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The real loci of the configuration space of six points on the projective line and a Picard modular 3-fold

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Contents

1 Introduction

The configuration space $X(n)$ of n points on the projective line is studied by many authors in different ways. The space $X(6)$ carries naturally the stucture of an algebraic variety over the rational number field.

In this paper, we study the real locus $X_{\mathbf{R}}(6)$ first combinatorially, then algebro-geometrically and finally show that this space carries the structure of a hyperbolic manifold; the two structures — algebraic and hyperbolic — relates through hypergeometric functions and theta functions.

Our method to find the hyperbolic structure in based on the study of the hypergeometric differential equations defined algebraically on the complex locus $X_{\mathbf{C}}(6)$ and the monodromy behavior of the solutions. A direct way can be found in [7].

Since I believe that this paper will serve a bridge between hyperbolic geometry and hypergeometric functions (modular interpretations of configuration spaces), I made this paper somewhat expository; known facts are stated under the title of **Facts**. A complementary arguments can be found in [7].

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2 The configuration space $X(n)$ of n points on the projective line

Let $M(2, n)$ be the space of real $2\times n$-matrices and $M^*(2, n)$ its subspaces consisting of matrices $x=(x_{ij})$ such that any 2×2-minor

$$D_x(jk) := \begin{vmatrix} x_{1j} & x_{1k} \\ x_{2j} & x_{2k} \end{vmatrix}, \quad 1\leq j\neq k\leq n$$

does not vanish. Regarding the n columns of x as homogeneous coordinates of n points on the projective line $\mathbf{P}^1$, we define the configuration space $X(n)$ of n distinct points on the projective line by the quotient space

$$X(n)=GL(2)\backslash M^*(2,n)/H(n),$$

where the group $GL(2)$ acts from the left and $H(n)$ is the subgroup of $GL(n)$ consisting of diagonal matrices acting from the right. Note that $X(n)$ admits a natural action of the symmetric group S_n permuting the n points.

When the base field is $\mathbf{R}$, since $\mathbf{P}^1$ is homeomorphic to a circle, each component can be coded (according to the order of the n points on the circle) by a juzu, which is by definition an equivalence class on the sequences of nemerals 1,..., n up to cyclic permutations and orientation reversion. In particular, this space is the union of the $(n-1)!/2$ connected components, — sometimes called the chambers — on which the group S_n acts transitively.

The space $X(n)(1\leq n\leq 3)$ is a point. The space $X(4)$ is isomorphic to $\mathbf{A}-\{0,1\}$, which can be embedded linearly into the projective plane by

$$X(4)\ni x\longmapsto D_x(12)D_x(34):D_x(13)D_x(24):D_x(14)D_x(23)\in \mathbf{P}^2;$$

so its compactification is just $\mathbf{P}^1$. The space $X(5)$ is isomorphic to $\{(x,y)\in \mathbf{A}^2|xy(1-x)(1-y)(x-y)\neq 0\}$; when the base field is $\mathbf{R}$, its S_5-equivariant minimal smooth compactification $\overline{X}(5)$ is the union of twelve pentagons ([1]). The space $X(6)$ — often denoted simply by X in this paper — is isomorphic to

$$\{(x,y,z)\in \mathbf{A}^3|xyz(1-x)(1-y)(1-z)(x-y)(y-z)(z-x)\neq 0\};$$

when the base field is $\mathbf{R}$, its S_6-equivariant minimal smooth compactification is the union of sixty polyhedrons called Terada3's ([14]), which are described in the next section in detail. In this paper, to simplify notation, we omit to put suffices $\mathbf{R}$, $\mathbf{C}$ or K when it is clear on which base field we are discussing and when the statement holds for any base field.

3 $\overline{X}(6)$ as the union of 60 Terada3's

The configuration space $X=X(6)$ of six points on the projective line is the disjoint union of sixty chambers, and its S_6-equivariant minimal smooth compactification $\overline{X}=\overline{X}(6)$ is the union of sixty chambers, each of which is isomorphic to the Terada3, coded by 6-juzus, for example, 123456. We will see how these sixty cells are glued together.

3.1 Coding the faces of the chamber 123456

The Terada3 coded by 123456 has six pentagonal faces coded by

(12)3456, ..., 1234(56), (61)2345,

and three square faces coded by

$123\times 456,\quad 234\times 561,\quad 345\times 612.$

There are six edges

(12)(34)56, 12(34)(56), (12)34(56); (23)(45)61, 23(45)(61), (23)45(61),

three edges

$$(12)3(45)6,\quad 1(23)4(56),\quad 2(34)5(61),$$

and twelve edges

$$((12)3)456,\quad (1(23))456,\quad 123((45)6),\quad 123(4(56)),$$
$$((23)4)561,\quad (2(34))561,\quad 234((56)1),\quad 234(5(61)),$$
$$((34)5)612,\quad (3(45))612,\quad 345((61)2),\quad 345(6(12)).$$

There are two vertices

$$(12)(34)(56),\quad (23)(45)(61),$$

and twelve verices

$$(12)3\times(45)6,\quad 1(23)\times(45)6,\quad 1(23)\times 4(56),\quad (12)3\times 4(56),$$
$$(23)4\times(56)1,\quad 2(34)\times(56)1,\quad 2(34)\times 5(61),\quad (23)4\times 5(61),$$
$$(34)5\times(61)2,\quad 3(45)\times(61)2,\quad 3(45)\times 6(12),\quad (34)5\times 6(12).$$

The boundary of the pentagon (12)3456 consists of the five edges

$$((12)3)456,\quad (12)3(45)6,\quad (6(12))345,\quad (12)(34)56,\quad (12)34(56)$$

in this order. The boundary of the square 123×456 consists of the four edges

$$((12)3)456,\quad 123(4(56)),\quad (1(23))456,\quad 123((45)6)$$

in this order. The edge (12)(34)56 has the two vertices (12)(34)(56) and $(34)5\times 6(12)$. The edge ((12)3)456 has two vertices $(12)3\times(45)6$ and $(12)3\times 4(56)$. The six pentagons and the three squares are glued according to the following intersection relations (see Figure 1)

$$(12)3456\cap 12(34)56=(12)(34)56,\quad (12)3456\cap 123\times 456=((12)3)456$$

and

$$(12)3456\cap 12(34)56\cap 1234(56)=(12)(34)(56),$$
$$(12)3456\cap 123\times 456\cap 1234(56)=(12)3\times 4(56).$$

3.2 Chambers sharing a common face

The sixty Terada3's are glued to form the compactification $\overline{X}$ of X. I describe the patchwork of the chambers. For example, the pentagonal face (12)3456 of the chamber 123456 and the pentagonal face (21)3456 of the chamber 213456 are glued ; I will abbreviate this explanation of

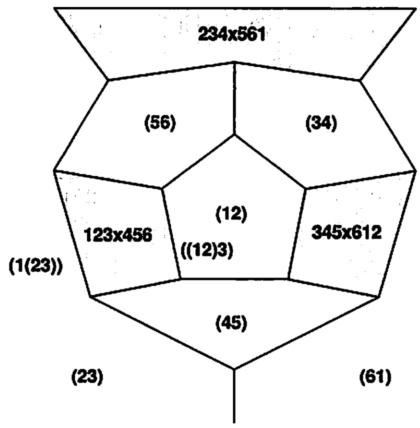

Figure 1 : The boundary of the chamber 123456, a Terada3, in $\overline{X}$

identification by saying that a pentagonal face is shared by two chambers, e.g.

$$(12)3456=(21)3456.$$

Applying this convention, I glue the boundary of the chambers as follows. A square face is ahared by two chambers, e.g.

$$123\times 456(=321\times 654)=321\times 456(=123\times 654).$$

Each edge is shared by four chambers, e.g.

$$(12)(34)56=(21)(34)56=(12)(43)56=(21)(43)56,$$
$$123(4(56))=123((65)4)=123((56)4)=123(4(65)).$$

Each vertex is shared by 8 chambers, e.g.

$$\begin{aligned} (12)(34)(56) &= (21)(34)(56)=\cdots=(21)(43)(65),\\ (12)3\times 4(56) &= (12)3\times(65)4=(12)3\times(56)4=(12)3\times 4(65)\\ =(21)3\times 4(56) &= (21)3\times(65)4=(21)3\times(56)4=(21)3\times 4(65). \end{aligned}$$

3.3 Divisors

There are fifteen divisors, corresponding to the degenerate arrangement $x_i=x_j$, formed by twelve pentagons. Each divisor is isomorphic to $\overline{X}(5)$. There are 10 divisors formed by nine squares. Each divisor is isomorphic to $\overline{X}(4)\times\overline{X}(4)$. I show below such a divisor by arranging nine rectangles in 3×3 matrix ; horizontal bonding edges are also shown.

(2(13))456 213×465 ((21)3)465	(2(13))456 213×456 ((21)3)456	(2(13)546 213×546 ((21)3)546
((12)3)465 123×465 (1(23))465	((12)3)456 123×456 (1(23))456	((12)3)546 123×546 (1(23))546
(1(32))465 132×465 ((13)2)465	(1(32))456 132×456 ((13)2)456	(1(32))546 132×546 ((13)2)546
((31)2)465	((31)2)456	((31)2)546

4 Projective embeddings

This section describes two known S_6-equivariant projective embeddings of X. Each has singularities in a different manner. The situation will be described in terms of the shapes of the chambers.

4.1 The first embedding

Let us consider the projective space $\mathbf{P}$ coordinatized by (labeled) triangles

$$ij\,;kl\,;mn={}_{kl}{}^{ij}{}_{mn},\quad \{i,j,k,l,m,n\}=\{1,\dots,6\},$$

where we make the following identification:

$$ij\,;kl\,;mn=ij\,;mn\,;kl=kl\,;ij\,;mn=-ji\,;kl\,;mn.$$

For each labeled triangle $T=ij\,;kl\,;mn$, we consider the polynomial function ψ_T as follows:

$$\psi_T : M(2,6)\ni x\to D_x(ij)D_x(kl)D_x(mn).$$

Note that $\{\psi_T\}_T$ do not vanish simultaneously on $M^*(2,6)$, and that, for every T, we have

$$\psi_T(gxh)=(\det g)^6(\det h)^2\psi_T(x),\quad g\in GL(2),\ h\in H(6).$$

Thus $\{\psi_T\}_T$ induces a map

$$\psi : X \longrightarrow \mathbf{P}.$$

The closure Y of the image of ψ is a subvariety of $\mathbf{P}$ defined by the linear and cubic equations

$$(L) \quad : \quad ij\,;kl\,;mn-ij\,;km\,;ln+ij\,;kn\,;lm=0,$$

$$(C) \quad : \quad ij\,;kl\,;mn\times ik\,;jn\,;lm\times im\,;jl\,;kn=ij\,;kn\,;lm\times ik\,;jl\,;mn\times im\,;jn\,;kl.$$

The variety Y admits the following stratifiaction. Let Y' be an open subdomain of Y defined by the inequalities

$$ij\,;kl\,;mn\neq 0.$$

For $\{i, j, k, l, m, n\}=\{1, \ldots, 6\}$, define subvarieties of Y:

$Y^{ij} \quad =\{(T)\in Y \mid$ every T with edge ij is 0, any other $T\neq 0\}$,

$Y^{ij;kl} \quad =\{(T)\in Y \mid$ every T with edge ij or kl is 0, any other $T\neq 0\}$,

$Y^{ij;kl;mn} =\{(T)\in Y \mid$ every T with edge ij, kl or mn is 0, any other $T\neq 0\}$,

$Y^{ijk;lmn} =\{(T)\in Y \mid$ every T with edge $pq(p\in\{i, j, k\}, q\in\{l, m, n\})$ is $\neq 0$, all other $T=0\}$.

Fact 1 ([2]) *The domain Y' of Y is a 3-dimensional affine variety isomorphic to $X(6)$; each Y^{ij} is isomorphic to $X(5)$; each $Y^{ij;kl}$ is isomorphic to $X(4)$; each $Y^{ij;kl;mn}$ is isomorphic to $X(3)$, which is a point; each $Y^{ijk;lmn}$ is isomorphic to $X(2)$, which is a point; the variety Y is irreducible and admits the following stratification:*

$$Y=Y'\cup\bigcup Y^{ij}\cup\bigcup Y^{ij;kl}\cup\bigcup Y^{ij;kl;mn}\cup\bigcup Y^{ijk;lmn}.$$

The variety Y has singularities only at the points $\cup\, Y^{ijk;lmn}$. Let an overline denote the closure in Y; then

$$\overline{Y^{ij}} \quad = Y^{ij}\cup\bigcup\overline{Y^{ij;kl}}\cong\mathbf{P}^2,$$

$$\overline{Y^{ij;kl}} \quad = Y^{ij;kl}\cup Y^{ij;kl;mn}\cup Y^{ijm;kln}\,Y^{ijn;klm}\cong\mathbf{P}^1.$$

By adding to X the degenerate configurations corresponding to the partitions

$$2+1+1+1+1,\quad 2+2+1+1,\quad 2+2+2,\quad 3+3,$$

we can extend the map ψ onto Y. As above we use indices $\{i, j, k, l, m, n\}=\{1, \ldots, 6\}$.

$$X^{ij}_{2B}=GL(2)\backslash\{x\in M(2, 6)\mid D_x(ij)=0, \text{ any other } D\neq 0\}/H(6);$$

(the set of arrangements that two points labeled i and j coincide and nothing further special occurs.) It is isomorphic to $X(5)$. There are $\binom{6}{2}=15$ of these.

$$X_{1\beta}^{ij;kl}=GL(2)\backslash\{x\in M(2,6)\,|\,D_x(ij)=0,\ D_x(kl)=0,\ \text{any other } D\neq 0\}/H(6);$$

(the set of arrangements that two points labeled i and j coincide, two points labeled k and l coincide and nothing further special occurs.) It is isomorphic to $X(4)$. There are $\binom{6}{2}\binom{4}{2}/2=45$ of these.

$$X_{0\beta}^{ij;kl;mn}=GL(2)\backslash\{x\in M(2,6)\,|\,D_x(ij)=0,\ D_x(kl)=0,\ D_x(mn)=0,\ \text{any other } D\neq 0\}/H(6);$$

(the set of arrangements that two points labeled i and j coincide, two points labeled k and l coincide, two points labeled m and n coincide and nothing further special occurs.) It is isomorphic to $X(3)$, a point. There are $\binom{6}{2}\binom{4}{2}/3!=15$ of these.

$$X_{0\beta}^{ijk;lmn}=GL(2)\backslash\{x\in M(2,6)\,|\,D_x(pq)\neq 0,\ p\in\{i,j,k\},\ q\in\{l,m,n\},\ \text{any other } D=0\}/H(6);$$

(the set of arrangements that three points labeled i, j and k coincide, three points labeled l, m and n coincide and nothing further special occurs.) It is isomorphic to $X(2)$, a point. There are $\binom{6}{3}/2=10$ of these. Finally we put

$$\dot{X}=X\cup\bigcup X_{2\beta}^{ij}\cup\bigcup X_{1\beta}^{ij;kl}\cup\bigcup X_{0\beta}^{ij;kl;mn}\cup\bigcup X_{0\beta}^{ijk;lmn},$$

which is considered as a topological subspace of the quotient space $GL(2)\backslash M(2,6)/H(6)$.

Fact 2 ([2]) *The map ψ gives a homeomorphism between $\dot{X}$ and Y; it preserves the stratifications, i.e. ψ gives isomorphisms between affine varieties*

$$X\longrightarrow Y',\quad X_{2\beta}^{ij}\longrightarrow Y^{ij},\quad X_{1\beta}^{ij;kl}\longrightarrow Y^{ij;kl},$$
$$X_{0\beta}^{ij;kl;mn}\longrightarrow Y^{ij;kl;mn},\quad X_{0\beta}^{ijk;lmn}\longrightarrow Y^{ijk;lmn}.$$

The chamber 123456 in $\dot{X}$ is a double tetrahedron as in shown in Figure 2 (in the figure, the symbol $X_{*\beta}$ is omitted).

Remark 1 *Though the subvariety Y^{ij} of Y is isomorphic to $X(5)$, its closure $\overline{Y^{ij}}$ in Y is not isomorphic to the S_5-equivariant compactification $\overline{X}(5)$ appeared in* [1]. *There is a birational morphism $\overline{X}(5)\to\overline{Y^{ij}}$, which blows down four disjoint non-singular rational curves to the four*

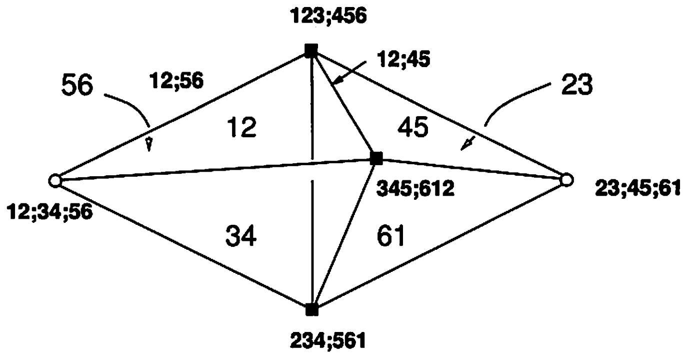

Figure 2 : The chamber 123456 in the first compactification $\dot{X}$

points

$$Y^{ijk;lmn},\quad Y^{ijl;mnk},\quad Y^{ijm;nkl},\quad Y^{ijn;klm}.$$

In Figure 3, we illustrate $\overline{Y}^{12}$ with its intersections with the remaining fourteen (in the figure, the symbol Y is omitted, only the indices are shown). One can see that

$$Y^{12} \quad \cong X(5), \quad Y^{12;34} \cong X(4), \quad \overline{Y}^{12} = Y^{12} \cup \bigcup \overline{Y}^{12;kl},$$
$$\overline{Y}^{12;34} \quad = Y^{12;34} \cup Y^{12;34;56} \cup Y^{125;346} \cup Y^{126;345}.$$

4.2 The isotropy subgroup of S_6 for the chamber 123456

Since the number of the chambers is sixty, the order of the isotropy subgroup of S_6 for a(ny) chamber must be twelve. More precisely, the isotropy subgroup *Isot* for the chamber 123456 is generated by the cyclic permutation $a=\langle 123456\rangle$ and the reversion $b=\langle 16\rangle\langle 25\rangle\langle 34\rangle$; this group consists of twelve permutations : *id*,

$$a=\langle 123456\rangle,\quad a^2=\langle 135\rangle\langle 246\rangle,\quad a^3=\langle 14\rangle\langle 25\rangle\langle 36\rangle,\quad a^4=a^{-2},\quad a^5=a^{-1},$$
$$b=\langle 16\rangle\langle 25\rangle\langle 34\rangle,\quad \langle 21\rangle\langle 36\rangle\langle 45\rangle,\quad \langle 32\rangle\langle 41\rangle\langle 56\rangle,\quad \langle 13\rangle\langle 46\rangle,\quad \langle 15\rangle\langle 24\rangle,\quad \langle 26\rangle\langle 35\rangle.$$

Thanks to this explicit description of *Isot*, we can readily see that each element keeps the triangle Δ with edges $X^{12;45}_{1\beta}$, $X^{34;61}_{1\beta}$, $X^{56;23}_{1\beta}$ and with vertices $X^{123;456}_{0\beta}$, $X^{234;561}_{0\beta}$, $X^{345;612}_{0\beta}$, and that only *id* and a^3 keep this triangle pointwise. Note that there is a unique point — call this point the center (of the chamber and of Δ) — which is fixed by every element of *Isot*. On Δ, a acts as a rotation of order three, and b as the reflection with respect to the line joyning $X^{123;456}_{0\beta}$ and the center. Now it is easy to see that *Isot* is generated by the reflections

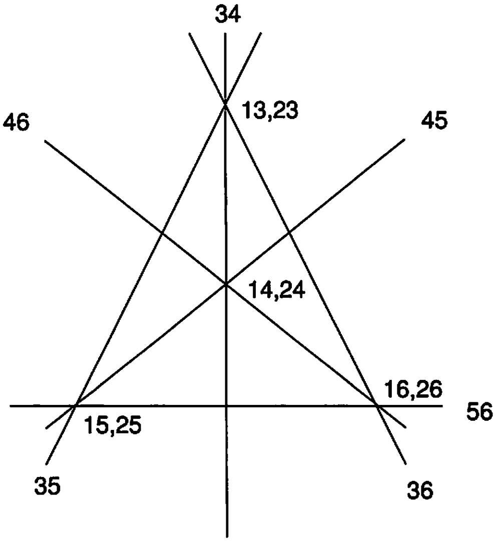

Figure 3 : The plane $\overline{Y}^{12}$

$$a^3, \quad b=\langle 16\rangle\langle 25\rangle\langle 34\rangle, \quad c:=aba^{-1}=\langle 21\rangle\langle 36\rangle\langle 45\rangle,$$

(note that $a^3bc=a$) which fix pointwise Δ, the plane supported by the center, $X_{0\beta}^{123;456}$ and $X_{0\beta}^{12;34;56}$, and the plane supported by the center, $X_{0\beta}^{234;361}$ and $X_{0\beta}^{12;34;56}$, respectively.

Thus the quotient $\dot{X}/S_6$ has the orbifold structure illustrated in Figure 4. In the figure, a thick (respectively, dotted thick and thin) edge indicates that the product of the two reflections fixing the edge is of order 4, 3 and 2, respectively.

4.3 the Segre cubic

Let us name the fifteen labeled triangles as follows:

$$\begin{array}{lll} a=12;34;56, & b=12;35;46, & c=12;36;45, \\ d=13;24;56, & e=13;25;46, & f=13;26;45, \\ g=14;23;56, & h=14;25;36, & t=14;26;35, \\ u=15;23;46, & v=15;24;36, & w=15;26;34, \\ x=16;23;45, & y=16;24;35, & z=16;25;34. \end{array}$$

Choose six coordinates:

$$t=14;26;35, \quad v=15;24;36, \quad z=16;25;34,$$

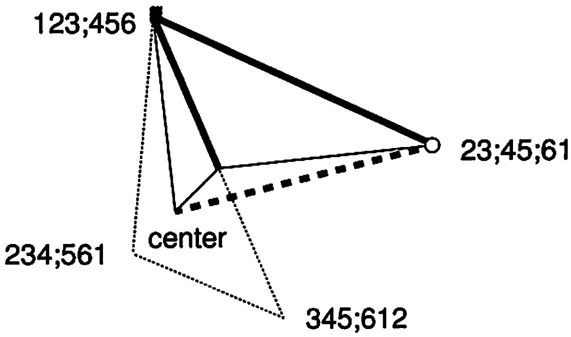

Figure 4 : The orbifold structure of $\dot{X}/S_6$

$$h=14;25;36, \quad w=15;26;34, \quad y=16;24;35.$$

Note that, by the linear relations (L), the remaining nine coordinates $a, b, c, \ldots$ are differences of the chosen six coordinates:

$$a=w-z, \quad b=t-y, \quad c=h-v,$$
$$d=v-y, \quad e=h-z, \quad f=t-w,$$
$$g=h-t, \quad u=v-w, \quad x=y-z.$$

Then the linear relations (L) and the cubic relations (C) boil down to

$$t+v+z=h+w+y, \quad tvz=hwy.$$

Thus our variety Y can be thought of as a cubic hypersurface in the 4-dimensional projective space $\mathbf{P}^4$. By the coordinate change

$$x_0+x_1=2t, \qquad x_1+x_2=2v, \qquad x_2+x_0=2z,$$
$$x_3+x_4=-2h, \quad x_4+x_5=-2w, \quad x_5+x_3=-2y,$$

these equalities change into

$$x_0+\cdots+x_5=0, \quad (x_0)^3+\cdots+(x_5)^3=0,$$

which defines a 3-fold known as the *Segre cubic* S. The following is known and is easy to check.

- There are ten ordinary double points on S: the S_6-orbit of $(1,1,1,-1,-1,-1)$. These are the $X_{0\beta}^{ijk;lmn}$.
- There are fifteen planes on S:

$$P_{ij;kl;mn}: x_i+x_j=x_k+x_l=x_m+x_n=0.$$

These are the $X_{2\beta}^{pq}$. Each plane carries four of the singular points (see Figure 5). For example, we have

$$X_{2\beta}^{12} \Longleftrightarrow a=b=c \Longleftrightarrow w=z, \quad t=y, \quad h=v$$
$$\Longleftrightarrow P_{05;13;24}: x_0+x_5=x_1+x_3=x_2+x_4=0.$$

Remark 2 *Such correspondence* (12)↦(05)(13)(24) *defines an outer automorphism of the symmetric group* S_6.

4.4 The second embedding

The configuration space $X(3, 6)$ of six points in the projective *plane* — this space is defined just as $X(6)$ in § 2 using $M^*(3, 6)$ and $GL(3)$ in place of $M^*(2, 6)$ and $GL(2)$ — and its compactifications are 4-dimensional and are studied in [15] and [12]. There is an involution $*$ on $X(3, 6)$; a point $x \in X(3, 6)$ is fixed under $*$ if and only if x is represented by six points on a conic. Let Q be the totality of fixed points of $*$. Since any nonsingular conic is isomorphic to $\mathbf{P}^1$ and since such conics are equivalent under projective transformations on $\mathbf{P}^2$, Q is naturally isomorphic to $X(6)$. There is an S_6-equivariant (open) embedding

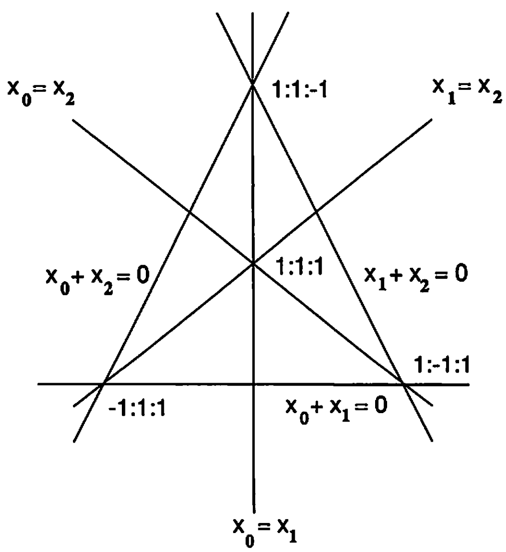

Figure 5 : $X_{2\beta}^{12}$ coordinatized by $x_0 : x_1 : x_2$

$$p_r : X(3,6)/\langle * \rangle \to \mathbf{P}^4.$$

(Note: If the base field is $\mathbf{C}$ then its image is dense, while in case of $\mathbf{R}$ its image is not dense.) Since Q is, by definition, the ramification locus of p_r, Q can be thought of a subset of $\mathbf{P}^4$. The closure $\bar{Q}$ in $\mathbf{P}^4$ is a quartic, known as the *Igusa guartic*. Through the natural isomorphism $Q \cong X(6)$, the configuration space $X(6)$ can be compactified accordingly, say to $\ddot{X}$. The variety $\ddot{X}$, which is isomorphic to the Igusa quartic, admits the following stratification. As above we use indices $\{i, j, k, l, m, n\} = \{1, \dots, 6\}$.

$X \cong Q$: The set of arrangements of six distinct points on a nonsingular conic.

$X_{2a}^{ijk;lmn}$: The set of arrangements where three distinct points labeled i, j and k lie on a line, and three distinct points labeled l,m and n lie on another line. This is isomorphic to $X(4) \times X(4)$.

X_{1a}^{ij}: The set of arrangements where two points labeled i and j coincide, and four other points are distinct and lie on a line which does not path the point labeled i. This is isomorphic to $X(4)$.

$X_{0a}^{ij;kl;mn}$: The set of arrangements where two points labeled i and j coincide, two points labeled k and l coincide, two points labeled m and l coincide, and these three points are not collinear. This is a point.

Fact 3 ([10]) Put

$$\ddot{X} := X \cup \bigcup X_{2a}^{ijk;lmn} \cup \bigcup X_{1a}^{ij} \cup \bigcup X_{0a}^{ij;kl;mn},$$

which is considered as a topological subspace of the quotient space $GL(3)\backslash M(3,6)/H(3)$. The map p_r induces a homeomorphism of $\ddot{X}$ and $\bar{Q}$.

Thanks to this proposition, we regard $\ddot{X}$ a projective variety. Then each stratum is an algebraic set, and $\ddot{X}$ has singularities exactly along

$$\bigcup X_{1a}^{ij} \cup \bigcup X_{0a}^{ij;kl;mn}.$$

Moreover we have

$$\begin{aligned} \partial X_{2a}^{ijk;lmn} &= \bigcup_{\{p,q\} \subset \{i,j,k\}} \overline{X}_{1a}^{pq} \cup \bigcup_{\{p,q\} \subset \{l,m,n\}} \overline{X}_{1a}^{pq}, \\ \partial X_{1a}^{ij} &= X_{0a}^{ij;kl;mn} \cup X_{0a}^{ij;km;ln} \cup X_{0a}^{ij;kn;lm}, \end{aligned}$$

where ∂ and the bar (over X) represent respectively the boundary and closure in $\ddot{X}$.

Let us study the shape of the chamber C of X, with the code 123456, in $\ddot{X}$. The degeneration diagram of arrangements in C can be given as follows.

1,2,3,4,5,6 in this order on a non-singular conic

$$\swarrow \qquad\qquad \downarrow \qquad\qquad \searrow$$

$$X_{2a}^{123;456} \qquad\qquad X_{2a}^{234;561} \qquad\qquad X_{2a}^{345;612}$$

$$\swarrow \qquad \searrow \quad \swarrow \qquad \searrow \quad \swarrow \qquad \searrow$$

$$X_{1a}^{12}, X_{1a}^{45} \qquad X_{1a}^{23}, X_{1a}^{56} \qquad X_{1a}^{34}, X_{1a}^{61} \qquad X_{1a}^{12}, X_{1a}^{45}$$

This implies that C is bounded by three rectangles (see Figure 6 ; in the figure, the symbol X_{*a} is omitted) ; we call such a body a 3-rectangular-face-body. Thus $\ddot{X}$ is the union of sixty such 3-bodies.

Remark 3 *This compactification $\ddot{X}$ is different from $\dot{X}$ and $\overline{X}$. The relation between these three compactifications will be clarified in* § *4.6 and* § *4.7.*

4.5 The Igusa quartic

The Igusa quartic $\mathcal{I}$ is the hypersurface in $\mathbf{P}^4$ defined (for example) by

$$R := (u_0u_1 + u_1u_2 + u_2u_0 - u_3u_4)^2 - 4u_0u_1u_2(u_0 + \cdots + u_4).$$

The following is well known (cf. [5]) and is easy to check

- Singularities of $\mathcal{I}$ lie on the fifteen lines (parameterized by $s : t$)

$$(0\ 0\ s\ 0\ t), \quad (0\ 0\ s\ t\ 0), \quad (-s\ -s\ t\ \ s\ s),$$
$$(0\ s\ 0\ 0\ t), \quad (0\ s\ 0\ t\ 0), \quad (-s\ t\ -s\ s\ s),$$

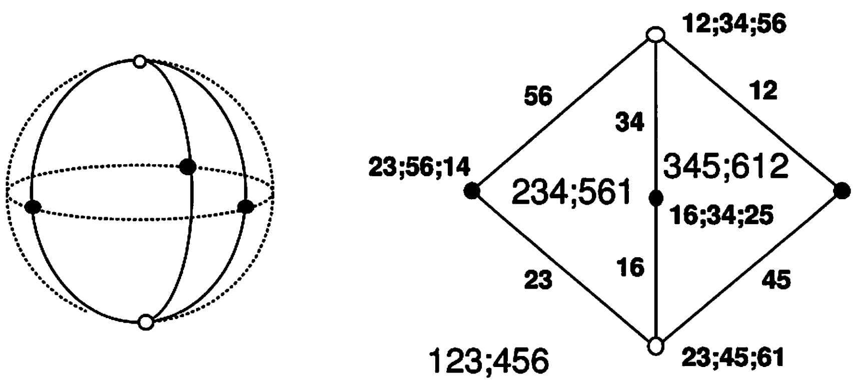

Figure 6 : The chamber 123456 in $\ddot{X}$, a 3-rectangular-face-body, and a stereographic projection of its boundary

$$(s\ 0\ 0\ 0\ t),\quad (s\ 0\ 0\ t\ 0),\quad (t\ \ -s\ -s\ s\ s),$$
$$(0\ \ -s\ -t\ s\ t),\quad (0\ \ -s\ -t\ t\ s),$$
$$(-s\ \ 0\ -t\ s\ t),\quad (-s\ \ 0\ -t\ t\ s),$$
$$(-s\ -t\ \ 0\ s\ t),\quad (-s\ -t\ \ 0\ t\ s).$$

- These fifteen lines meet three by three at fifteen points of the following types:

$$(0, 0, 0, 0, 1),\ (0, 0, 1, 0, 0,),\ (0, 0, -1, 0, 1),\ (0, -1, -1, 1, 1),\ (-1, -1, -1, 1, 1).$$

Figure 7 shows part of these (in the figure, $\dot{s}$, $\dot{t}$ and $\dot{1}$ stand for $-s$, $-t$ and -1).

Remark 4 *Around a segment, there are (eight squares and) eight chambers. Around a vertex, there are 24 squares. Around a vertex, there are twenty chambers.*

Remark 5 *The quartic* $Q \subset \mathbf{P}^4$ *is defined in* [15, p.154] *in terms of the coordinates* $z_{ij} (= p_i - n_j)$ *by*

$$R' := (-z_{13}z_{31} - z_{31}z_{32} + z_{32}z_{33} + z_{33}z_{21} - z_{21}z_{13})^2 - 4z_{21}z_{12}z_{13}z_{31} = 0,$$

where $z_{12} + z_{21} + z_{33} = z_{21} + z_{32} + z_{13}$. *If we change the coordinates, for example, as*

$$z_{31} = u_0,\quad z_{33} = u_1,\quad z_{21} = -u_2,\quad z_{12} = u_2 + u_4,\quad z_{13} = u_1 + u_3,$$

then R' is tranformed into the quartic R.

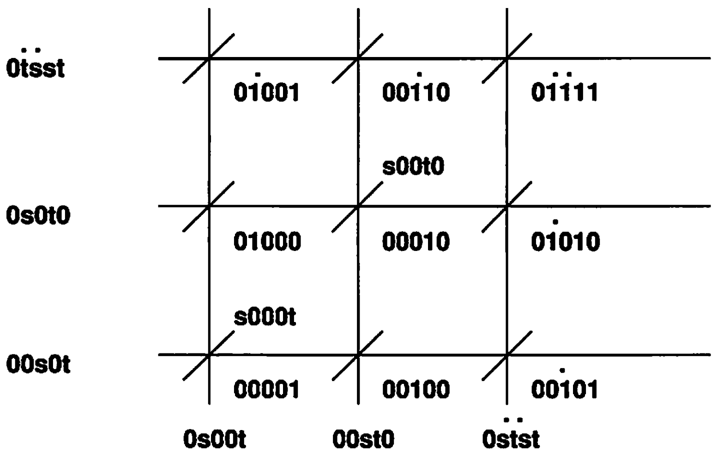

Figure 7 : Fifteen lines on the Igusa quartic

4.6 A birational map between the Igusa quartic and the Segre cubic

The Igusa quartic and the Segre cubic are known to be dual each other (cf. [15]). Since we need an explicit expression of the dualizing map later, we rediscover this fact in this subsection. The dual variety of the Segre cubic is, by definition, a hypersurface of the 4-dimensional projective space with coordinates y parametrized by x in the Segre Cubic as follows:

$$y_j=\partial g/\partial x_j, \quad j=1, \cdots, 5,$$

where $g=(x_1)^3+\cdots+(x_5)^3-(x_1+\cdots+x_5)^3$. That is,

$$y_j=x_j^2-x_0^2, \quad j=1, \cdots, 5,$$

where x satisfies $x_0+\cdots+x_5=0$, $(x_0)^3+\cdots+(x_5)^3=0$. In order to see that the image of this birational map is projectively equivalent to the Igusa quartic, we proceed as follows.
The fifteen planes

$$x_i+x_j=x_k+x_l=x_m+x_n=0 \quad (\{i, j, k, l, m\}=\{0, ..., 5\})$$

are mapped to fifteen lines: the S_5-orbit of

$$(y_1, ..., y_5)=(0, s, s, t, t).$$

These lines meet three by three at the fifteen points: the S_5-orbits of

$$(0, 1, 1, 1, 1) \text{ and } (0, 0, 0, 1, 1).$$

Figure 8 shows part of these.

The arrangement of the five points in y-space (left-bottom of Figure 8) and that in u-space (left-bottom of Figure 7)

(0, 1, 0, 0, 1)		(1, 1, 0, 1, 1)		(0, 1, 0, 0, 0)		(0, 0, 0, 1, 0)
	(0, 1, 0, 1, 0)		and		(1, 0, 0, 0, 0)	
(0, 1, 1, 1, 1)		(0, 0, 0, 1, 1)		(0, 0, 0, 0, 1)		(0, 0, 1, 0, 0)
	y-space				u-space	

suggests the projective transformation

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix}.$$

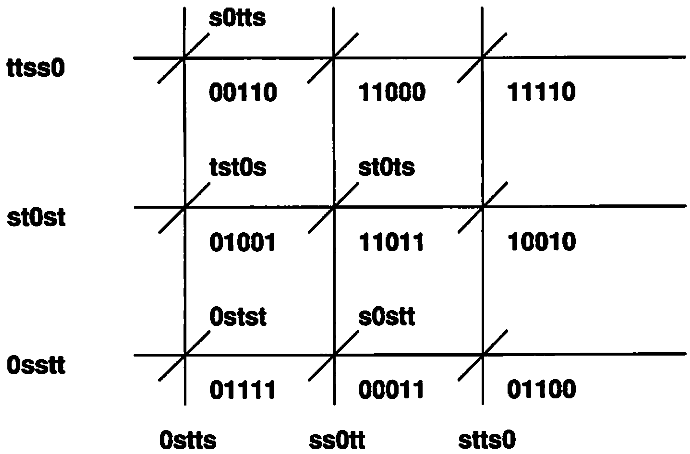

Figure 8 : Fifteen lines on the dual of the Segre cubic

Solving this system, we have

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}.$$

Putting this expression into $R(u)$, the defining equation of the Igusa quadric, and pulling back under the dualizing birational map $x \mapsto y$, we get the expression $R(u(y(x)))$ of degree eight in x. One can checked that $R(u(y(x)))$ has a factor

$$(x_1)^3 + \cdots + (x_5)^3 - (x_1 + \cdots + x_5)^3.$$

This proves that the dual of the Segre cubic can be transformed by the projective transformation above into the Igusa quartic $R(u)=0$.

4.7 Relations between $\overline{X}$, $\dot{X}$ and $\ddot{X}$ in terms of transformations of chambers

Under the (birational) dualizing map

$$y_j = x_j^2 - x_0^2, \quad j = 1, \ldots, 5,$$

where $x_0 + \cdots + x_5 = 0$, $(x_0)^3 + \cdots + (x_5)^3 = 0$, each chamber — a double tetrahedron — in the Segre cubic is transformed into a chamber — a 3-rectangular-face-body — in the Igusa quartic. Since we know now the coordinates and the equations for all the vertices, lines and planes in question, this fact can be readily checked. For example, the double tetrahedron coded by 6-juzu 123456 is bounded by the six walls (planes)

$$X_{2\beta}^{12} \Longleftrightarrow x_0+x_5=x_1+x_3=x_2+x_4=0,$$
$$X_{2\beta}^{34} \Longleftrightarrow x_4+x_5=x_2+x_0=x_1+x_3=0,$$
$$X_{2\beta}^{56} \Longleftrightarrow x_1+x_3=x_0+x_4=x_2+x_5=0,$$
$$X_{2\beta}^{23} \Longleftrightarrow x_2+x_5=x_0+x_3=x_1+x_4=0,$$
$$X_{2\beta}^{45} \Longleftrightarrow x_0+x_5=x_2+x_3=x_1+x_4=0,$$
$$X_{2\beta}^{61} \Longleftrightarrow x_0+x_2=x_5+x_3=x_1+x_4=0,$$

and has five vertices

$$X_{0\beta}^{12;34;56} \Longleftrightarrow (0,1,0,-1,0,0), \qquad X_{0\beta}^{23;45;61} \Longleftrightarrow (0,1,0,0,-1,0),$$
$$X_{0\beta}^{123;456} \Longleftrightarrow (1,1,1,-1,-1,-1), \qquad X_{0\beta}^{234;561} \Longleftrightarrow (1,1,-1,-1,-1,1),$$
$$X_{0\beta}^{345;612} \Longleftrightarrow (-1,1,1,-1,-1,1),$$

The edges $X_{1\beta}^{12;34}$ and $X_{1\beta}^{12;45}$ are mapped to the points

$$(1,0,1,0,0) \text{ and } (1,1,1,1,0),$$

respectively, and the triangle $X_{2\beta}^{12}$ is mapped to the edge joining these two points. In this way, we can describe the birational map in terms of the transformation of the shape of chambers. The birational map $\dot{X}\rightarrow\ddot{X}$ can be best understood by the birational morphisms $\overline{X}\rightarrow\dot{X}$ and $\overline{X}\rightarrow\ddot{X}$, where the latter is studied in [12]. These morphisms can be expressed by the correspondence of the faces of the chambers.

$\dot{X}$	$\longleftarrow$	$\overline{X}$	$\longrightarrow$	$\ddot{X}$
double tetrahedron	$\longleftarrow$	Terade3	$\longrightarrow$	3-rec-face-body
triangle $X_{2\beta}^{12}$	$\longleftarrow$	pentagon (12)3456	$\longrightarrow$	edge $X_{1\alpha}^{12}$
point □ $X_{0\beta}^{123;456}$	$\longleftarrow$	square 123×456	$\longrightarrow$	square $X_{0\alpha}^{123;456}$
edge $X_{0\beta}^{12;34}$	$\longleftarrow$	edge (12)(34)56	$\longrightarrow$	point ○ $X_{0\alpha}^{12;34;56}$
point □ $X_{0\beta}^{123;456}$	$\longleftarrow$	edge ((12)3)456	$\longrightarrow$	edge $X_{1\alpha}^{12}$
edge $X_{0\beta}^{12;45}$	$\longleftarrow$	edge (12)3(45)6	$\longrightarrow$	point ● $X_{0\alpha}^{12;45;36}$
point ○ $X_{0\beta}^{12;34;56}$	$\longleftarrow$	point (12)(34)(56)	$\longrightarrow$	point ○ $X_{0\alpha}^{12;34;56}$
point □ $X_{0\beta}^{123;456}$	$\longleftarrow$	point (12)3×(45)6	$\longrightarrow$	point ● $X_{0\alpha}^{12;45;36}$

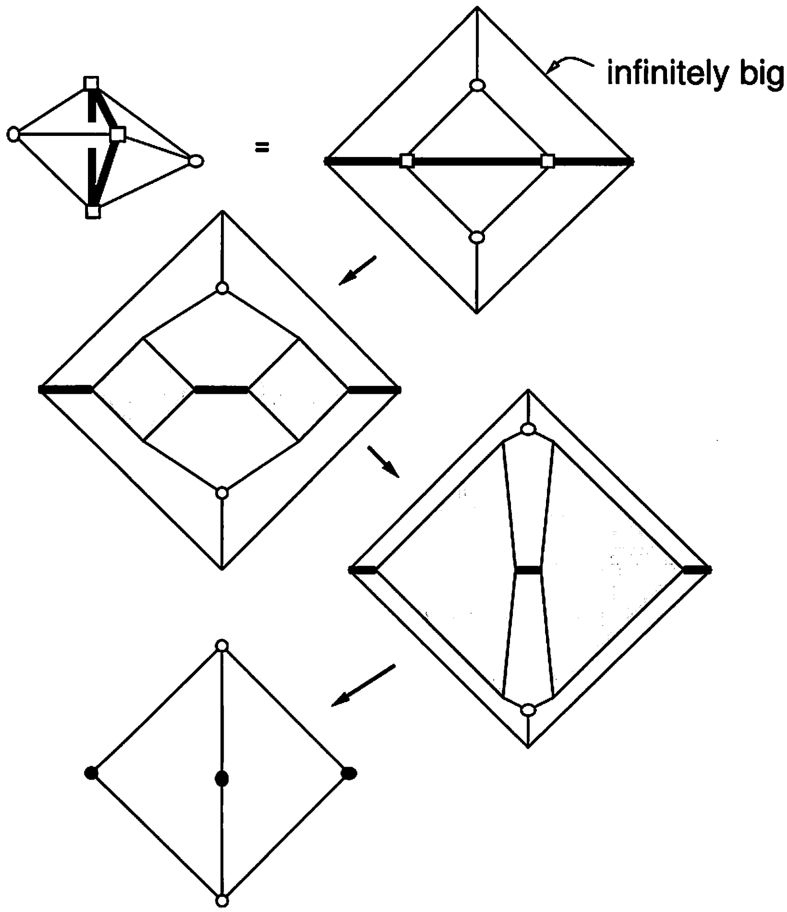

Figure 9 : From a double tetrahedron to a 3-rectangle-face-body

We visualize, in Figure 9, the rational map $\dot{X} \to \overline{X} \to \ddot{X}$ by the deformation of the chambers from a double tetrahedron via Terada3 to a 3-rectangular-face body. I hope the symbols □, ● and ○ will help the reader.

5 Modular interpretations of the configuration spaces

For a point $x \in X_C$ represented by a point set

$$0, 1, \infty, x_1, x_2, x_3 \in \mathbf{P}^1,$$

we consider a curve

$$C_x : t^3=x(x-1)(s-x_1)(s-x_2)(s-x_3)$$

and its periods

$$F_j(x):=\int_{\gamma_j}\{s(s-1)(s-x_1)(s-x_2)(s-x_3)\}^{-1/3},$$

where $\gamma_j (j=1, \dots 4)$ are cycles on C_x such that the intersection matrix is given as

$$(\gamma_j, \gamma_k)_{j,k}=H:=\mathrm{diag}(1, 1, 1, -1).$$

Note that the F_j are linearly independent solutions of the so-called *Appell-Lauricella hypergeometric differential equation* defined on X_C. The correspondence

$$x\longmapsto F_1(x):\cdots:F_4(x)\in \mathbf{P}^3$$

defines a multi-valued map

$$\phi : X\longrightarrow \mathbf{B}_3:=\{v\in \mathbf{C}^4-\{0\}\mid v^*Hv<0\}/\mathbf{C}^\times\subset \mathbf{P}^3;$$

its monodromy group is a reflection group (see the next subsection), which happens to be equal to the congruence subgroup

$$\Gamma(1-\omega):=\{g\in\Gamma\mid g\equiv I_4 \bmod (1-\omega)\}$$

of the modular group

$$\Gamma:=\{g\in GL_4(\mathbf{Z}[\omega])\mid g^*Hg=H\},$$

where $\omega=\exp(2\pi\sqrt{-1}/3)$. The map ϕ induces an open inclusion $X\to \mathbf{B}_3/\Gamma(1-\omega)$ and the isomorphism

$$\varphi : \dot{X}\xrightarrow{\cong}\bar{\mathbf{B}}_3/\Gamma(1-\omega),$$

where the right-hand side denotes the Satake compactification.

Remark 6 *The inverse map of φ can be expressed in terms of theta functions (see* [3]*).*

Remark 7 *By considering the double cover*

$$t^2=s(s-1)(s-x_1)(s-x_2)(s-x_3)$$

instead of the triple one, one gets the isomorphism

$$\ddot{X} \xrightarrow{\cong} \bar{\mathbf{H}}_2/\Gamma_2(2),$$

where $\mathbf{H}_2$ *is the Siegel upper-half space of genus* 2, $\bar{\mathbf{H}}_2$ *is its rational closure, and* $\Gamma_2(2)$ *is the Siegel modular group of genus* 2 *of level* 2 *(cf.* [6], [10]*).*

5.1 Discrete groups acting on the complex 3-ball

Let $\langle u, v\rangle = u^*Hv$ be a Hermitian form on $\mathbf{C}^4$. The group of automorphisms of $\mathbf{B}_3$ is given by $U(3,1)=\{g \in GL_4(\mathbf{C}) \mid g^*Hg=H\}$. Our group Γ can be expressed as $\Gamma = GL_4(\mathbf{Z}[\omega]) \cap U(3,1)$. For $v \in \mathbf{C}^4$, $\langle v, v\rangle \neq 0$ and a primitive m-th root of unity ε, we define the reflection with respect to a root v and an exponent ε as

$$r_\varepsilon(v) = I_4 - \frac{(1-\varepsilon)}{\langle v, v\rangle} vv^*H,$$

which is of order m and belongs to $U(3,1)$. For 15 vectors

$$\begin{array}{lll} v(12)={}^t(1,0,0,0), & v(13)={}^t(-1,1,0,1), & v(14)={}^t(-1,-1,0,1), \\ v(15)={}^t(1,0,-1,1), & v(16)={}^t(1,0,1,1), & v(23)={}^t(1,1,0,1), \\ v(24)={}^t(1,-1,0,1), & v(25)={}^t(-1,0,-1,1), & v(26)={}^t(-1,0,1,1), \\ v(34)={}^t(0,1,0,0), & v(35)={}^t(0,-1,1,1), & v(36)={}^t(0,-1,-1,1), \\ v(45)={}^t(0,1,1,1), & v(46)={}^t(0,1,-1,1), & v(56)={}^t(0,0,1,0), \end{array}$$

we have thirty reflections

$$r_\omega(ij) = r_\omega(v(ij)), \quad r_\rho(ij) = r_\rho(v(ij)),$$

where $1 \leq i < j \leq 6$ and $\rho = \exp(2\pi\sqrt{-1}/6)$. Since $\langle v(ij), v(ij)\rangle = 1$, we have

$$r_\omega(ij) \in \Gamma(1-\omega), \quad r_\rho(ij) \in \Gamma.$$

Fact 4 ([4],[8]) *(a) The monodromy group for* ϕ *is the congruence subgroup* $\Gamma(1-\omega)$.

(b) The group $\Gamma(1-\omega)$ *is a reflection group generated by reflections* $r_\omega(ij)$ *with roots* $v(ij)(1 \leq i < j \leq 6)$.

(b) The group Γ *is generated by* $r_\rho(i, i+1)$*'s,* $1 \leq i \leq 5$.

(c) The quotient group $\Gamma/\Gamma(1-\omega)$ *is isomorphic to the symmetric group* S_6.

(d) The group $\Gamma(1-\omega)$ *has ten cusps, while the group* Γ *has only one cusp.*

Thus we have the commutative diagram

$$\begin{array}{ccc} \dot{X}_{\mathbf{C}} & \xrightarrow{\cong} & \bar{\mathbf{B}}_3/\Gamma(1-\omega) \\ \downarrow & & \downarrow \\ \dot{X}_{\mathbf{C}}/S_6 & \xrightarrow{\cong} & \bar{\mathbf{B}}_3/\Gamma \end{array}$$

5.2 Discrete groups acting on the real 3-ball and the hyperbolic structure on $X(6)$

Let us restrict the action of the groups Γ and $\Gamma(1-\omega)$ to the totally real locus

$$\mathbf{B}_3^{\mathbf{R}} := \{v \in \mathbf{R}^4 - \{0\} \mid {}^t vHv < 0\}/\mathbf{R}^\times \subset \mathbf{P}_{\mathbf{R}}^3$$

of $\mathbf{B}_3$, which is the Klein model of the 3-dimensional *real hyperbolic space*; its group of automorphisms is given by $O(3,1) = \{g \in GL_4(\mathbf{R}) \mid {}^t gHg = H\}$. So we consider the groups

$$G := \Gamma \cap O(3,1) = \{g \in GL_4(\mathbf{Z}) \mid {}^t gHg = H\},$$
$$G(3) := \Gamma(1-\omega) \cap O(3,1) = \{g \in G \mid g \equiv I_4 \bmod 3\}.$$

Proposition 1

(a) The group G is a Coxeter group generated by the four reflections with root

$$v_0 = (1,0,0,0), \quad v_1 = (0,1,-1,0), \quad v_2 = (1,0,-1,0), \quad v_3 = (1,1,1,1).$$

Its graph is given as

$$v_1 \text{——} v_2 \text{══} v_0 \text{══} v_3.$$

The Weyl chamber W is a tetrahedron (see Figure 10) with the vertex $(0,1,0,1)$ on the bundary and the three vertives $(0,0,0,1)$, $(1/3,1/3,1/3,1)$, $(0,1/2,1/2,1)$ in the ball. In particular, G has only one cusp.

(b) The quotient group $G/G(3)$ is isomorphic to the symmetric group S_6.

(c) Let Stab be the subgroup of G which fixes the cusp $(0,1,0,1)$. Then

$$Stab/Stab \cap G(3) \cong (S_3 \times S_3) \cdot \mathbf{Z}/2\mathbf{Z}.$$

Since the order of this group is 72 and $|S_6| = 720$, the group $G(3)$ has ten cusps.

Proof.

1) Let $r_0, \ldots, r_3$ be the reflections with root $v_0, \ldots, v_3$. Note the inclusion

$$\langle r_0, \ldots, r_3 \rangle / \langle r_0, \ldots, r_3 \rangle \cap G(3) \subset G/G(3) = G/G \cap \Gamma(1-\omega) \subset \Gamma/\Gamma(1-\omega) \cong S_6,$$

where $\langle r_0, \ldots \rangle$ denotes the group generated by $r_0, \ldots$ Put $T_{ij} := r_i \cdot r_j$. Then their orders are given as

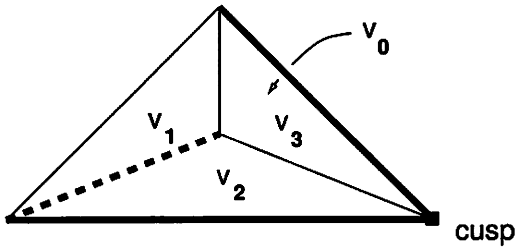

Figure 10 : The Weyl chamber W of G

$$(T_{01})^2=(T_{02})^4=(T_{03})^4=(T_{12})^3=(T_{13})^2=(T_{23})^2=1.$$

Put $T_{ijk}:=r_i\cdot r_j\cdot r_k$. Then T_{023} has infinite order, but

$$(T_{023})^6\equiv 1 \bmod 3,\quad (T_{013})^4=(T_{012})^6=(T_{123})^6=1.$$

The correspondence

$$r_0\longmapsto\langle 12\rangle\langle 34\rangle\langle 56\rangle,\quad r_1\longmapsto\langle 56\rangle,\quad r_2\longmapsto\langle 45\rangle,\quad r_3\longmapsto\langle 23\rangle$$

gives an isomorphism $G/G(3)\rightarrow S_6$. In fact,

$$R0:=r_1,\quad R1:=T_{023}\cdot R0\cdot T_{023}^{-1},\quad \ldots,\quad R5:=T_{023}^5\cdot R0\cdot T_{023}^{-5}$$

form a complete Dynkin diagram (of type A_5)

$$R0-R1-R2-R3-R4-R5-R0.$$

2) The stabilizer *Stab* at the cusp is given by $\langle r_0, r_2, r_3\rangle$. We have

$$1\longrightarrow\langle\langle 14\rangle,\langle 45\rangle\rangle\times\langle\langle 23\rangle,\langle 36\rangle\rangle\longrightarrow Stab \bmod 3\longrightarrow \mathbf{Z}/2\mathbf{Z}\longrightarrow 1.$$

Now the proof of the proposition is immediate.

The Weyl chamber W (see Figure 10) viewed as an orbifold uniformized by $(\bar{\mathbf{B}}_3^{\mathbf{R}}/G, S_6)$ coincides with the orbifold $\dot{X}_{\mathbf{R}}/S_6$ studied in § 4.2. (The apace $X(6)/S_6$ is often called the configuration space of *non-colored* six points.) Therefore we get

Proposition 2 *We have the isomorphism*

$$\dot{X}_{\mathbf{R}}\xrightarrow{\cong}\bar{\mathbf{B}}_3^{\mathbf{R}}/G(3);$$

in particular, $\dot{X}$ *has a hyperbolic structure with cusps at the ten points* $X_{0\beta}^{ij;kl;mn}$.

We can summerize a story in this paper by the following diagram.

$$
\begin{array}{ccccccc}
\text{configuration} & & & \text{hypergeometric} & & & \text{hyperbolic} \\
\text{spaces} & & & \text{functions} & & & \text{geometry} \\
\dot{X}_{\mathbf{C}}(6) & & & \overset{\cong}{\longrightarrow} & & & \mathbf{B}_3^{\mathbf{C}}/\Gamma(1-\omega) \\
 & \nwarrow & & & & \nearrow & \\
 & & \dot{X}_{\mathbf{R}}(6) & \overset{\cong}{\longrightarrow} & \mathbf{B}_3^{\mathbf{R}}/G(3) & & \\
\downarrow & & \downarrow & \text{real} & \downarrow & & \downarrow \\
 & & \dot{X}_{\mathbf{R}}(6)/S_6 & \overset{\cong}{\longrightarrow} & \mathbf{B}_3^{\mathbf{R}}/G & & \\
 & \swarrow & & & & \searrow & \\
\dot{X}_{\mathbf{C}}(6)/S_6 & & & \overset{\cong}{\longrightarrow} & & & \mathbf{B}_3^{\mathbf{C}}/\Gamma
\end{array}
$$

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