A note on the localization of $|N, p_n|_k$
summability of Fourier series

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Abstract

Localization problem for absolute summability of Fourier series has been examined.

1. Let $\sum a_n$ be a given infinite series with $(s_n)$ as the sequence of its $n$-th partial sums. Let $(p_n)$ be a sequence of positive numbers such that $P_n=p_0+p_1+\cdots+p_n \to \infty$ as $n \to \infty$. The series $\sum a_n$ is said to be summable $|N, p_n|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{k-1} |T_n - T_{n-1}|^k < \infty,$$

(1.1)

where

$$T_n = \frac{1}{P_n} \sum_{p=0}^{n} p \frac{a_{p}}{p}.$$ 

For $k=1$, the summability $|N, p_n|_k$ reduces to the summability $|N, p_n|$ which is the same as summability $|C, 1|$ for $p_n=1$ and is equivalent to the summability $|R, \log n, 1|$ for $p_n=\frac{1}{n+1}$.

Let $f(t)$ be a periodic function with period $2\pi$ and integrable (L) in $(-\pi, \pi)$ and let $f(t) \sim \frac{a_0}{2} + \sum (a_n \cos nt + b_n \sin nt) = \sum A_n(t)$. It is well known (ref. [15]) that convergence of a Fourier series at a point is a local property, that is to say however small $\delta > 0$ may be, the behaviour of $(s_n(x))$, the $n$-th partial sum of the series $\sum A_n(x)$, depends upon the nature of the generating function in the interval $(x-\delta, x+\delta)$ only and is not affected by the values it takes outside the interval. On the other hand it is known that absolute convergence of a Fourier series is not a local property. In 1939 Bosanquet and Kestelman [9] showed that even summability $|C, 1|$ is not a local property. Subsequently Mohanty [14] observed that summability $|C, 1|$ of the series $\sum \frac{A_n(x)}{\log (n+1)}$ is not a local property. Since summability $|C, 1|$ implies summability $|R, \log n, 1|$, Mohanty [14] and Izumi [10] investigated this problem for this summability and concluded that summability $|R, \log n, 1|$ of a Fourier series at a point is

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not a local property. Mohanty [14] proved that the summability \(|R, \log n, 1|\) of the series \(\sum \frac{A_n(x)}{\log (n+1)}\) is a local property. Matsumoto [12] improved the result of Mohanty by replacing the series \(\sum \frac{A_n(x)}{\log (n+1)}\) by the series \(\sum \frac{A_n(x)}{(\log \log (n+1))^{1+\varepsilon}}, \varepsilon > 0\). Bhatt [1] further generalized the above results by proving the following:

**Theorem A.** If \(\{\lambda_n\}\) is a convex sequence (that is if \(\Delta^2 \lambda_n \geq 0\), where \(\Delta^2 \lambda_n = \Delta (\Delta \lambda_n)\) and \(\Delta \lambda_n = \lambda_n - \lambda_{n+1}\)) such that \(\sum \frac{\lambda_n}{n} < \infty\), then the summability \(|R, \log n, 1|\) of the series \(\sum A_n(t) \lambda_n \log n\) at a point can be ensured by a local property.

Mishra [12] with a view to obtain a general result proved the following theorem:

**Theorem B.** Let \(\{p_n\}\) be a sequence such that

\[
P_n = O(n p_n) \quad (1.2)
\]

\[
P_n \Delta p_n = O(p_n p_{n+1}). \quad (1.3)
\]

Then the summability \(|\overline{N}, p_n|\) of the series \(\sum A_n(t) \lambda_n \frac{P_n}{np_n}\), where \(\{\lambda_n\}\) is a convex sequence such that \(\sum \frac{\lambda_n}{n} < \infty\), can be ensured by a local property.

Theorem B was extended by Bor [2] who proved that under the conditions of Theorem B the result also holds for the summability \(|\overline{N}, p_n| \cdot k \geq 1\). Recently he [7] further generalized his result in the following way:

**Theorem C.** Let \(\{p_n\}\) and \(\{\lambda_n\}\) be sequences such that

\[
\Delta X_n = O\left(\frac{1}{n}\right), \quad X_n = \frac{P_n}{np_n}, \quad (1.4)
\]

\[
\sum_{n=1}^\infty \frac{X_n^{k-1} (|\lambda_n|^k + |\lambda_{n+1}|^k)}{n} < \infty, \quad (1.5)
\]

\[
\sum_{n=1}^\infty (X_n^{k+1} + 1) |\Delta \lambda_n| < \infty, \quad (1.6)
\]

then the summability \(|\overline{N}, p_n| \cdot k\) of the \(\sum A_n(t) X_n \lambda_n\) at a point can be ensured by a local property.

It is known that if \(\{\lambda_n\}\) is a convex sequence such that \(\sum \frac{\lambda_n}{n} < \infty\), then \(\{\lambda_n\}\) is decreasing and \(\sum \log n \Delta \lambda_n < \infty\). Thus Theorem C for \(k=1\) generalizes Theorem A and other earlier results. However for \(k \geq 2\) the corresponding extension to the summability \(|\overline{N}, \frac{1}{n+1}| \cdot \lambda\) does not hold for the series \(\sum \frac{A_n(t)}{\log (n+1)}\). The series in (1.6) becomes divergent. Thus the condition (1.6) does
not seem to be an appropriate condition. Also the condition (1.5) involves restriction on \(|\lambda_n|\) and \(||\lambda_{n+1}||\).

2. In what follows we prove the following theorem which generalizes Theorem C and also has a shorter proof besides being in a more compact form.

**Theorem.** Let \(\{p_n\}\) and \(\{\lambda_n\}\) be sequences such that

\[
\Delta(P_{n-1}X_n) = O\left(\frac{P_n}{n}\right), \quad X_n = \frac{P_n}{np_n} \quad (2.1)
\]

\[
\sum_{n=1}^{\infty} \frac{X_{n+1}^k|\lambda_n|^k}{n} < \infty, \quad k \geq 1 \quad (2.2)
\]

\[
\sum_{n=1}^{\infty} X_{n+1} |\Delta \lambda_n| < \infty, \quad (2.3)
\]

then the summability \(\sum_{n} p_n |x|\) of the series \(\sum A_n(t)X_n\lambda_n\) at a point can be ensured by a local property.

In view of

\[
\Delta(P_{n-1}X_n) = -p_nX_n + P_n\Delta X_n
\]

\[
= -\frac{P_n}{n} + P_n\Delta(X_n)
\]

\[
= P_n\left(\Delta X_n - \frac{1}{n}\right).
\]

it is clear that (1.4) holds if and only if (2.1) holds. Also if (1.6) holds, then \(\sum |\Delta \lambda_n| < \infty\), and hence

\[
\sum_{n=1}^{\infty} X_n |\lambda_n| \leq \left(\sum_{n=1}^{\infty} X_n^k |\Delta \lambda_n| \right)^{\frac{1}{k}} \left(\sum_{n=1}^{\infty} |\Delta \lambda_n| \right)^{\frac{1}{k'}} < \infty
\]

and in view of (1.4)

\[
X_{n+1} = (X_{n+1} - X_n) + X_n
\]

\[
\leq |\Delta X_n| + X_n
\]

\[
\sum_{n=1}^{\infty} X_{n+1} |\Delta \lambda_n| = O(1) \sum_{n=1}^{\infty} \frac{|\Delta \lambda_n|}{n} + \sum_{n=1}^{\infty} X_n |\Delta \lambda_n|
\]

\[
= O(1).
\]

Thus (1.4) and (1.6) imply (2.3).
3. Proof of Theorem:

As mentioned in the beginning, the convergence of Fourier series at a point is a local property. Therefore in order to prove the theorem it is sufficient to prove that if \( \{s_n\} \) is bounded, then under the conditions of our theorem \( \sum a_n X_n \lambda_n \) is summable \( \|N, p_n\| \), \( k \geq 1 \).

Now using Abel's transformation

\[
T_n - T_{n-1} = \frac{p_n}{p_n p_{n-1}} \sum_{v=1}^{n-1} a_v \lambda_v X_v
\]

\[
= \frac{p_n}{p_n p_{n-1}} \sum_{v=1}^{n-1} s_v \Delta(P_{v-1}X_v) + \frac{p_n}{p_n p_{n-1}} \sum_{v=1}^{n-1} s_v \Delta(P_{v-1}X_v + \frac{p_n}{p_n p_{n-1}} \sum_{v=1}^{n-1} s_v \Delta(P_{v-1}X_v + \Delta \lambda_v + \frac{p_n}{p_n p_{n-1}} \sum_{v=1}^{n-1} s_v \Delta(P_{v-1}X_v
\]

\[
= L_1 + L_2 + L_3, \quad \text{say.}
\]

In view of Minkowski's inequality it is enough to prove that

\[
\sum_{n=1}^{\infty} \left( \frac{p_n}{p_n} \right)^{k-1} |L_r|^k < \infty, \quad r = 1, 2, 3.
\]

Now since \( s_n = O(1) \), in view of (2.1),

\[
\sum_{n=1}^{\infty} \left( \frac{p_n}{p_n} \right)^{k-1} |L_r|^k = O(1) \sum_{n=1}^{\infty} \frac{p_n}{p_n p_{n-1}} \left( \sum_{v=1}^{n-1} |\lambda_v| |\Delta(P_{v-1}X_v)| \right)^k
\]

\[
= O(1) \sum_{n=1}^{\infty} \frac{p_n}{p_n p_{n-1}} \left( \sum_{v=1}^{n-1} |\lambda_v| \right)^k
\]

\[
= O(1) \sum_{n=1}^{\infty} \frac{p_n}{p_n p_{n-1}} \left( \sum_{v=1}^{n-1} |\lambda_v| X_v p_v \right)^k
\]

\[
= O(1) \sum_{n=1}^{\infty} \frac{p_n}{p_n p_{n-1}} \left( \sum_{v=1}^{n-1} |\lambda_v| |X_v|^k \right) \left( \sum_{v=1}^{n-1} p_v \right)^{k-1}
\]

\[
= O(1) \sum_{v=1}^{\infty} \frac{|\lambda_v|^k X_v^k}{p_v}
\]

\[
= O(1) \sum_{v=1}^{\infty} \frac{|\lambda_v|^k X_v^k}{p_v}
\]

\[
= O(1)
\]

in view of (2.2). Also
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\[ \sum_k \left( \frac{P_n}{p_n} \right)^{k-1} |L_k|^k = O(1) \sum_k \frac{P_n}{p_n} \left( \sum_{n=1}^{\infty} p_n X_{p_n-1} |\Delta \lambda_n| \right)^k \]

\[ = O(1) \sum_k \frac{P_n}{p_n} \left( \sum_{n=1}^{\infty} p_n X_{p_n-1} |\Delta \lambda_n| \right)^{k-1} \]

\[ = O(1) \sum_k \frac{P_n}{p_n} X_{p_n-1} |\Delta \lambda_n| \sum_k \frac{p_n}{p_n} \frac{p_n}{p_n} \]

\[ = O(1) \sum_k X_{p_n-1} |\Delta \lambda_n| = O(1) \]

in view of (2.3). Finally

\[ \sum_k \left( \frac{P_n}{p_n} \right)^{k-1} |L_3|^k = O(1) \sum_k \frac{P_n}{p_n} |\lambda_n|^k X_n^k \]

\[ = O(1) \sum_k \frac{X_n^k |\lambda_n|^k}{nX_n} = O\left( \sum_k X_n^{k-1} |\lambda_n|^k \right) \]

\[ = O(1). \]

This proves our theorem.


**Theorem D.** Let the sequences $\{\lambda_n\}$ and $\{p_n\}$ satisfy the conditions

\[ \sum\frac{P_n}{p_n} |\lambda_n| < \infty \quad \text{(4.1)} \]

\[ \sum |\Delta \lambda_n| < \infty. \quad \text{(4.2)} \]

If $\{s_n\}$ is bounded, then the series $\sum a\lambda_n$ is summable $|N, p_n|$.

This generalizes a result of Bor [4]. Later Bor [6] extended Theorem D to the summability $|N, p_n|_k$, $k \geq 1$ in the following way.

**Theorem E.** Let $\{\lambda_n\}$ and $\{p_n\}$ satisfy the condition (4.2) and

\[ \sum\frac{P_n}{p_n} |\lambda_n|^k < \infty. \quad \text{(4.3)} \]

If $\{s_n\}$ is bounded, then $\sum a\lambda_n$ is summable $|N, p_n|_k$. 
In view of (4.2), \( \lambda_n \in B \) so \( \sum \frac{|\lambda_n|^k}{P_n} p_n \leq C \sum \frac{|\lambda_n| p_n}{P_n} < \infty \). Thus (4.1) and (4.2) imply (4.3).

This theorem also generalizes two previous results of Bor [3, 5]. As a consequence of Theorem F, he deduced the following result on the local property of the summability \( \vec{N}, p_n \) of the series \( \sum A_n(t) \lambda_n \).

**Theorem F.** Under the conditions (4.2) and (4.3), the summability \( \vec{N}, p_n \) of the series \( \sum A_n(t) \lambda_n \), at a point, can be ensured by a local property.

It is therefore desirable to compare our theorem with Theorem F which can be restated as:

**Theorem F*. If \( \{\lambda_n\} \) and \( \{p_n\} \) satisfy the conditions

\[
\sum_{n=1}^{\infty} X_n^{-1} |\lambda_n|^k \frac{P_n^{1/p}}{n} < \infty, \quad X_n = \frac{P_n}{np_n}, \quad k \geq 1
\]

\[
\sum_{n=1}^{\infty} |\Delta(X_n \lambda_n)| < \infty,
\]

then the summability \( \vec{N}, p_n \) of the series \( \sum A_n(t) X_n \lambda_n \) at a point can be ensured by a local property.

Choosing \( X_n = \log n \log \log n, \lambda_n = \frac{1}{\log n (\log \log n)^{1+\varepsilon}} (0 < \varepsilon < 1) \), we observe that the conditions (4.4) and (4.5) are satisfied for \( k \geq 1 \) but \( \Delta X_n \neq O\left(\frac{1}{n}\right) \) and \( \sum x_{n+1} |\Delta \lambda_n| = \infty \). Hence (2.1) and (2.3) are not satisfied. Thus the hypotheses of Theorem F* do not imply those of our theorem.

Again choosing \( \lambda_n = 1 \) and \( X_n = \frac{e^{(1-\varepsilon) x}}{n} \) we find that

\[ \Delta X_n = O\left(\frac{1}{n}\right) \]

and

\[
\sum_{n=1}^{\infty} X_n^{-1} |\lambda_n|^k \frac{1}{n} = O(1) \sum_{n=1}^{\infty} \frac{1}{n^k} < \infty \text{ if } k > 1.
\]

Also \( \sum x_{n+1} |\Delta \lambda_n| < \infty \). However
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$$\sum_{n=1}^{\infty} |\Delta(X_n l_n)| = \sum_{n=1}^{\infty} \left| \Delta \frac{e^{i-1/n}}{n} \right| > C \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$ 

This shows for $k > 1$, the hypotheses of our theorem do not imply that of Theorem F*. Hence Theorem F and our theorem are independent of each other for $k > 1$.

References


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