $\lambda_{\beta\alpha}=1$ on $U_\alpha\cap U_\beta$. By the definition of $\lambda_{\beta\alpha}$, $f_\alpha=f_\beta$. We can define a global function $\tau':M\longrightarrow\mathbb{R}^+$ to be $\tau'(x)=f_\alpha^{-1}(x)$ if $x\in U_\alpha$. If we note that $\varphi_\alpha{}^*g_0=g_\alpha$ for each $\alpha\in\Lambda$, and that $\widehat{\mathrm{dev}}\,|\,U_\alpha=P\circ\varphi_\alpha$,

$$\begin{aligned} \widehat{\mathrm{dev}}^*\hat{g}_0\,|\,U_\alpha &= \varphi_\alpha^*P^*\hat{g}_0\,|\,U_\alpha \\ &= \varphi_\alpha^*g_0\,|\,U_\alpha \\ &= g_\alpha=f_\alpha^{-1}\cdot g \\ &= \tau'\cdot g\,|\,U_\alpha. \end{aligned}$$

Thus we have $\widehat{\mathrm{dev}}^*\hat{g}_0=\tau'\cdot g$. Hence (M, g) is conformally equivalent to the euclidean space form $(\mathbb{R}^{2n}/\Gamma, \hat{g}_0)$.
Since g is Hermitian with respect to J, conjugate by $\widehat{\mathrm{dev}}$, there is a complex structure on $\mathbb{R}^{2n}/\Gamma$ invariant under $\hat{g}_0$. Then Γ sits inside $\mathbb{C}^n\rtimes\mathrm{U}(n)$ so that $\mathbb{R}^{2n}/\Gamma$ is isometric to $\mathbb{C}^n/\Gamma$.

Case 2. $(\mathrm{O}(2n)\times\mathbb{R}^+, S^{2n-1}\times\mathbb{R}^+)$. We have the family of local Kähler metrics $\{V_\alpha, h_\alpha\}_{\alpha\in\Lambda}$ and the Hermitian metric h on the infra-Hopf manifold $S^{2n-1}\underset{F}{\times}S^1$ satisfying that:

$$h\,|\,V_\alpha=s_\alpha\cdot h_\alpha,\quad h_\beta=a_{\beta\alpha}\cdot h_\alpha,\quad a_{\beta\alpha}=\text{const},$$
$$a_{\beta\alpha}=s_\alpha\cdot s_\beta^{-1},\quad g_0\,|\,\tilde{V}_\alpha=P^*h_\alpha.$$

If we note that

$$\begin{aligned}\widehat{\text{dev}}^* h_\alpha | U_\alpha &= (\varphi_\alpha^* P^*) h_\alpha | U_\alpha \\ &= \varphi_\alpha^*(g_0 | \tilde{V}_\alpha) = g_\alpha,\end{aligned}$$

and from the property that $g_\beta = \lambda_{\beta\alpha} \cdot g_\alpha$ on $U_\alpha \cap U_\beta$, we obtain that $a_{\beta\alpha} = \lambda_{\beta\alpha}$. As s_α is defined on $V_\alpha = \widehat{\text{dev}}(U_\alpha)$, the equality $a_{\beta\alpha} = \lambda_{\beta\alpha}$ implies that $\widehat{\text{dev}}^* s_\beta \cdot f_\beta^{-1} = \widehat{\text{dev}}^* s_\alpha \cdot f_\alpha^{-1}$ on $U_\alpha \cap U_\beta$. So there is a global function $\mu' : M \longrightarrow \mathbb{R}^+$ such that $\mu' | U_\alpha = \widehat{\text{dev}}^* s_\alpha \cdot f_\alpha^{-1}$. Then,

$$\begin{aligned}\widehat{\text{dev}}^* h | U_\alpha &= \widehat{\text{dev}}^*(s_\alpha \cdot h_\alpha) = \widehat{\text{dev}}^* s_\alpha \cdot g_\alpha \\ &= \widehat{\text{dev}}^* s_\alpha \cdot f_\alpha^{-1} \cdot g | U_\alpha \\ &= \mu' \cdot g | U_\alpha.\end{aligned}$$

Since $\widehat{\text{dev}}(U_\alpha) = V_\alpha$ for each α, $\widehat{\text{dev}}^* h = \mu' \cdot g$. Hence (M, g) is conformally equivalent to the infra-Hopf manifold $(S^1 \underset{F}{\times} S^{2n-1}, h)$. As above, the complex structure J on M defines a complex structure on $S^1 \underset{F}{\times} S^{2n-1}$ which makes h Hermitian, yielding that $F \subset \mathrm{U}(n) \times S^1 \subset \mathrm{O}(2n) \times S^1$. As a consequence, $\widehat{\text{dev}}$ is a conformal diffeomophism of M onto $S^1 \underset{F}{\times} S^{2n-1}$ preserving the complex structure on each. □

3. Conformally flat l.c. Kähler manifolds of dimension 4

Let W be the Weyl curvature tensor as a Riemannian manifold M. If dim $M = 4$, there is the further decomposition of Weyl curvature tensor $W = W_+ + W_-$. (See [23] for example.) Suppose that (M, g, J) is a Kähler 4-manifold. Then there is the Bochner curvature tensor B on M. It is well known that $B = W_-$ for Kähler metrics in dimension 4. A Riemannian 4-manifold with $W_- = 0$ is said to be *self dual*. As a consequence a self dual Kähler 4-manifold is a Bochner flat Kähler 4-manifold.

Proposition 7. *Let (M, g, J) be an l.c. Kähler 4-manifold. If (M, g, J) is conformally flat, then M is uniformizable with respect to either the complex similarity geometry* $(\mathbb{C}^2 \rtimes (\mathrm{U}(2) \times \mathbb{R}^+), \mathbb{C}^2)$ *or the projective-hyperbolic geometry* $(\mathrm{PU}(1,1) \times \mathrm{PU}(2), \mathbb{H}_{\mathbb{C}}^1 \times \mathbb{CP}^1)$.

Proof. If (M, g) is conformally flat, then the Weyl curvature tensor $W(g) = 0$. Since the Weyl curvature tensor is a conformal invariant, $W(g_\alpha) = 0$ for each α. In particular, $W_-(g_\alpha) = 0$ from the decomposition of Weyl curvature tensor. By the above remark, as $B = W_-$, we have $B(g_\alpha) = 0$. Again by the conformal invariance of the Bochner curvature tensor, $B(g) = 0$. Hence, (M, g, J) is a Bochner flat l.c. Kähler manifold. So it is uniformizable with respect to the four geometries of Uniformization Theorem 1. Moreover, the uniformization $\{U_\alpha, \varphi_\alpha\}_{\alpha \in \Lambda}$ of $(M, g,$

$J\}$ satisfies that $c_\alpha \cdot g_\alpha = \varphi_\alpha^* g_Y$ for some constant c_α. On the other hand, as above $W(g_\alpha)=0$ for each α. Each (U_α, g_α) is also conformally flat. By the tensor calculus, the scalar curvature of a conformally flat Kähler manifold is zero (for example, cf. [29]). The scalar curvature of each (U_α, g_α) is zero. The above relation that $c_\alpha \cdot g_\alpha = \varphi_\alpha^* g_Y$ implies that the scalar curvature of our model space $(Y_{\mathbb{C}}^2, g_Y)$ is also zero. Hence the geometry $(Y_{\mathbb{C}}^2, g_Y)$ must be either one of the complex similarity geometry $(\mathbb{C}^2 \rtimes (\mathrm{U}(2) \times \mathbb{R}^+), \mathbb{C}^n)$ or the projective-hyperbolic geometry $(\mathrm{PU}(1,1) \times \mathrm{PU}(2), \mathbb{H}_{\mathbb{C}}^1 \times \mathbb{CP}^1)$. The only thing is left to check that each geometry admits a conformally flat geometry. In fact, let $(\mathrm{Conf}(S^n), S^n)$ be the conformal geometry where $\mathrm{Conf}(S^n) = \mathrm{PO}(n+1, 1)$. Then it is known that $(\mathrm{Conf}(\mathbb{R}^n), \mathbb{R}^n)$ is the similarity geometry $(\mathrm{Sim}(\mathbb{R}^n), \mathbb{R}^n) = (\mathbb{R}^n \rtimes (\mathrm{O}(n) \times \mathbb{R}^+), \mathbb{R}^n)$ where $S^n - \{0\} = \mathbb{R}^n$ by the stereographic projection. The complex similarity geometry is a subgeometry of the similarity geometry $(\mathrm{Sim}(\mathbb{R}^n), \mathbb{R}^n)$. The sphere complement $S^n - S^{m-1}$ is conformally equivalent to the product of the real hyperbolic space and the sphere $\mathbb{H}_{\mathbb{R}}^m \times S^{n-m}$ in which the conformal subgroup $\mathrm{Conf}(S^n - S^{m-1}) = \mathrm{PO}(m, 1) \times \mathrm{O}(n-m+1)$, which is the group of isometries of the the product $\mathbb{H}_{\mathbb{R}}^m \times S^{n-m}$. (See [11].) So if $n=4$, $m=2$, the conformally flat geometry $(\mathrm{Conf}(S^4 - S^1), S^4 - S^1)$ is identified with $(\mathrm{PO}(2,1) \times \mathrm{O}(3), \mathbb{H}_{\mathbb{R}}^2 \times S^2)$. If we note that $\mathbb{H}_{\mathbb{C}}^1$ is isometric to $\mathbb{H}_{\mathbb{R}}^2$ under the low dimensional Lie group identification, we have the canonical isomorphism:

$$(\mathrm{PU}(1,1) \times \mathrm{PU}(2), \mathbb{H}_{\mathbb{C}}^1 \times \mathbb{CP}^1) = (\mathrm{PO}(2,1)^0 \times \mathrm{O}(3)^0, \mathbb{H}_{\mathbb{R}}^2 \times S^2).$$

Hence both geomotry admits a conformally flat structure.

□

Theorem 8. *Let (M, g) be a compact conformally flat l.c. Kähler manifold of dimension 4. Then M is conformally equivalent to one of the following l.c. Kähler manifolds preserving the complex structures.*

(*i*) *A complex euclidean space form $T_{\mathbb{C}}^2/F$ $(F \subset \mathrm{U}(2))$.*

(*ii*) *A fiber space $\mathbb{H}_{\mathbb{C}}^1 \underset{\Gamma}{\times} \mathbb{CP}^1$ $(\Gamma \subset \mathrm{PU}(1,1) \times \mathrm{PU}(2))$.*

(*iii*) *An infra-Hopf manifold $S^3 \underset{F}{\times} S^1$ $(F \subset \mathrm{U}(2) \times S^1)$.*

Proof. If (M, g) is conformally flat, then (M, g) is a Bochner flat l.c. Kähler manifold which is uniformizable over $(\mathbb{C}^2 \rtimes (\mathrm{U}(2) \times \mathbb{R}^+), \mathbb{C}^n)$ or $(\mathrm{PU}(1,1) \times \mathrm{PU}(2), \mathbb{H}_{\mathbb{C}}^1 \times \mathbb{CP}^1)$ by Proposition 7. As in the proof of [9] (also the arguments of § 1, 2), (M, g) is conformally equivalent to either a complex euclidean space form $\mathbb{C}^2/\Gamma$ $(\Gamma \subset \mathrm{E}_{\mathbb{C}}(2))$, or an infra-Hopf manifold $S^3 \underset{F}{\times} S^1$ $(F \subset \mathrm{U}(2) \times S^1)$ according as whether M is complete or not, or (M, g) is conformally equivalent to the S^2 bundle over hyperbolic orbifold; $\mathbb{H}_{\mathbb{C}}^1 \underset{\Gamma}{\times} \mathbb{CP}^1$. Note that in each case, some finite cover of M is either a complex torus $T_{\mathbb{C}}^2$, a Hopf manifold $S^3 \times S^1$ or the product $S^2 \times \Sigma_g$ where Σ_g is a closed orientable surface of genus $g \geqq 2$.

□

Remark 9. *We notice that* (Conf(S^4-S^2), S^4-S^2)=(PO(3, 1)×O(2), $\mathbb{H}^3_{\mathbb{R}}\times S^1$). *Choosing a torsionfree discrete uniform subgroup* $\Gamma\subset$PO(3, 1), *we have a compact hyperbolic 3-manifold. Since* $\mathbb{H}^3_{\mathbb{R}}/\Gamma$ *is of dimension three,* $\mathbb{H}^3_{\mathbb{R}}/\Gamma$ *is a CR-manifold but not admitting a circle action. The construction of* [9] *gives a locally conformal symplectic structure on the product* $\mathbb{H}^3_{\mathbb{R}}/\Gamma\times S^1$. *So we obtain a conformally flat locally conformal symplectic 4-manifold* $\mathbb{H}^3_{\mathbb{R}}/\Gamma\times S^1$. *By our construction, a complex structure on the CR-manifold* $\mathbb{H}^3_{\mathbb{R}}/\Gamma$ *provides an almost complex structure on* $\mathbb{H}^3_{\mathbb{R}}/\Gamma\times S^1$, *which is not integrable because the characteristic vector field on CR-manifold* $\mathbb{H}^3_{\mathbb{R}}/\Gamma$ *is not Killing, (equivalently the CR-manifold* $\mathbb{H}^3_{\mathbb{R}}/\Gamma$ *is not K-contact.) We donot know whether there exists an exotic l.c. Kähler structure on* $\mathbb{H}^3_{\mathbb{R}}/\Gamma\times S^1$. Compare [20].

4. Transformation groups of l.c. Kähler manifolds

Let $(M^{2n}, J, \{U_\alpha, \Omega_\alpha\}_{\alpha\in\Lambda})$ be an l.c. Kähler manifold associated with the canonical Hermitian metric g. We denote by Aut $_{l.c.K}(M)$ the group of locally conformal Kähler transformations, *i. e.*, an element $f: M\to M$ is a diffeomorphism such that $f_*\circ J=J\circ f_*$, and when $f(x)\in U_\beta$ for each $x\in U_\alpha$, $f^*\Omega_\beta=\mu_{\beta\alpha}\cdot\Omega_\alpha$ for some constant $\mu_{\beta\alpha}>0$. In addition, if $f^*\Omega_\beta=\Omega_\alpha$ for each element $\alpha\in\Lambda$, then f is said to be a locally Kählerian transformation, its group denoted by Aut $_{l.K}(M)$. Since $\mu_{\beta\alpha}$ is constant, each element f of Aut $_{l.c.K}(M)$ is also locally homothetric. By the property that $f_*\circ J=J\circ f_*$, Aut $_{l.c.K}(M)$ is a subgroup of holomorphic transformations of the complex manifold (M, J). On the other hand, it is known that the group of conformal transformations of a Riemannian manifold (M, g), Conf(M), is a Lie group provided that dim $M\geqq 3$ (cf. [13]). We shall prove that Aut $_{l.c.K}(M)$ is a Lie group. In order to do so, first recall the following (cf. [29]).

Lemma 10. *A conformal transformation preserving complex structures between Kähler manifolds* $(M^{2n}, g, \Omega, J), (N^{2n}, g', \Omega', J')$ *is necessarily homothetic, provided that* $n>1$. *Namely, if* $h^*g'=\lambda\cdot g$ *and* $h_*\circ J=J'\circ h_*$ *(equivalently,* $h^*\Omega'=\lambda\cdot\Omega$, $h_*\circ J=J'\circ h_*$*), then* λ *is a constant.*

Proposition 11. *Let* Conf(M) *be the group of conformal transformations of an l.c. Kähler manifold* M *with the associated Hermitian metric* g *as before. If* dim $M=2n>2$, *then* Aut $_{l.c.K}(M)$ *is a closed subgroup of* Conf(M) *whose elements preserve the complex structure* J.

Proof. Let $h\in$ Aut $_{l.c.K}(M)$. For $x\in U_\alpha$, if $h(x)\in U_\beta$, then $(h^*g_\beta)_x=\mu_{\beta\alpha}\cdot(g_\alpha)_x$ and $h_*\circ J=J\circ h_*$ by the definition. Suppose that $h(x)\in U_\gamma$ for some γ. The property $g_\beta=\lambda_{\beta\gamma}\cdot g_\gamma$ implies that $\mu_{\beta\alpha}\cdot g_\alpha=\lambda_{\beta\gamma}\cdot\mu_{\gamma\alpha}\cdot g_\alpha$. Thus

$$\mu_{\beta\alpha}=\lambda_{\beta\gamma}\cdot\mu_{\gamma\alpha} \text{ on } U_\beta\cap U_\gamma.$$

For $h(x)\in U_\beta\cap U_\gamma$ and $x\in U_\alpha$,

$$\begin{aligned} f_\gamma(h(x))\cdot\mu_{\gamma\alpha}\cdot f_\alpha(x)^{-1} &= \lambda_{\beta\gamma}\cdot f_\beta(h(x))\cdot\mu_{\gamma\alpha}\cdot f_\alpha(x)^{-1} \\ &= f_\beta(h(x))\cdot\mu_{\beta\alpha}\cdot f_\alpha(x)^{-1}. \end{aligned}$$

Hence the function $f_\beta(h(x))\cdot\mu_{\beta\alpha}\cdot f_\alpha(x)^{-1}$ does not depend on the domain of $h(x)$. Put

$$\tau_\alpha(x)=f_\beta(h(x))\cdot\mu_{\beta\alpha}\cdot f_\alpha(x)^{-1} \text{ on } U_\alpha.$$

If $x\in U_\alpha\cap U_\delta$, then $h^*g_\beta=\mu_{\beta\delta}\cdot g_\delta$ as above. Since $g_\delta=\lambda_{\delta\alpha}\cdot g_\alpha$ and $h^*g_\beta=\mu_{\beta\alpha}\cdot g_\alpha$, we have that

$$\mu_{\beta\alpha}=\mu_{\beta\delta}\cdot\lambda_{\delta\alpha} \text{ on } U_\alpha\cap U_\delta.$$

As $f_\alpha=\lambda_{\delta\alpha}\cdot f_\delta$,

$$\begin{aligned} \tau_\alpha(x) &= f_\beta(h(x))\cdot\mu_{\beta\alpha}\cdot f_\alpha(x)^{-1} \\ &= f_\beta(h(x))\cdot\mu_{\beta\delta}\cdot\lambda_{\delta\alpha}\cdot f_\alpha(x)^{-1} \\ &= f_\beta(h(x))\cdot\mu_{\beta\delta}\cdot f_\delta(x)^{-1}=\tau_\delta(x). \end{aligned}$$

So, there is a positive function $\tau: M\longrightarrow\mathbb{R}$ such that $\tau\,|\,U_\alpha=\tau_\alpha$. Since

$$\begin{aligned} (h^*g)_x &= (h^*(f_\beta\cdot g_\beta))_x \\ &= f_\beta(h(x))\cdot(h^*g_\beta)_x \\ &= f_\beta(h(x))\cdot\mu_{\beta\alpha}\cdot(g_\alpha)_x \\ &= f_\beta(h(x))\cdot\mu_{\beta\alpha}\cdot f_\alpha(x)^{-1}\cdot g_x \\ &= \tau(x)\cdot g_x, \end{aligned}$$

h is a conformal map with respect to g, *i.e.*, $h\in \mathrm{Conf}(M)$.
Conversely, let $h\in\mathrm{Conf}(M)$. Then $(h^*g)_x=\mu(x)\cdot g_x$ for some positive function μ. The property $g\,|\,U_\beta=f_\beta\cdot g_\beta$ shows that for $x\in U_\alpha$, $h(x)\in U_\beta$,

$$\begin{aligned} (h^*g_\beta)_x &= f_\beta(h(x))^{-1}\cdot\mu(x)\cdot g_x \\ &= f_\beta(h(x))^{-1}\cdot\mu(x)\cdot f_\alpha(x)^{-1}\cdot(g_\alpha)_x. \end{aligned}$$

By Lemma 10 and if h preserves J, $f_\beta(h(x))^{-1}\cdot\mu(x)\cdot f_\alpha(x)^{-1}$ is a constant $\mu_{\beta\alpha}$ on U_α. In other words, the above formula becomes

$$(h^*\Omega_\beta)_x=\mu_{\beta\alpha}\cdot(\Omega_\alpha)_x.$$

Therefore, $h \in \mathrm{Aut}_{l.c.K}(M)$.

□

Theorem 12. *Let* $\mathrm{Aut}_{l.c.K}(M)$ *be the group of locally conformal Kähler transformations of an l.c. Kähler manifold* M *of dimension* $2n$. *If* M *is compact and* $n>1$, *then* $\mathrm{Aut}_{l.c.K}(M)$ *is a compact Lie group.*

Proof. If $\mathrm{Aut}_{l.c.K}(M)$ is noncompact, then so is $\mathrm{Conf}(M)$ by Proposition 11. Applying the Obata and Lelong-Ferrand's result to (M, g) (cf. [21], [15], [10]) yields that (M, g) is conformally equivalent to the standard sphere S^{2n}. In particular, (M, g) is conformally flat. Since M is a compact conformally flat l.c. Kähler manifold of dimension $2n$ $(n \geq 2)$, this contradicts the conclusion of Corollary 6, Theorem 8.

□

5. Structure of l.c. Kähler transformation groups

Let (M, J) be a locally conformal Kähler manifold equipped with a locally conformal Kähler structure $\{U_\alpha, \Omega_\alpha\}_{\alpha \in \Lambda}$. There is a 2-form Ω such that $\Omega | U_\alpha = \lambda_\alpha \cdot \Omega_\alpha$ for each $\alpha \in \Lambda$. Recall that Ω is not necessarily a Kähler form on M. Let $P: \tilde{M} \to M$ be the covering map where $\tilde{M}$ is the universal covering space of M. There exists a 2-form $\tilde{\Omega}$ on $(\tilde{M}, \tilde{J})$ such that $P^*\Omega = \tilde{\Omega}$ and let $\tilde{J}$ be a lift of J. As $f_\alpha = \lambda_{\beta\alpha} \cdot f_\beta$ on $U_\alpha \cap U_\beta$, letting $\theta | U_\alpha = d \log f_\alpha$, there is a global 1-form θ on M such that $d\theta = 0$. Then we have $d\Omega = \theta \wedge \Omega$. If $\tilde{\theta}$ is a lift of θ to $\tilde{M}$, then $d\tilde{\theta} = 0$ implies that there exists a function f on $\tilde{M}$ such that $df = \tilde{\theta}$. Moreover, as $d\tilde{\Omega} = \tilde{\theta} \wedge \tilde{\Omega}$, $d\tilde{\Omega} - df \wedge \tilde{\Omega} = 0$. Put

$$\bar{\Theta} = e^{-f} \cdot \tilde{\Omega}.$$

Then,

$$d\bar{\Theta} = -e^{-f} df \wedge \tilde{\Omega} + e^{-f} d\tilde{\Omega} = e^{-f}(-df \wedge \tilde{\Omega} + d\tilde{\Omega}) = 0.$$

Since $\Omega^n | U_\alpha = f_\alpha^n \cdot \Omega_\alpha^n \neq 0$, and $\tilde{J}$ preserves $\bar{\Theta}$, $\bar{\Theta}$ becomes a Kähler form on $\tilde{M}$. Put

$$\mathrm{Aut}_{h.K}(\tilde{M}, \bar{\Theta}, \tilde{J}) = \{f : \tilde{M} \to \tilde{M} \text{ is a holomorphic transformation with respect to } \tilde{J} \mid f^*\bar{\Theta} = c_f \cdot \bar{\Theta} \text{ for some constant } c_f\}.$$

Then $\mathrm{Aut}_{h.K}(\tilde{M}, \bar{\Theta}, \tilde{J})$ is the group of homothetic Kähler transformations. Note that $c_f \neq 0$ is uniquely determined by f. Let ρ be the map which assigns to f a number c_f. Then the correspondence:

$$\rho : \mathrm{Aut}_{h.K}(\tilde{W}, \bar{\Theta}) \longrightarrow \mathbb{R}^+$$

is a continuous homomorphism. Put $\pi_1(M)=\pi$.

Lemma 13. *If $\alpha\in\pi$, then $\alpha^*\bar{\Theta}=c\cdot\bar{\Theta}$ for some constant c. In particular, $\pi\subset \mathrm{Aut}_{h.K}(\tilde{M},\bar{\Theta},\tilde{J})$.*

Proof. Calculate

$$\begin{aligned}\alpha^*\bar{\Theta}&=\alpha^*(e^{-f}\cdot\tilde{\Omega})\\ &=e^{-\alpha^*f}\cdot\alpha^*\tilde{\Omega}\\ &=e^{-\alpha^*f}\cdot\tilde{\Omega}\\ &=e^{-\alpha^*f}\cdot e^f\bar{\Theta}\\ &=e^{f-\alpha^*f}\cdot\bar{\Theta}.\end{aligned}$$

As $\tilde{\theta}$ is the lift to $\tilde{M}$, note that each α leaves $\tilde{\theta}$ invariant, so $d\alpha^*f=df$. Thus $f-\alpha^*f=\bar{c}$ for some constant $\bar{c}$ on $\tilde{M}$. If we put $c=e^{\bar{c}}$, then we have $\alpha^*\bar{\Theta}=c\cdot\bar{\Theta}$.

□

Note that the Kähler form $\bar{\Theta}$ does not come from a lift of a 2-form of M. However, if $\rho(\pi)=1$, then $\bar{\Theta}$ induces a Kähler form Θ on M.

Proposition 14. A locally conformal Kähler structure on a complex manifold M becomes a Kähler structure if and only if $\rho:\pi\longrightarrow\mathbb{R}$ is trivial.

According to [3], [14], there is the exact sequence: $1\to\pi\to N_{\mathrm{Diff}(\tilde{M})}(\pi)\xrightarrow{\nu}\mathrm{Diff}(M)\to 1$ where $N_{\mathrm{Diff}(\tilde{M})}(\pi)$ is the normalizer of π in the diffemorphism group $\mathrm{Diff}(\tilde{M})$. We have a refinement about the group $\mathrm{Aut}_{l.c.K}(M)$. First we prove the following.

Proposition 15. There is an exact sequence:

$$1\to\pi\to N_{\mathrm{Aut}_{h.K}(\tilde{M},\bar{\Theta})}(\pi)\xrightarrow{\nu}\mathrm{Aut}_{l.c.K}(M)\to 1.$$

Proof. If $\tilde{h}\in N_{\mathrm{Aut}_{h.K}(\tilde{M},\bar{\Theta})}(\pi)$, then $\tilde{h}^*\bar{\Theta}=c\cdot\bar{\Theta}$. Assume that $\tilde{h}:\tilde{U}_\alpha\to\tilde{U}_\beta$. By the commutativity, $P\circ\tilde{h}=\nu(\tilde{h})\circ P$. Put $\nu(\tilde{h})=h\in\mathrm{Diff}(W)$. Define a 2-form Θ_α on U_α to be

$$P^*\Theta_\alpha=\bar{\Theta}\,|\,\tilde{U}_\alpha.$$

Since each Θ_α is invariant under J, we have a family of Kähler forms $\{U_\alpha,\Theta_\alpha\}_{\alpha\in\Lambda}$ on M. As $\Omega\,|\,U_\alpha=f_\alpha\cdot\Omega_\alpha$,

$$\bar{\Theta}\,|\,\tilde{U}_\alpha=e^{-f}\cdot\tilde{\Omega}\,|\,\tilde{U}_\alpha=e^{-f}\cdot P^*f_\alpha\cdot\tilde{\Omega}_\alpha.$$

Since both $\bar{\Theta}$ and $\tilde{\Omega}_\alpha$ are Kähler forms, $e^{-f}\cdot P^*f_\alpha$ is a constant number μ_α. We have that $\Theta_\alpha=\mu_\alpha\cdot\Omega_\alpha$ on U_α for each α. Then calculate

$$\begin{aligned}\tilde{h}^*\bar{\Theta}\,|\,\tilde{U}_\alpha&=c\cdot\bar{\Theta}\,|\,\tilde{U}_\alpha\\&=c\cdot P^*\Theta_\alpha\\&=c\cdot\mu_\alpha\cdot P^*\Omega_\alpha\\&=\tilde{h}^*(\bar{\Theta}\,|\,\tilde{U}_\beta)\,|\,\tilde{U}_\alpha\\&=\tilde{h}^*(P^*\Theta_\beta)\,|\,\tilde{U}_\alpha\\&=P^*h^*\Theta_\beta\,|\,\tilde{U}_\alpha=P^*h^*\mu_\beta\cdot\Omega_\beta.\end{aligned}$$

Hence $c\cdot\mu_\alpha\cdot\mu_\beta^{-1}\cdot\Omega_\alpha=h^*\Omega_\beta$ so that $h\in \mathrm{Aut}_{\,l.c.K}(M)$.
Conversely, let $h\in\mathrm{Aut}_{\,l.c.K}(M)$. Suppose that $h: U_\alpha\to U_\beta$. Then $h^*\Omega_\beta=\lambda_{\beta\alpha}\cdot\Omega_\alpha$ for some constant $\lambda_{\beta\alpha}$. As $\Theta_\alpha=\mu_\alpha\cdot\Omega_\alpha$, we have $h^*\Theta_\beta=\mu_{\beta\alpha}\cdot\Theta_\alpha$ for a constant $\mu_{\beta\alpha}$. Consider a lift $\tilde{h}: \tilde{W}\to\tilde{W}$. Let $\tilde{x}\in\gamma\cdot\tilde{U}_\alpha$ and suppose that $\tilde{h}:\gamma\cdot\tilde{U}_\alpha\to\gamma'\cdot\tilde{U}_\beta$ for some $\gamma,\gamma'\in\pi$.
Note that $P(\tilde{x})=x\in U_\alpha$, $hx\in U_\beta$. Suppose that $X, Y\in T_{\tilde{x}}(\gamma\cdot\tilde{U}_\alpha)$. As $\gamma_*(T_{\gamma^{-1}\tilde{x}}\tilde{U}_\alpha)=T_{\tilde{x}}(\gamma\cdot\tilde{U}_\alpha)$, put $\gamma_*X'=X$, $\gamma_*Y'=Y$. Similarly, as $\gamma'_*(T_{\gamma'^{-1}\tilde{h}\tilde{x}}\tilde{U}_\beta)=T_{\tilde{h}\tilde{x}}(\gamma'\cdot\tilde{U}_\beta)$, put $\gamma'_*A=\tilde{h}_*X$, $\gamma'_*B=\tilde{h}_*Y$.

$$\begin{aligned}(\tilde{h}^*\bar{\Theta})_{\tilde{x}}(X, Y)&=\tilde{h}^*\bar{\Theta}\,|\,\gamma\cdot\tilde{U}_\alpha(X, Y)\\&=\bar{\Theta}_{\tilde{h}\tilde{x}}\,|\,\gamma'\cdot\tilde{U}_\beta(\tilde{h}_*X, \tilde{h}_*Y)\\&=\bar{\Theta}_{\tilde{h}\tilde{x}}\,|\,\gamma'\cdot\tilde{U}_\beta(\gamma'_*A, \gamma'_*B)\\&=\gamma'^*\bar{\Theta}_{\gamma'^{-1}\tilde{h}\tilde{x}}\,|\,\tilde{U}_\beta(A, B)\\&=c_\beta\cdot\bar{\Theta}_{\gamma'^{-1}\tilde{h}\tilde{x}}\,|\,\tilde{U}_\beta(A, B)\\&=c_\beta\cdot P^*\Theta_\beta(A, B)=c_\beta\cdot\Theta_\beta(P_*A, P_*B)\\&=c_\beta\cdot\Theta_\beta(h_*P_*X, h_*P_*Y)\\&=c_\beta\cdot h^*\Theta_\beta(P_*X, P_*Y)\\&=c_\beta\cdot\mu_{\beta\alpha}\cdot\Theta_\alpha(P_*X, P_*Y)\\&=c_\beta\cdot\mu_{\beta\alpha}\cdot\Theta_\alpha(P_*X', P_*Y')\\&=c_\beta\cdot\mu_{\beta\alpha}\cdot P^*\Theta_\alpha(X', Y')\\&=c_\beta\cdot\mu_{\beta\alpha}\cdot\bar{\Theta}\,|\,\tilde{U}_\alpha(X', Y')\\&=c_\beta\cdot\mu_{\beta\alpha}\cdot\bar{\Theta}\,|\,\tilde{U}_\alpha(\gamma_*^{-1}X, \gamma_*^{-1}Y)\\&=c_\beta\cdot\mu_{\beta\alpha}\cdot c_\alpha^{-1}\cdot\bar{\Theta}\,|\,\gamma\cdot\tilde{U}_\alpha(X, Y)\\&=c_\beta\cdot\mu_{\beta\alpha}\cdot c_\alpha^{-1}\cdot\bar{\Theta}_{\tilde{x}}(X, Y).\end{aligned}$$

Hence, $\tilde{h}^*\bar{\Theta}=c_\beta\cdot\mu_{\beta\alpha}\cdot c_\alpha^{-1}\cdot\bar{\Theta}$ at a neighborhood of $\tilde{x}$. Now, $c_\beta\cdot\mu_{\beta\alpha}\cdot c_\alpha^{-1}$ is a constant λ in the neighborhood of $\tilde{x}$. Moreover this is true for every neighborhood, thus $\tilde{h}^*\bar{\Theta}=\lambda\cdot\bar{\Theta}$. Since $h_*\circ J=J\circ h_*$, it is easy to see that $\tilde{h}_*\circ\tilde{J}=\tilde{J}\circ\tilde{h}_*$. Hence $\tilde{h}\in \mathrm{N}_{\mathrm{Aut}_{h.K}(\tilde{M},\bar{\Theta})}(\pi)$.

□

There is also an exact sequence:

$$1\to C(\pi)\to C_{\mathrm{Aut}_{h.K}(\tilde{M},\bar{\Theta})}(\pi)\xrightarrow{\ \nu\ }\mathrm{Aut}_{l.c.K}(M)^0\to 1$$

where $C_{\mathrm{Aut}_{h.K}(\tilde{M},\bar{\Theta})}(\pi)$ is the centralizer of π in $\mathrm{Aut}_{h.K}(\tilde{M},\bar{\Theta})$.
Finally we can show the structure on the identity component of the group of locally conformal Kähler transformations $\mathrm{Aut}_{l.c.K}(M)^0$. An element h of $\mathrm{Aut}_{l.c.K}(M)$ is *essential* if the lift of h to $\tilde{M}$ acts as a nontrivial homothetic Kähler transformation of $\tilde{M}$ with respect to $\bar{\Theta}$. Recall that $\mathrm{Aut}_{l.K}(M)$ is the group of locally Kählerian transformations.

Theorem 16. *Let M be a compact locally conformal Kähler $2n$-manifold $(n>1)$. Suppose that a circle S^1 acts as essential locally conformal Kähler transformations. Then,* $\mathrm{Aut}_{l.c.K}(M)^0$ *is isomorphic to* $\mathrm{Aut}_{l.K}(M)^0\times S^1$.

Proof. Let $\rho:\mathrm{Aut}_{h.K}(\tilde{M},\bar{\Theta})\longrightarrow\mathbb{R}^+$ be the homomorphism as above. Its kernel $\mathrm{Aut}_K(\tilde{M},\bar{\Theta})$ is the usual group of Kähler transformations of $\tilde{M}$ with respect to $\bar{\Theta}$. Then there is the commutative diagram from Proposition 15:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] \\
1 \arrow[r] & C(\pi)\cap C_{\mathrm{Aut}_K(\tilde{M})}(\pi) \arrow[r] \arrow[d] & C(\pi) \arrow[r, "\rho"] \arrow[d] & \rho(C(\pi)) \arrow[d] \\
1 \arrow[r] & C_{\mathrm{Aut}_K(\tilde{M},\bar{\Theta})}(\pi) \arrow[r] \arrow[d] & C_{\mathrm{Aut}_{h.K}(\tilde{M},\bar{\Theta})}(\pi) \arrow[r, "\rho"] \arrow[d, "\tau"'] & \mathbb{R}^+ \\
1 \arrow[r] & \mathrm{Aut}_{l.K}(M)^0 \arrow[r] \arrow[d] & \mathrm{Aut}_{l.c.K}(M)^0 \arrow[d] & \\
1 \arrow[r] & 1 \arrow[r] & 1 &
\end{tikzcd}$$

Note that $\mathrm{Aut}_{l.c.K}(M)^0\neq\mathrm{Aut}_{l.K}(M)^0$. By the structure theorem of compact Lie group theory, $\mathrm{Aut}_{l.c.K}(M)^0=T\underset{F}{\times}H$ where T is a toroid, H is a semisimple Lie group, and F is a finite central subgroup. (See [7] for example.)
Associated to the covering: $\pi\to\tilde{M}\longrightarrow M$, there is a covering map of groups as in the above remark:

$$C(\pi)\to C_{\mathrm{Aut}_{h.K}(\tilde{M},\bar{\Theta})}(\pi)\xrightarrow{\ \tau\ }\mathrm{Aut}_{l.c.K}(M)^0.$$

The group $C_{\mathrm{Aut}_{h.K}(\tilde{M},\bar{\Theta})}(\pi)$ has the form $\tilde{T}\times_{\tilde{F}}\tilde{H}$ for which $\tilde{T}$, $\tilde{H}$, $\tilde{F}$ are corresponding covering groups of T, H, F. Since H is semisimple, $\tilde{H}$ is still compact. So if $C_{\mathrm{Aut}_{h.K}(\tilde{M},\bar{\Theta})}(\pi)$ is compact, then it lies in $C_{\mathrm{Aut}_{K}(\tilde{M},\bar{\Theta})}(\pi)$ by the above diagram, which contradicts the hypothesis. Thus, $\tilde{T}=\mathbb{R}^k\times T^l$ $(k\geqq 1)$. Moreover, there exists a one-parameter group $\mathcal{R}\subset\mathbb{R}^k$ such that $\rho(\mathcal{R})=\mathbb{R}^+$. $\mathcal{R}$ is the essential subgroup. Hence $C_{\mathrm{Aut}_{h.K}(\tilde{M},\bar{\Theta})}(\pi)$ is the semidirect product of $C_{\mathrm{Aut}_{K}(\tilde{M},\bar{\Theta})}(\pi)$ with $\mathcal{R}$. Since $\mathbb{R}^k$ belongs to the center of $C_{\mathrm{Aut}_{h.K}(\tilde{M},\bar{\Theta})}(\pi)$, we obtain that $C_{\mathrm{Aut}_{h.K}(\tilde{M},\bar{\Theta})}(\pi)=C_{\mathrm{Aut}_{K}(\tilde{M},\bar{\Theta})}(\pi)\times\mathcal{R}$. Passing to the above diagram, $\mathrm{Aut}_{l.c.K}(M)^0=\mathrm{Aut}_{l.K}(M)^0\times S^1$.

□

We give examples of compact l.c. Kähler manifolds with large group $\mathrm{Aut}_{l.c.K}(M)^0$.

Suppose that (M^{2n+1},ω) is a regular pseudo-Hermitian manifold. Then there is a principal circle bundle: $S^1\to M\xrightarrow{\pi}W$ satisfying that

(1) S^1 induces the characteristic vector field Z, *i.e.*, $\omega(Z)=1$, $d\omega(Z,v)=0$ for all $v\in TM$.

(2) (W,Ω) is a Kähler manifold with $\pi^*\Omega=d\omega$. The CR structure Ker ω maps isomorphically onto TW at each point of M. As there exists a complex structure $\hat{J}$ for which $\hat{g}(X,Y)=\Omega(X,\hat{J}Y)$ is a Kähler metric, Ker ω supports a complex structure J which commutes with $\hat{J}$.

(3) There is a Riemannian metric g on M defined by $g(X,Y)=\omega(X)\cdot\omega(Y)+d\omega(X,JY)$. Moreover, $(M,g)\xrightarrow{\pi}(W,\hat{g})$ is a Riemannian submersion.

(4) Let $C(S^1,M)$ be the centralizer of S^1 in the group of pseudo-Hermitian transformations $\mathrm{Psh}(M)=\{f: M\to M$ is a diffeomorphism; $f^*\omega=\omega, f_*\circ J=J\circ f_*\}$. Obviously $C(S^1,M)$ preserves ω and g.

Suppose that we have two compact regular pseudo-Hermitian manifolds (M_i,ω_i,g_i) $(i=1,2)$ with $\dim M_1=2m+1$, $\dim M_2=2n+1$. Consider the commutative diagram of principal bundles:

$$\begin{tikzcd}
\triangle S^1 \arrow[r, equal] \arrow[d] & \triangle S^1 \arrow[d] & \\
S^1\times S^1 \arrow[r] \arrow[d] & M_1\times M_2 \arrow[r, "\pi_1\times\pi_2"] \arrow[d, "\nu"] & W_1\times W_2 \arrow[d, equal] \\
S^1 \arrow[r] & M \arrow[r, "\pi"] & W_1\times W_2,
\end{tikzcd}$$

where $\triangle S^1=\{(t,t^{-1})\}$ which acts on the product $M_1\times M_2$. Then $\triangle S^1$ induces the vector field Z_1-Z_2 which is annihilated by the form $\omega_1+\omega_2$. Obviously the form $\omega_1+\omega_2$ is invariant under the group $C(S^1,M_1)\times C(S^1,M_2)$. Therefore it induces a 1-form ω on the $(2m+2n+1)$-dimensional orbit space M such that $\nu^*\omega=\omega_1+\omega_2$. There is the Kähler form $\Omega=\Omega_1+\Omega_2$ on $W_1\times W_2$ equipped with the complex structure $\hat{J}=\hat{J}_1+\hat{J}_2$. Then $d\omega=\pi^*\Omega$ from the above diagram. It is easy to check that

1. ω is a contact form on M. There is a complex structure J on Ker ω so that (ω,J) is

a pseudo-Hermitian structure and $g(X, Y)=\omega(X)\cdot\omega(Y)+d\omega(X, JY)$ defines a Riemannian metric on M. Moreover, the quotient group $G=\nu(\mathcal{C}(S^1, M_1)\times\mathcal{C}(S^1, M_2))$ preserves ω and g.

2. Put $\nu_*(Z_1+Z_2)=\nu_*(Z_i)=Z$. Then Z is the characteristic vector field of the pseudo-Hermitian manifold (M, ω), where it generates a free action of S^1 which induces the above principal bundle: $S^1\to M \xrightarrow{\pi} W_1\times W_2$. The circle S^1 lies in $\nu(\mathcal{C}(S^1, M_1)\times\mathcal{C}(S^1, M_2))$ acting as pseudo-Hermitian transformations of M.

By this remark, (M, ω, g) is a compact regular pseudo-Hermitian manifold invariant under the group of pseudo-Hermitian transformations G. It is noted that G is isomorphic to $\mathcal{C}(S^1, M_1)/S^1\times\mathcal{C}(S^1, M_2)$.

The idea of this construction has been already seen in [2].

We denote by $M(M_1, M_2)$ the resulting manifold obtained in the above method. Let M_1 be a compact regular pseudo-Hermitian manifold of dimension $2m+1$. For example, Choose $M_2=S^{2n+1}$ with the standard contact form, *that is* , the contact form induces the Kähler form (equivalently, Fubini-Study metric) on $\mathbb{CP}^n$ for the Hopf bundle: $S^1\to S^{2n+1}\longrightarrow\mathbb{CP}^n$. In this case, the centralizer $\mathcal{C}(S^1, S^{2n+1})$ is the centralizer of diagonal S^1 in $U(n+1)$, which is isomorphic to T^{n+1}. Thus $G=\mathcal{C}(S^1, M_1)/S^1\times T^{n+1}$. The resulting regular pseudo-Hermitian manifold $M(M_1, S^{2n+1})$ is S^{2n+1}-bundle over the Kähler manifold W_1. $M(M_1, S^{2n+1})$ admits a subgroup of pseudo-Hermitian diffeomorphisms T^{n+1} at least. Among them, the regular pseudo-Hermitian manifold $M(S^{2m+1}, S^{2n+1})$ supports the maximal group $G=T^{m+1}/S^1\times T^{n+1}=T^{m+n+1}$ acting as pseudo-Hermitian transformations.

Choose such $M(M_1, M_2)$. Let $W=M(M_1, M_2)\times S_1$ be a closed l.c. Kähler manifold obtained in [9]. (See also [12].) It is easy to check that $\mathrm{Psh}(M(M_1, M_2))=\mathrm{Aut}_{l.K}(W)$. Then

$$\begin{aligned}\mathrm{Aut}_{l.c.K}(W)^0&=\mathrm{Aut}_{l.K}(W)^0\times S^1\\&=\mathrm{Psh}^0(M(M_1, M_2))\times S^1\\&=C(S^1, M_1)/S^1\times C(S^1, M_2)\times S^1\end{aligned}$$

In particular, if $W=M(S^{2m+1}, S^{2n+1})\times S^1$, then $\mathrm{Aut}_{l.c.K}(W)^0=T^{m+n+2}$.

6. Concluding Remark

It is far from valid to classify 4-dimensional compact l.c. Kähler manifolds which are not conformal Kähler. There is a class of (1) generalized Hopf manifolds. (See [26] for the definition of generalized Hopf manifolds.) It is also shown by Tricerri that the Inoue surface is l.c. Kähler but not conformal Kähler. The class (2) of Inoue surfaces are characterized as a compact complex solvmanifold with $b_1=1$ (equivalently a T^3-bundle over S^1).

If we consider another class of compact l.c. Kähler manifolds which are not conformal Kähler, we first notice from [24] that a generalized Hopf manifold admits a k-dimensional torus action T^k, at least $k \geq 2$. We note that if a circle acts as essential l.c. Kähler transformations of a compact l.c. Kähler manifold M, then M will be a generalized Hopf manifold. (See [9].) Suppose that M is a closed aspherical manifold (*that is, its universal covering space is contractible*). If a torus T^k acts nontrivially on M, then it is known that the orbit map induces an injective homomorphism: $\mathbb{Z}^k \longrightarrow \pi_1(M)$ such that its image of $\mathbb{Z}^k$ belongs to the center $C(\pi_1(M))$ of $\pi_1(M)$. (See [14].) So if a closed aspherical l.c. Kähler manifold M is a generalized Hopf manifold, then the above remark shows that $C(\pi_1(M))$ contains at least $\mathbb{Z}^2$. In view of this, in order to obtain l.c. Kähler manifold but not generalized Hopf, it is sufficient to construct closed aspherical l.c. Kähler manifolds with the rank of center $C(\pi_1(M))$ at most 1. In fact we can show that

Proposition 17. *There exists a class* (3) *of closed aspherical l.c. Kähler manifolds with center* $C(\pi_1(M))=\mathbb{Z}$.

One of such a manifold is a compact complex infranilmanifold. More precisely, let N/Δ be a three dimensional Heisenberg nilmanifold. Then our manifold M is obtained from $N/\Delta \times I$ by gluing their boundaries by a periodic diffeomorphism of order 4. So the 4-fold covering of M is an S^1-bundle over T^3. As M admits a locally Kähler S^1-action $\mathrm{Aut}_{l.c.K}(M)^0=\mathrm{Aut}_{l.K}(M)^0=S^1$, we have that $C(\pi_1(M))=\mathbb{Z}$ or $b_1=1$. We shall discuss the details of this section in the future paper.

References

[1] D. A. Alekseevskii, "Riemannian spaces with exceptional holonomy groups," *Functional Anal. Appl.* **2** (1968), 97-105.

[2] D. Blair, G. Ludden and K. Yano, "Geometry of complex manifolds similar to Calabi-Eckmann manifolds," *J. Diff. Geom* **9** (1974), 263-274.

[3] G. Bredon, 'Introduction to compact transformation groups,' *Academic Press, New York*, 1972.

[4] B. Y. Chen, "Some topological obstructions to Bochner-Keahler metrics and their applications," *J. Diff. Geom.* **13** (1978), 547-558.

[5] B. Y. Chen and K. Ogiue, "Some characterizations of complex space forms in terms of Chern classes," *Quart. J. of Math. (Oxford)* **26** (1975), 459-464.

[6] D. Fried, "Closed similarity manifolds," *Comment. Math. Helv.* **55** (1980), 576-582.

[7] G. Hochschild, 'The structure of Lie groups,' *Holden-Day, Inc. San Francisco, London, Amsterdam*, 1965.

[8] Y. Kamishima, "Uniformization of Kähler manifolds with vanishing Bochner tensor," *Acta. Math.* **172** (1994), 299-308.

[9] Y. Kamishima, "A geometry of vanishing Bochner tensor on complex manifolds with a geometric structure and their transformations," *Preprint* (1996).

[10] Y. Kamishima, "Geometric flows on compact manifolds and Global rigidity," *Topology* **35** (1996), 439-450.

[11] Y. Kamishima, "Conformally flat manifolds whose development maps are not surjective I," *Trans. A. M. S.* 38 (1986), 697-621.

[12] Y. Kamishima, "Locally conformally Kählerian structures and uniformization," *Geometry, Topology and Physics, Proceedings, 1996, B. Apanasov, S. Bradlow, W. Rodrigues, K. Uhlenbeck (eds.)*, Proceedings of the first Brazil-USA Workshop in Campinas, Walter de Gruyter & Co., Berlin, New York (1997), 174-190.

[13] S. Kobayashi, 'Transformation Groups in Differential Geometry,' Ergebnisse Math. vol. 70, Springer-Verlag, 1970.

[14] K. B. Lee and F. Raymond, "Topological, affine and isometric actions on flat Riemannian manifolds," *J. Diff. Geom.* **16** (1981), 255-269.

[15] J. Lelong-Ferrand, "Transformations conformes et quasi conformes des varietes riemanniennes compactes," *Acad. Roy. Belgique Sci. Mem. Coll.* 8 (1971), 1-44.

[16] R. Cheing Liu and S. Tachibana, "Notes on Kählerian metrics with vanishing Bochner tensor," *Kodai Math J. Rep.* **22** (1970), 313-321.

[17] M. Matsumoto and S. Tanno, "Kählerian spaces with parallel or vanishing Bochner tensor," *Tensor* **27** (1973), 291-294.

[18] A. Madsen, H. Pedersen, Y. Poon and A. Swann, "Compact Einstein-Weyl manifolds with large symmetry group," *Duke Math. J.* **88** (1997), 407-434.

[19] L. Ornea and P. Piccinni, "Locally conformal Kähler structures in quaternionic geometry," *Trans. A. M. S.* **349** (1997), 641-655.

[20] S. Dragomir and L. Ornea, "Locally conformal Kähler geometry," *Progress in Math.*, 155, Birkhäuser 1998.

[21] M. Obata, "The conjectures on conformal transformations of Riemannian manifolds," *J. Diff. Geom.* **6** (1971), 247-258.

[22] S. Salamon, 'Riemannian geometry and holonomy groups,' *Pitman research notes in Math., 201, Longman Scientific*, 1989.

[23] F. Tricerri and L. Vanhecke, "Curvature tensor on almost Hermitian manifolds," *Trans. Amer. Math. Soc.* **267** (1981), 365-398.

[24] K. Tsukada, "The canonical foliation of a compact generalized Hopf manifold," (to appear).

[25] I. Vaisman, "Some curvature properties of locally conformal Kähler manifolds," *Trans.*

Amer. Math. Soc. **259** (1980), 439-447.

[26] I. Vaisman, "Generalized Hopf manifolds," *Geometriae Dedicata* **13** (1982), 231-255.

[27] I. Vaisman, "On locally conformal almost Kähler manifolds," *Israel J. Math.* **24** (1976), 338-351.

[28] I. Vaisman, "Locally conformal symplectic manifolds," *Internat. J. Math. & Math. Sci.* 8 (3) (1985), 521-536.

[29] K. Yano, 'Differential geometry on complex and almost complex spaces,' *Pergamon Press, New York 49*, 1965.

Department of Mathematics,
Kumamoto University,
Kumamoto 860-8555, JAPAN,
MSC 1991 : 57s25, 51m10,
yoshi@sci. kumamoto-u. ac. jp.