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# A note on the localization of $|\bar{N}, p_n|_k$ summability of fourier series[1]

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**Abstract**

Localization problem for absolute summability of Fourier series has been examined.

1. Let $\Sigma a_n$ be a given infinite series with $\{s_n\}$ as the sequence of its $n$-th partial sums. Let $\{p_n\}$ be a sequence of positive numbers such that $P_n=p_0+p_1+\cdots+p_n \longrightarrow \infty$ as $n \longrightarrow \infty$. The series $\Sigma a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|T_n-T_{n-1}|^k<\infty, \tag{1.1}$$

where

$$T_n=\frac{1}{P_n}\sum_{v=0}^{n} p_v s_v.$$

For $k=1$, the summability $|\bar{N}, p_n|_k$ reduces to the summability $|\bar{N}, p_n|$ which is the same as summability $|C, 1|$ for $p_n=1$ and is equivalent to the summability $|R, \log n, 1|$ for $p_n=\frac{1}{n+1}$.

Let $f(t)$ be a periodic function with period $2\pi$ and integrable (L) in $(-\pi, \pi)$ and let $f(t)\sim \frac{a_0}{2}+\Sigma(a_n \cos nt+b_n \sin nt)=\Sigma A_n(t)$. It is well known (ref. [15]) that convergence of a Fourier series at a point is a local property, that is to say however small $\delta>0$ may be, the behaviour of $\{s_n(x)\}$, the $n$-th partial sum of the series $\Sigma A_n(x)$, depends upon the nature of the generating function in the interval $(x-\delta, x+\delta)$ only and is not affected by the values it takes outside the interval. On the other hand it is known that absolute convergence of a Fourier series is not a local property. In 1939 Bosanquet and Kestalman [9] showed that even summability $|C, 1|$ is not a local property. Subsequently Mohanty [14] observed that summability $|C, 1|$ of the series $\Sigma \frac{A_n(x)}{\log(n+1)}$ is not a local property. Since summability $|C, 1|$ implies summability $|R, \log n, 1|$, Mohanty [14] and Izumi [10] investigated this problem for this summability and concluded that summability $|R, \log n, 1|$ of a Fourier series at a point is

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not a local property. Mohanty [14] proved that the summability $|R, \log n, 1|$ of the series $\sum \frac{A_n(x)}{\log(n+1)}$ is a local property. Matsumoto [12] improved the result of Mohanty by replacing the series $\sum \frac{A_n(x)}{\log(n+1)}$ by the series $\sum \frac{A_n(x)}{(\log\log(n+1))^{1+\varepsilon}}$ $(\varepsilon>0)$. Bhatt [1] further generalized the above results by proving the following:

**Theorem A.** If $\{\lambda_n\}$ is a convex sequence (that is if $\Delta^2\lambda_n \geq 0$, where $\Delta^2\lambda_n = \Delta(\Delta\lambda_n)$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$) such that $\sum \frac{\lambda_n}{n} < \infty$, then the summability $|R, \log n, 1|$ of the series $\sum A_n(t)\lambda_n \log n$ at a point can be ensured by a local property.

Mishra [12] with a view to obtain a general result proved the following theorem:

**Theorem B.** Let $\{p_n\}$ be a sequence such that

$$P_n = O(np_n) \tag{1.2}$$

$$P_n \Delta p_n = O(p_n p_{n+1}). \tag{1.3}$$

Then the summability $|\bar{N}, p_n|$ of the series $\sum A_n(t)\lambda_n \frac{P_n}{np_n}$, where $\{\lambda_n\}$ is a convex sequence such that $\sum \frac{\lambda_n}{n} < \infty$, can be ensured by a local property.

Theorem B was extended by Bor [2] who proved that under the conditions of Theorem B the result also holds for the summability $|\bar{N}, p_n|_k \geq 1$. Recently he [7] further generalized his result in the following way:

**Theorem C.** Let $\{p_n\}$ and $\{\lambda_n\}$ be sequences such that

$$\Delta X_n = O\left(\frac{1}{n}\right), \quad X_n = \frac{P_n}{np_n}, \tag{1.4}$$

$$\sum_{n=1}^{\infty} \frac{X_n^{k-1}(|\lambda_n|^k + |\lambda_{n+1}|^k)}{n} < \infty, \tag{1.5}$$

$$\sum_{n=1}^{\infty} (X_n^k + 1)|\Delta\lambda_n| < \infty, \tag{1.6}$$

then the summability $|\bar{N}, p_n|_k$ of the $\sum A_n(t) X_n \lambda_n$ at a point can be ensured by a local property.

It is known that if $\{\lambda_n\}$ is a convex sequence such that $\sum \frac{\lambda_n}{n} < \infty$, then $\{\lambda_n\}$ is decreasing and $\sum \log n \Delta\lambda_n < \infty$. Thus Theorem C for $k=1$ generalizes Theorem A and other earlier result. However for $k \geq 2$ the corresponding extension to the summability $\left|\bar{N}, \frac{1}{n+1}\right|_k$ dose not hold for the series $\sum \frac{A_n(t)}{\log(n+1)}$. The series in (1.6) becomes divergent. Thus the condition (1.6) does not seem to be an appropriate condition. Also the condition (1.5) involves restriction on $\{|\lambda_n|\}$ and $\{|\lambda_{n+1}|\}$.

2. In what follows we prove the following theorem which generalizes Theorem C and also has a shorter proof besides being in a more compact form.

**Theorem.** Let $\{p_n\}$ and $\{\lambda_n\}$ be sequences such that

$$\Delta(P_{n-1}X_n)=O\left(\frac{P_n}{n}\right), \quad X_n=\frac{P_n}{np_n} \tag{2.1}$$

$$\sum_{n=1}^{\infty} X_n^{k-1}\frac{|\lambda_n|^k}{n}<\infty, \quad k\geq 1 \tag{2.2}$$

$$\sum_{n=1}^{\infty} X_{n+1}|\Delta\lambda_n|<\infty, \tag{2.3}$$

then the summability $|\bar{N}, p_n|_k$ of the series $\sum A_n(t)X_n\lambda_n$ at a point can be ensured by a local property.

In view of

$$\begin{aligned}\Delta(P_{n-1}X_n)&=-p_nX_n+P_n\Delta X_n\\ &=-\frac{P_n}{n}+P_n\Delta(X_n)\\ &=P_n\left(\Delta X_n-\frac{1}{n}\right),\end{aligned}$$

it is clear that (1.4) holds if and only if (2.1) holds. Also if (1.6) holds, then $\sum|\Delta\lambda_n|<\infty$, and hence

$$\sum_{n=1}^{\infty} X_n|\Delta\lambda_n|\leq\left(\sum_{n=1}^{\infty} X_n^k|\Delta\lambda_n|\right)^{\frac{1}{k}}\left(\sum_{n=1}^{\infty}|\Delta\lambda_n|\right)^{\frac{1}{k'}}<\infty$$

and in view of (1.4)

$$\begin{aligned}X_{n+1}&=(X_{n+1}-X_n)+X_n\\ &\leq|\Delta X_n|+X_n\\ \sum_{n=1}^{\infty} X_{n+1}|\Delta\lambda_n|&=O(1)\sum_{n=1}^{\infty}\frac{|\Delta\lambda_n|}{n}+\sum_{n=1}^{\infty} X_n|\Delta\lambda_n|\\ &=O(1).\end{aligned}$$

Thus (1.4) and (1.6) imply (2.3).

## 3. Proof of The Theorem :

As mentioned in the beginig, the convergence of Fourier series at a point is a local property. Therefore in order to prove the theorem it is sufficient to prove that if $\{s_n\}$ is bounded, then under the conditions of our theorem $\sum a_n X_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

Now using Abel's transformation

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v X_v \\ &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v \Delta(P_{v-1} \lambda_v X_v) + p_n \frac{s_n P_{n-1} \lambda_n X_n}{P_n P_{n-1}} \\ &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v \lambda_v \Delta(P_{v-1} X_v) + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v P_v X_{v+1} \Delta \lambda_v + \frac{p_n s_n \lambda_n X_n}{P_n} \\ &= L_1 + L_2 + L_3, \quad \text{say.} \end{aligned}$$

In view of Minkowski's unequality it is enough to prove that

$$\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{k-1} |L_r|^k < \infty, \quad r = 1, 2, 3.$$

Now since $s_n = O(1)$, in view of (2.1),

$$\begin{aligned} \sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{k-1} |L_r|^k &= O(1) \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}^k} \left( \sum_{v=1}^{n-1} |\lambda_v| |\Delta(P_{v-1} X_v)| \right)^k \\ &= O(1) \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}^k} \left( \sum_{v=1}^{n-1} \frac{|\lambda_v| P_v}{v} \right)^k \\ &= O(1) \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}^k} \left( \sum_{v=1}^{n-1} |\lambda_v| X_v p_v \right)^k \\ &= O(1) \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}^k} \left( \sum_{v=1}^{n-1} p_v |\lambda_v|^k X_v^k \right) \left( \sum_{v=1}^{n-1} p_v \right)^{k-1} \\ &= O(1) \sum_{v=1}^{\infty} p_v |\lambda_v|^k X_v^k \sum_{n=v+1}^{\infty} \frac{p_n}{P_n P_{n-1}} \\ &= O(1) \sum_{v=1}^{\infty} \frac{|\lambda_v|^k X_v^k}{P_v} p_v \\ &= O(1) \sum_{v=1}^{\infty} \frac{X_v{}^{k-1} |\lambda_v|^k}{v} \\ &= O(1) \end{aligned}$$

in view of (2.2). Also

$$\sum_1^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|L_2|^k=O(1)\sum_{v=1}^{\infty}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{v=1}^{n-1}p_vX_{v+1}|\Delta\lambda_v|\right)^k$$

$$=O(1)\sum_{v=1}^{\infty}\frac{p_n}{P_nP_{n-1}^k}\left(\sum_{v=1}^{n-1}p_v^kX_{v+1}|\Delta\lambda_v|\right)\left(\sum_1^{n-1}X_{v+1}|\Delta\lambda_v|\right)^{k-1}$$

$$=O(1)\sum_{v=1}^{\infty}P_v^kX_{v+1}|\Delta\lambda_v|\sum_{n=v+1}^{\infty}\frac{p_n}{P_nP_{n-1}^k}$$

$$=O(1)\sum_{v=1}^{\infty}X_{v+1}|\Delta\lambda_v|=O(1)$$

in view of (2.3). Finally

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|L_3|^k=O(1)\sum_{n=1}^{\infty}\frac{p_n}{P_n}|\lambda_n|^kX_n^k$$

$$=O(1)\sum_{n=1}^{\infty}\frac{X_n^k|\lambda_n|^k}{nX_n}=O\left(\sum_{n=1}^{\infty}X_n^{k-1}\frac{|\lambda_n|^k}{n}\right)$$

$$=O(1).$$

This proves our theorem.

4. Lal [11] in 1971 and Borwein [8] in 1992 proved the following result on the summability of $|\bar{N}, p_n|$ to study localization problem for Fourier series.

**Theorem D.** Let the sequences $\{\lambda_n\}$ and $\{p_n\}$ satisfy the conditions

$$\sum_{n=1}^{\infty}\frac{p_n}{P_n}|\lambda_n|<\infty \tag{4.1}$$

$$\sum_{n=1}^{\infty}|\Delta\lambda_n|<\infty. \tag{4.2}$$

If $\{s_n\}$ is bounded, then the series $\sum a_n\lambda_n$ is summable $|\bar{N}, p_n|$.

This generalizes a result of Bor [4]. Later Bor [6] extended Theorem D to the summability $|\bar{N}, p_n|_k$, $k\geq 1$ in the following way.

**Theorem E.** Let $\{\lambda_n\}$ and $\{p_n\}$ satisfy the condition (4.2) and

$$\sum_{n=1}^{\infty}\frac{p_n}{P_n}|\lambda_n|^k<\infty. \tag{4.3}$$

If $\{s_n\}$ is bounded, then $\sum a_n\lambda_n$ is summable $|\bar{N}, p_n|_k$.

In view of (4.2), $\lambda_n \in B$ so $\sum \frac{|\lambda_n|^k}{P_n} p_n \leq C \sum \frac{|\lambda_n| p_n}{P_n} < \infty$. Thus (4.1) and (4.2) imply (4.3).

This theorem also generlizes two previous results of Bor [3, 5]. As a cosequence of Theorem E, he deduced the following result on the local property of the summability $|\bar{N}, p_n|_k$ of the series $\sum A_n(t)\lambda_n$.

**Theorem F.** Under the conditoins (4.2) and (4.3), the summability $|\bar{N}, p_n|_k$ of the series $\sum A_n(t)\lambda_n$, at a point, can be ensured by a local property.

It is therefore desirable to compare our theorem with Theorem F which can be restated as:

**Theorem F*.** If $\{\lambda_n\}$ and $\{p_n\}$ satisfy the conditions

$$\sum_{n=1}^{\infty} X_n^{k-1} \frac{|\lambda_n|^k}{n} < \infty, \quad X_n = \frac{P_n}{np_n}, \quad k \geq 1 \tag{4.4}$$

$$\sum_{n=1}^{\infty} |\Delta(X_n \lambda_n)| < \infty, \tag{4.5}$$

then the summability $|\bar{N}, p_n|_k$ of the series $\sum A_n(t) X_n \lambda_n$ at a point can be ensured by a local property.

Choosing $X_n = \log n \log \log n$, $\lambda_n = \frac{1}{\log n (\log \log n)^{1+\varepsilon}}$ $(0 < \varepsilon < 1)$, we observe that the conditions (4.4) and (4.5) are satisfied for $k \geq 1$ but $\Delta X_n \neq O\left(\frac{1}{n}\right)$ and $\sum X_{n+1} |\Delta \lambda_n| = \infty$. Hence (2.1) and (2.3) are not satisfied. Thus the hypotheses of Theorem F* do not imply those of our theorem.

Again choosing $\lambda_n = 1$ and $X_n = \frac{e^{(-1)^n}}{n}$ we find that

$$\Delta X_n = O\left(\frac{1}{n}\right)$$

and

$$\sum_{n=1}^{\infty} X_n^{k-1} \frac{|\lambda_n|^k}{n} = O(1) \sum_{n=1}^{\infty} \frac{1}{n^k} < \infty \text{ if } k > 1.$$

Also $\sum X_{n+1} |\Delta \lambda_n| < \infty$. However

$$\sum_{n=1}^{\infty}|\Delta(X_n\lambda_n)|=\sum_{n=1}^{\infty}\left|\Delta\frac{e^{(-1)^n}}{n}\right|>C\sum_{n=1}^{\infty}\frac{1}{n}=\infty.$$

This shows for $k>1$, the hypotheses of our theorem do not imply that of Theorem F*. Hence Theorem F and our theorem are independent of each other for $k>1$.

## References

[1] S. N. Bhatt, An espect of local property of $|R, \log n, 1|$ summability of the Fourier series, Proc. Nat. Inst. Sci. India, **26** (1960), 67-73.

[2] H. Bor, Local property of $|\bar{N}, p_n|$ summability of factored Fourier series, Bull. Inst. Math. Acad. Sinica, **17** (1989), 165-170.

[3] H. Bor, A note on absolute summability factors of infinite series. SEA Bull. Math. **13** (1989), 123-126.

[4] H. Bor, On $|\bar{N}, p_n|$ summability of some numerical series, Ann. Soc. Math. Polonae, **28** (1989), 171-174.

[5] H. Bor, On absolute summability factors, Ann. Soc. Math. Polonae **32** (1192), 1-4.

[6] H. Bor, A note on $|\bar{N}, p_n|_k$ summability factors, Rend. Mat. **12** Roma (1992), 937-942.

[7] H. Bor, On the local property of $|\bar{N}, p_n|_k$ summability of the factored Fourier series, Jour. Math. Anal. Appl. **163** (1992), 220-226.

[8] D. Borwein, The non-local nature of the summability of Fourier series by certain absolute Riesz summability methods, Proc. Amer. Math. Soc. **114** (1992), 89-94.

[9] L. S. Bosanquet and H. Kestalman, The absolute convergence of series of integrals, Proc. London Math. Soc. **45** (1939), 88-97.

[10] S. Izumi, Notes on Fourier analysis (viii), Local properties of Fourier series, Tohoku Math J. **1** (1949-50), 136-143.

[11] S. N. Lal, On the absolute summability factors of infinite series, Math. Vesnik **23** (1971), 101-112.

[12] K. Matsumoto, Local property of the summability $|R, \lambda_n|$, Tohoku Math J. **8** (1956), 114-124.

[13] K. N. Mishra, Multipliers for $|\bar{N}, p_n|$ summability of Fourier series, Bull. Inst. Math. Acad. Sinica **14** (1986) 431-438.

[14] R. Mohanty, On the summability $|R, \log n, 1|$ of a Fourier series, Jour. London Math. Soc. **25** (1950), 67-72.

[15] E. C. Titchmarsh, The Theory of Functions, Oxford Univ. Press, London, 1961.

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