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Twisted homology of the configuration spaces of n-points with applications to hypergeometric functions

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Abstract

Algebraic topology of local systems on the configuration space of n-points is developed systematically, with emphasis on its simplicial aspects. Comparisons between simplicial theory and singular theory are given. As an application, an exterior power structure of the hypergeometric functions is derived over a rather general base ring.

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I. Hypergeometric Functions — A Motivation

1. Introduction

A hypergeometric function in the sense of K. Aomoto [3] and I.M. Gel'fand [8] is defined to be the pairing between twisted homology and twisted cohomology of the complements of hyperplanes in a complex projective space, where arrangements of hyperplanes are regarded as the independent variables of the hypergeometric function. So it is important to investigate the structure of twisted homology and cohomology groups of the complements of hyperplanes in a projective space.

In the previous paper [16], we investigated twisted de Rham cohomology groups of those spaces and established an exterior power structure on them. In this paper we investigate twisted simplicial (singular) homology groups of those spaces. We also discuss a reduction of the base ring of local systems. Previously, local systems, homology and cohomology groups, etc. were defined over the complex number field $\mathbf{C}$. In this paper the base ring will be a more general ring R. We establish an exterior power structure on the twisted homology groups over R associated with hypergeometric functions. This result is obtained as a simple consequence of a more general theory, i.e., algebraic topology of the configuration spaces of n-points. Accordingly this paper aims at developing a twisted simplicial (singular) theory for the configuration spaces of n-points.

In Chapter I we discuss hypergeometric functions, giving a motivation to Chapters II and III. The exterior power structure of the twisted homology groups associated with hypergeometric functions, which is the main result of Chapter I, is stated in Theorem 3.3.3, though its proof is postponed until Chapter III, § 22. Chapters II and III are devoted to simplicial theory and singular theory for the configuration space of n-points, respectively. The main theorems of Chapters II and III are Theorem 15.3.3 and Theorem 21.3.3, respectively.

2. Hypergeometric functions

Following [3] [8], we formulate the hypergeometric functions of several variables in a manner suitable for our purpose.

(2.1) A fibration $\pi : \bar{E} \to \bar{M}$.

Definition 2.1.1. For a pair (m, n) of positive integers with $m>n$, let $\bar{M}=\bar{M}(m+1, n+1)$ denote the set of all $(m+1)\times(n+1)$ complex matrices of *full rank*. A matrix $z\in\bar{M}$ is said to be *in general position if none of the* $(n+1)$*-minors of* z *vanishes.* Let $M=M(m+1, n+1)$ be the set of all matrices in general position.

We consider $\mathbf{C}^{m+1}$ and $\mathbf{C}^{n+1}$ as column vector spaces with the coordinates $x={}^t(x_0, x_1, \cdots, x_m)$ and $u={}^t(u_0, u_1, \cdots, u_n)$, respectively. These coordinates are also regarded as the homogeneous coordinates of the complex projective spaces $\mathbf{P}^m$ and $\mathbf{P}^n$, respectively.

Definition 2.1.2. We define a fibration $\pi : \bar{E} \to \bar{M}$ by

$$\bar{E}=\bar{E}(m+1, n+1):=\{(z, u)\in\bar{M}\times\mathbf{P}^n ; \prod_{i=0}^{m} x_i(zu)\neq 0\}$$

where $\pi : \bar{E} \to \bar{M}$ is the projection onto the first component. Let $E_z=E_z(m+1, n+1)$ be the fiber of $\bar{E}$ over $z\in\bar{M}$, and $\bar{E}=E(m+1, n+1):=\bar{E}|_M$ be the restriction to M of the base space of $\bar{E}$.

Lemma 2.1.3. ([13] Proposition 2.1) $\pi : E \to M$ is a C^∞ fiber bundle. In particular it is topologically locally trivial.

(2.2) Local systems on $\bar{E}$.

Definition 2.2.1. Let $A=A(m+1, n+1)$ be an affine space defined by

$$A:=\{\alpha={}^t(\alpha_0, \alpha_1, \cdots, \alpha_m)\in\mathbf{C}^{m+1} ; \sum_{i=0}^{m}\alpha_i=-(n+1)\}.$$

For any $\alpha\in A$, let f be a multi-valued section of $\mathcal{O}_{\bar{E}/\bar{M}}(-n-1)$ defined by

$$f=f(z, u)=f(z, u; m+1, n+1, \alpha):=\prod_{i=0}^{m} x_i(zu)^{\alpha_i}.$$

We shall consider the following local systems:

Definition 2.2.2. Let $\mathscr{L}=\mathscr{L}(m+1, n+1, \alpha)$ be the local system on $\bar{E}$ such that each branch of f determines a horizontal local section of $\mathscr{L}$, and $\mathscr{L}^\vee=\mathscr{L}^\vee(m+1, n+1, \alpha)$ be the dual local system of $\mathscr{L}$ on $\bar{E}$. Let

$$\mathscr{L}_z=\mathscr{L}_z(m+1,\ n+1,\ \alpha):=\mathscr{L}|_{E_z},$$
$$\mathscr{L}_z^\vee=\mathscr{L}_z^\vee(m+1,\ n+1,\ \alpha):=\mathscr{L}^\vee|_{E_z}$$

be the restriction of $\mathscr{L}$, $\mathscr{L}^\vee$ to the fiber E_z, respectively.

Remark 2.2.3. For the time being, local systems are assumed to be defined over the complex number field $\mathbf{C}$. Reduction of the base ring will be discussed in (3.2).

(2.3) Twisted homology and twisted cohomology.

Definition 2.3.1. Let $\mathscr{H}^\bullet_{\mathrm{DR}}=\mathscr{H}^\bullet_{\mathrm{DR}}(m+1,\ n+1,\alpha)=\mathscr{H}^\bullet_{\mathrm{DR}}(E,\ L):=\mathscr{R}^\bullet\pi^*\mathscr{L}$ be the twisted de Rham cohomology group of $(E,\ \mathscr{L})$ along the fibers of $\pi: E\to M$, and

$$\mathscr{H}^\bullet_{\mathrm{DR}}=\mathscr{H}^\bullet_{\mathrm{DR}}(m+1,\ n+1,\ \alpha)=\mathscr{H}^\bullet_{\mathrm{DR}}(E,\ \mathscr{L}^\vee):=\bigcup_{z\in M}H_\bullet(E_z,\ \mathscr{L}_z^\vee)$$

be the twisted singular homology group of $(E,\ \mathscr{L}^\vee)$ along the fibers of $\pi: E\to M$.

There are natural projections $\pi: \mathscr{H}^\bullet_{\mathrm{DR}}\to M$ and $\pi: \mathscr{H}^\vee_\bullet\to M$.

Definition 2.3.2. The fiber $H^\bullet_{\mathrm{DR}}(E_z,\ \mathscr{L}_z)$ of $\pi: \mathscr{H}^\bullet_{\mathrm{DR}}\to M$ over z is the de Rham chomology group

$$H^\bullet_{\mathrm{DR}}(E_z,\ \mathscr{L}_z)=\frac{\mathrm{Ker}\,[d: \mathscr{E}^\bullet(E_z,\ \mathscr{L}_z)\to \mathscr{E}^{\bullet+1}(E_z,\ \mathscr{L}_z)]}{\mathrm{Im}\,[d: \mathscr{E}^{\bullet-1}(E_z,\ \mathscr{L}_z)\to \mathscr{E}^\bullet(E_z,\ \mathscr{L}_z)]},$$

where $\mathscr{E}^\bullet(E_z,\ \mathscr{L}_z)$ is the set of all C^∞ differential forms on E_z with values in the flat line bundle L_z. The twisted C^∞ de Rham cohomology group formulated in the previous paper [16] is slightly different and given by

$$H^\bullet_f(E_z)=\frac{\mathrm{Ker}\,[\nabla_f: \mathscr{E}^\bullet(E_z)\to \mathscr{E}^{\bullet+1}(E_z)]}{\mathrm{Im}\,[\nabla_f: \mathscr{E}^{\bullet-1}(E_z)\to \mathscr{E}^\bullet(E_z)]},$$

where $\nabla_f=d+df/f$ and $\mathscr{E}^\bullet(E_z)$ is the set of all C^∞ differential forms on E_z. But the canonical isomorphism $\mathscr{E}^\bullet(E_z)\to \mathscr{E}^\bullet(E_z, L_z)$, $u\mapsto fu$ induces an isomorphism $H^\bullet_f(E_z)\to H^\bullet_{\mathrm{DR}}(E_z,\ \mathscr{L}_z)$. So we can identify those two de Rham groups. In this paper we consider $H^\bullet_{\mathrm{DR}}(E_z,\ \mathscr{L}_z)$ rather than $H^\bullet_f(E_z)$.

By Lemma 2.1.3., we have the following:

Lemma 2.3.3. *$\pi : \mathscr{H}^{\bullet}_{\mathrm{DR}} \to M$ and $\pi : \mathscr{H}^{\vee}_{\bullet} \to M$ admit natural structures of local system on M.*

(2.4) Hypergeometric functions.

We denote by

$$\mathscr{H}^{\vee}_{\bullet} \otimes \mathscr{H}^{\bullet}_{\mathrm{DR}} \to \mathbf{C}_M, \quad (c, \varphi) \mapsto \int_c \varphi$$

the fiberwise pairing of the twisted homology group and the twisted cohomology group, where $\mathbf{C}_M$ is the constant system on M with fiber $\mathbf{C}$. Let $du := du_0 \wedge du_1 \wedge \cdots \wedge du_n$ be the standard volume form on $\mathbf{C}^{n+1}$. The interior product of du by the Euler vector field $e = \sum_{i=0}^{n} u_i \partial/\partial u_i$ defines an $\mathscr{O}_{\mathbf{P}^n}(n+1)$-valued n-form $\omega := \iota_e du$ on $\mathbf{P}^n$. The pull-back of ω by the canonical projection $\bar{E} \to \mathbf{P}^n$ is an $\mathscr{O}_{\bar{E}/\bar{M}}(n+1)$-valued n-form along the fibers of $\pi : \bar{E} \to \bar{M}$, which is also denoted by ω. Put $\varphi(z) = \varphi(z; m+1, n+1, \alpha) := f(z, u; m+1, n+1, \alpha)\omega$. For each $z \in M$, $\varphi(z)$ determines an element of $H^n_{\mathrm{DR}}(E_z, \mathscr{L}_z)$.

Definition 2.1.1. A *hypergeometric function* of the type $(m+1, n+1, \alpha)$ is a (germ of) function of the form

$$F(z) = F(z; m+1, n+1, \alpha) := \int_{c(z)} \varphi(z),$$

where $c(z)$ is a horizontal local section of $\pi : \mathscr{H}_n^{\vee} \to M$. If α is not specified, then $F(z)$ is said to be of the type $(m+1, n+1)$.

The following lemma is well known.

Lemma 2.4.2. *The hypergeometric function $F(z)$ is (continued to) a multi-valued holomorphic function on M with regular singularities along $\bar{M} \backslash M$.*

Remark 2.4.3. The hypergeometric functions of the type $(m+1, 2)$ are essentially the same as the Lauricella hypergeometric functions F_D in $(m-2)$-variables ([24]). The Lauricella functions F_D has been investigated extensively, since they were defined in 1893, (see e.g., [4] [7]).

3. Exterior power structure

(3.1) The Veronese embedding.

Definition 3.1.1. The *Veronese embedding* $M(m+1, 2) \xrightarrow{\text{Vero}} M(m+1, n+1)$ is defined by

$$w=\begin{pmatrix} \cdots & \cdots \\ w_{i0} & w_{i1} \\ \cdots & \cdots \end{pmatrix} \mapsto z=\begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ w_{i0}^{n} & w_{i0}^{n-1}w_{i1} & w_{i0}^{n-2}w_{i1}^{2} & \cdots & w_{i1}^{n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Remark 3.1.2. This is indeed an embedding, because we have

$$z(i_0, i_1, \cdots, i_n)=\text{nonzero const.} \prod_{p<q} w(i_p, i_q),$$

where $z(i_0, i_1, \cdots, i_n)$ is the $(n+1)$-minor of z determined by the i_0th, i_1th, $\cdots$, i_nth columns of z, $w(i_p, i_q)$ being defined in a similar manner.

The pulls-back of the local systems $\mathscr{H}^{\bullet}_{\mathrm{DR}}(m+1, n+1; \alpha)$ and $\mathscr{H}^{\vee}_{\bullet}(m+1, n+1; \alpha)$ on $M(m+1, n+1)$ by the Veronese embedding are local systems on $M(m+1, 2)$. This observation, together with Remark 2.4.3, leads us to the following:

Problem 3.1.3. Are there any relations between the hypergeometric funtions of the type $(m+1, n+1)$ restricted to the Veronese image and the Lauricella hypergeometric functions F_D?

(3.2) Reduction of the base ring.

So far $\mathscr{L}=\mathscr{L}_\alpha(\alpha\in A)$ has been regarded as a local system over the complex number field $\mathbb{C}$. Now let us discuss reduction of the base ring. We put

$$e_i=\exp(2\pi\sqrt{-1}\alpha_i)\in\mathbb{C}^{\times}, \qquad (i=0, 1, \cdots, m)$$

and $e:=(e_0, e_1, \cdots, e_m)$. Since $\sum\alpha_i=-(n+1)$, we have

$$(*) \qquad e_0e_1\cdots e_m=1.$$

If R is a *subring* of $\mathbb{C}$ such that $\mathbb{Q}[e_0^{\pm1}, e_1^{\pm1}, \cdots, e_m^{\pm1}]\subseteq R\subseteq\mathbb{C}$, then the local system $\mathscr{L}=\mathscr{L}_\alpha$ can be defined over R. Hereafter we assume that $\mathscr{L}$ is defined over R. This reduction of the base ring would allow us to study hypergeometric functions more closely.

(3.3) The exterior power structure.

Let I_e be the ideal of R generated by $1-e_0, 1-e_1, \cdots, 1-e_m$, i.e.,

$$I_e:=\sum_{i=0}^{m}R(1-e_i).$$

Remark 3.3.1. In fact, I_e is generated by $1-e_1, 1-e_2, \cdots, 1-e_m$, because $(*)$ implies

$$e_0-1=\sum_{i=1}^{m}\frac{1-e_i}{e_1e_2\cdots e_i}.$$

To state the main theorem of Chapter I (Theorem 3.3.3), we establish some notations.

Notation 3.3.2. (1) Let

$$\mathscr{H}^{\bullet}=\mathscr{H}^{\bullet}(m+1, n+1; \alpha):=\bigcup_{z\in M}H^{\bullet}(E_z, \mathscr{L}_z)$$

be the twisted singular cohomology group of $(E, \mathscr{L})$ along the fibers $\pi: E\to M$. Note that there is an isomorphism of $\mathbf{C}$-vector spaces: $\mathscr{H}^{\bullet}\otimes\mathbf{C}\simeq\mathscr{H}^{\bullet}_{\mathrm{DR}}$.

(2) Let $H^q(m+1, n+1; \alpha)$ and $H^{\vee}_q(m+1, n+1; \alpha)$ be arbitrary fibers of the projections

$$\pi: \mathscr{H}^q(m+1, n+1; \alpha)\to M(m+1, n+1)$$
$$\pi: \mathscr{H}^{\vee}_q(m+1, n+1; \alpha)\to M(m+1, n+1).$$

(3) Let V_e be the R-module defined by

$$V_e:=\{r={}^t(r_1, r_2, \cdots, r_m)\in R^m;\ \sum_{i=1}^{m}r_i(1-e_i)=0\}.$$

Theorem 3.3.3. Assume that $I_e=R$.

(i) There exist canonical isomorphisms of R-modules:

$$\mathrm{Vero}^*\,\mathscr{H}^q(m+1, n+1; \alpha)\simeq\begin{cases}\bigwedge^{n}\mathscr{H}^1(m+1, 2; \tilde{\alpha}) & (q=n)\\ 0 & (q\neq n),\end{cases}$$

$$\mathrm{Vero}^*\,\mathscr{H}^{\vee}_q(m+1, n+1; \tilde{\alpha})\simeq\begin{cases}\bigwedge^{n}\mathscr{H}^{\vee}_1(m+1, 2; \tilde{\alpha}) & (q=n)\\ 0 & (q\neq n),\end{cases}$$

where $\tilde{\alpha}=(\tilde{\alpha}_0, \tilde{\alpha}_1, \cdots, \tilde{\alpha}_m)$ is any element of $A(m+1, 2)$ such that $\alpha_i-\tilde{\alpha}_i\in\mathbf{Z}$ for any $i=0, 1, \cdots, m$.

(ii) *We have*

$$H^q(m+1, 2; \tilde{\alpha})=0=H^{\vee}_q(m+1, 2; \tilde{\alpha})\qquad(q\neq 1),$$
$$H^1(m+1, 2; \tilde{\alpha})\simeq V_e\simeq H^{\vee}_1(m+1, 2; \tilde{\alpha})\qquad(q=1),$$

where the above R-isomorphisms are not canonical.

A proof of this theorem will be given in Chapter III, § 22 (cf. Theorem 22.2.3).

Remark 3.3.4. (1) The R-module structure of arbitrary fibers of

$$\pi : \mathscr{H}^q(m+1, n+1; \alpha) \to M(m+1, n+1),$$
$$\pi : \mathscr{H}_q^\vee(m+1, n+1; \alpha) \to M(m+1, n+1)$$

is determined by Theorem 3.3.3, since they are local systems of R-modules on $M(m+1, n+1)$.

(2) If $\mathscr{L}$ is *trivial*, i.e., $e_i=1$ for all i, then $I_e=\{0\}\neq R$. In this case Theorem 3.3.3 does not hold.

Next we give two examples for which $I_e=R$ holds.

Example 3.3.5. If $\mathscr{L}$ is not trivial, i.e., there exists an i $(1 \leq i \leq m)$ such that $e_i\neq 1$, then the ring

$$R := \mathbf{Q}\left[e_1^{\pm 1}, e_2^{\pm 1}, \cdots, e_m^{\pm 1}, \frac{1}{1-e_i}\right]$$

satisfies $I_e=R$. In this case, V_e is a free R-module of rank $m-1$, and therefore $\mathscr{H}^n(m+1, n+1; \alpha)$ and $\mathscr{H}_n^\vee(m+1, n+1; \alpha)$ are local systems of *free* R-modules of rank

$$\binom{m-1}{n}$$

Examples 3.3.6. If there exists a polynomial $p(x_1, x_2, \cdots, x_m)$ over $\mathbf{Q}$ such that $p(e_1, e_2, \cdots, e_m)=0$ and $p(1, 1, \cdots, 1)\neq 0$, then the ring

$$R := \mathbf{Q}[e_1^{\pm 1}, e_2^{\pm 1}, \cdots, e_m^{\pm 1}]$$

satisfies $I_e=R$. Indeed, for $i=1, 2, \cdots, m$, let $q_i(x_1, x_2, \cdots, x_m)$ be the polynomial over $\mathbf{Q}$ defined by

$$q_i(x_1, x_2, \cdots, x_m) := \frac{p(1, \cdots, 1, x_i, \cdots, x_m) - p(1, \cdots, 1, x_{i+1}, \cdots, x_m)}{x_i - 1}.$$

Since $p(1, 1, \cdots, 1)$ is a nonzero rational number, we have

$$u_i := \frac{q_i(e_1, e_2, \cdots, e_m)}{p(1, 1, \cdots, 1)} \in R.$$

By the definition of the $q_i(x_1, x_2, \cdots, x_m)$'s, we have

$$p(x_1, x_2, \cdots, x_m)=\sum_{i=1}^{m}(x_i-1)q_i(x_1, x_2, \cdots, x_m)+p(1, 1, \cdots, 1).$$

Substituting $x_i=e_i$ $(i=1, 2, \cdots, m)$ into the above equality, we obtain $\sum_{i=1}^{m}u_i(1-e_i)=1$, i.e., $I_e=R$.

(3.4) Concluding remarks.

Roughly speaking, Theorem 3.3.3 states that the hypergeometric functions of type $(m+1, n+1)$ restricted to the Veronese image is the nth "exterior product" of the Lauricella hypergeometric functions F_D. See [11] for the exterior products of differential equations.

$$\boxed{\mathrm{HGF}(m+1, n+1)|_{\mathrm{Veronese}}=\overset{n}{\wedge}F_D}$$

To know the global behaviour of a hypergeometric function, we have to find their monodromy groups. For this it is convenient to take a point on the Veronese image as the base point of the fundamental group. Finding the monodromy has been done by K. Matsumoto, T. Sasaki, N. Takayama, M. Yoshida [26] and others.

II. Simplicial Local Systems

4. Categories of simplicial complexes

(4.1) The category S of simplicial complexes.

Throughout the paper, a simplicial complex is an *abstract* simplicial complex ([28] Chapter 3, § 1). We briefly recall the category S of simplicial complexes.

Definition 4.1.1. An object of S, called a *simplicial complex*, is a set K, together with a set V, such that the following conditions hold:

(1) each element of K is a nonempty finite subset of V,

(2) for any $a\in V$, $\{a\}\in K$, and

(3) if $\sigma\in K$, then any nonempty subset of σ belongs to K.

An element of K is called a *simplex* of K. Since V is uniquely determined by K, we put $V=V_K$. An element of V_K is called a *vertex* of K. A morphism $f: K \to L$ of S, called a *simplicial map*, is a set-theoretical map $f: V_K \to V_L$ such that $f(\sigma):=\{f(a); a\in\sigma\}\in L$ for any simplex $\sigma\in K$.

Unless otherwise stated explicitly, the terminology concerning the simplicial complexes is always cited from [28].

(4.2) The locally injective category $\mathbf{S}^{li}$ of simplicial complexes.

A simplicial map $f: K \to L$ is said to be *locally injective* if, for any simplex $\sigma \in K$, the set-theoretical map $f: V_K \to V_L$ is injective on σ.

Definition 4.2.1. The locally injective category $\mathbf{S}^{li}$ of simplicial complexes is the subcategory of $\mathbf{S}$ whose morphism is a locally injective simplicial map.

(4.3) The category $\mathbf{S}_{ord}$ of ordered simplicial complexes.

Definition 4.3.1. An object of $\mathbf{S}_{ord}$, called an *ordered simplicial complex* is a pair $(K, <)$ such that K is a simplicial complex and $<$ is a partial order on V_K which induces a total order on every simplices of K. A morphism $f: (K, <) \to (L, <)$ of $\mathbf{S}_{ord}$ is merely a simplical map $f: K \to L$, i.e., f may not be order-preserving.

Remark 4.3.2. There is a natural forgetful functor $\mathbf{S}_{ord} \to \mathbf{S}$.

For every ordered simplicial complexes, the partial order on the vertices is always denoted by $<$. Hence we use K etc. instead of $(K, <)$ etc. to denote an ordered simplicial complex.

(4.4) The category $\mathbf{S}^{li}_{ord}$.

Definition 4.4.1. Let $\mathbf{S}^{li}_{ord}$ be the subcategory of $\mathbf{S}^{li}_{ord}$ whose morphism is a locally injective simplicial map.

Remark 4.4.2. There is a natural forgetful functor $\mathbf{S}^{li}_{ord} \to \mathbf{S}^{li}$ with a commutative diagram of functors:

$$\begin{tikzcd} \mathbf{S}^{li}_{ord} \arrow[r, "\text{subcategory}"] \arrow[d, "\text{forgetful}"'] & \mathbf{S}_{ord} \arrow[d, "\text{forgetful}"] \\ \mathbf{S}^{li} \arrow[r, "\text{subcategory}"'] & \mathbf{S} \end{tikzcd}$$

5. Categories of local systems

(5.1) The category $\mathbf{L}(K)$ of local systems on a simplicial complex K.

Let R be a commutative ring with unity, K a simplicial complex. Let $K^{(i)}$ be the ith skeleton of K, i.e., $K^{(i)}=\{\sigma\in K; \#\sigma\leq i+1\}$ $(i=0, 1, 2, \cdots)$. We define the category $\mathbf{L}(K)$ of local systems of R-modules on K.

Definition 5.1.1. An object $\mathscr{L}=\{\mathscr{L}_a, \xi_{ba}\}$ of $\mathbf{L}(K)$, called a *local system* of R*-modules on* K, is an assignment:

(1) $V_K\ni a\mapsto \mathscr{L}_a$: module,

(2) $K^{(1)}\ni\{a, b\}\mapsto \xi_{ba}: \mathscr{L}_a\rightarrow\mathscr{L}_b$: R-isomorphism,

such that

(i) for any $a\in V_K$, ξ_{aa} is the identity homomorphism on $\mathscr{L}_a$, and

(ii) for any $\{a, b, c\}\in K^{(2)}$, $\xi_{cb}\circ\xi_{ba}=\xi_{ca}$.

Let $\mathscr{L}=\{\mathscr{L}_a, \xi_{ba}\}$ and $\mathscr{M}=\{\mathscr{M}_a, \eta_{ba}\}$ be local systems on K. A morphism $\varphi=\{\varphi_a\}: \mathscr{L}\rightarrow\mathscr{M}$ of $\mathbf{L}(K)$ is a collection of R-homomorphisms $\varphi_a: \mathscr{L}_a\rightarrow\mathscr{M}_a$, $(a\in V_K)$ such that, for any $\{a, b\}\in K^{(1)}$, the following diagram is commutative:

$$\begin{tikzcd} \mathscr{L}_a \arrow[r, "\varphi_a"] \arrow[d, "\xi_{ba}"'] & \mathscr{M}_a \arrow[d, "\eta_{ba}"] \\ \mathscr{L}_b \arrow[r, "\varphi_b"'] & \mathscr{M}_b \end{tikzcd}$$

Remark 5.1.2. From conditions (i) and (ii), we can easily deduce $\xi_{ab}=(\xi_{ba})^{-1}$ for any $\{a, b\}\in K^{(1)}$.

Hereafter, unless otherwise stated explicitly, any local system is a local system of R*-modules*, R being fixed throughout the paper.

(5.2) The pull-back functor.

A simplicial map $f: K\rightarrow L$ induces a covariant functor $f^*: \mathbf{L}(L)\rightarrow\mathbf{L}(K)$, called the *pull-back functor*.

Definition 5.2.1. Given a local system $\mathscr{L}=\{\mathscr{L}_a, \xi_{ba}\}$ on L, put

$$(f^*\mathscr{L})_a=\mathscr{L}_{f(a)}, \qquad (a\in V_K),$$
$$(f^*\xi)_{ba}=\xi_{f(b)f(a)}, \qquad (\{a, b\}\in K^{(1)}).$$

Then $f^*\mathscr{L}=\{(f^*\mathscr{L})_a, (f^*\xi)_{ba}\}$ becomes a local system on K, called the *pull-back* of $\mathscr{L}$ by f. Given a morphism $\varphi=\{\varphi_a\}: \mathscr{L}\rightarrow\mathscr{M}$ of $\mathbf{L}(L)$, put

$$(f^*\varphi)_a=\varphi_{f(a)}:(f^*\mathscr{L})_a \to (f^*\mathscr{M})_a, \quad (a\in V_K).$$

Then $f^*\varphi=\{(f^*\varphi)_a\}:f^*\mathscr{L} \to f^*\mathscr{M}$ becomes a morphism of $\mathbf{L}(K)$, called the *pull-back* of φ by f.

(5.3) The category $\mathbf{L}$ of local systems.

We introduce the category $\mathbf{L}$ of local systems.

Definition 5.3.1. An object of $\mathbf{L}$ is a pair $(K, \mathscr{L})$ of a simplicial complex K and a local system $\mathscr{L}$ on K. A morphism of $\mathbf{L}$ is a pair $(f, \varphi):(K, \mathscr{L}) \to (L, \mathscr{M})$, where $f:K \to L$ is a simplicial map and $\varphi=\{\varphi_a\}$ is a collection of R-homomorphisms $\varphi_a:\mathscr{L}_a \to \mathscr{M}_{f(a)}$, $(a\in V_K)$ such that for each $\{a, b\}\in K^{(1)}$, the following diagram is commutative:

$$\begin{tikzcd}
\mathscr{L}_a \arrow[r, "\varphi_a"] \arrow[d, "\xi_{ba}"'] & \mathscr{M}_{f(a)} \arrow[d, "\eta_{f(b)f(a)}"] \\
\mathscr{L}_b \arrow[r, "\varphi_b"'] & \mathscr{M}_{f(b)}
\end{tikzcd}$$

This diagram is called the *compatibility condition.*

Remark 5.3.2. A morphism of $\mathbf{L}$ is nothing but a pair $(f, \varphi):(K, \mathscr{L}) \to (L, \mathscr{M})$, where $f:K \to L$ is a simplicial map and $\varphi:\mathscr{L} \to f^*\mathscr{M}$ is a morphism of $\mathbf{L}(K)$.

The group of all isomorphisms of $(K, \mathscr{L})$ is denote by $\mathrm{Aut}(K, \mathscr{L})$.

Remark 5.3.3. There is a natural covariant functor $\mathbf{L} \to \mathbf{S}$, i.e., the functor sending $(K, \mathscr{L})$ to $\mathscr{L}$.

(5.4) The locally injective category $\mathbf{L}^{li}$ of local systems.

A morphism $(f, \varphi):(K, \mathscr{L}) \to (L, \mathscr{M})$ of $\mathbf{L}$ is said to be *locally injective* if f is locally injective, (see (4.2)).

Definition 5.4.1. The locally injective category $\mathbf{L}^{li}$ of simplicial complexes is the subcategory of $\mathbf{L}$ whose morphisms are locally injective morphisms of $\mathbf{L}$.

Remark 5.4.2. There is a natural forgetful functor $\mathbf{L}^{li} \to \mathbf{S}^{li}$, (see Remark 5.3.2).

(5.5) The local section functor.

Let K be a simplicial complex, $\mathscr{L}=\{\mathscr{L}_a, \xi_{ba}\}$ a local system on K.

Definition 5.5.1. For any simplex σ of K, a *section* of $\mathscr{L}$ on σ is a map

$$u : \sigma \ni a \mapsto u(a) \in \mathscr{L}_a$$

such that, for any $\{a, b\} \in \sigma$, $u(b)=\xi_{ba}u(a)$. A section of $\mathscr{L}$ on some simplex is called a *local section* of $\mathscr{L}$. The set of all sections of $\mathscr{L}$ on σ is denoted by $\mathscr{L}_\sigma$.

$\mathscr{L}_\sigma$ admits a structure of R-modules in an obvious manner.

Definition 5.5.2. Let $(f, \varphi):(K, \mathscr{L}) \to (L, \mathscr{M})$ be a morphism of $\mathbf{L}^{li}$, i.e., a locally injective morphism of $\mathbf{L}$. For any simplex σ of K, let $f|_\sigma : \sigma \to f(\sigma)$ be the restriction of f to σ; $f|_\sigma$ is bijective. An R-homomorphism $\varphi_\sigma : \mathscr{L}_\sigma \to \mathscr{M}_{f(\sigma)}$, $u \mapsto \varphi_\sigma(u)$ is defined by $(\varphi_\sigma(u))(a)=(\varphi_{(f|_\sigma)^{-1}(a)} \circ u \circ (f|_\sigma)^{-1})(a)$ $(a \in f(\sigma))$.

This is well-defined, i.e., $\varphi_\sigma(u) \in \mathscr{M}_{f(\sigma)}$. Indeed, for a, $b \in f(\sigma)$, we have

$$\begin{aligned}(\varphi_\sigma(u))(b)&=(\varphi_{(f|_\sigma)^{-1}(b)} \circ u \circ (f|_\sigma)^{-1})(b)\\ &=(\varphi_{(f|_\sigma)^{-1}(b)} \circ \xi_{(f|_\sigma)^{-1}(b),\,(f|_\sigma)^{-1}(a)} \circ u \circ (f|_\sigma)^{-1})(a)\\ &=(\eta_{ba} \circ \varphi_{(f|_\sigma)^{-1}(a)} \circ u \circ (f|_\sigma)^{-1})(a)\\ &=(\eta_{ba}(\varphi_\sigma(u)))(a).\end{aligned}$$

This shows that $\varphi_\sigma(u) \in \mathscr{M}_{f(\sigma)}$.

Lemma 5.5.3. *Let $(f, \varphi):(K, \mathscr{L}) \to (L, \mathscr{M})$ and $(g, \psi):(L, \mathscr{M}) \to (M, \mathscr{N})$ be morphisms of $\mathbf{L}^{li}$ and put $(h, \chi)=(g, \psi) \circ (f, \varphi)$. Then for any simplex σ of K,*

$$\chi_\sigma = \psi_{f(\sigma)} \circ \varphi_\sigma.$$

Proof. First note that $f|_\sigma$, $g|_{f(\sigma)}$ and $h|_\sigma$ are bijective. For simplicity of notation, we put $x=(f|_\sigma)^{-1}$, $y=(g|_{f(\sigma)})^{-1}$ and $z=(h|_\sigma)^{-1}$. Since $h|_\sigma=(g|_{f(\sigma)}) \circ (f|_\sigma)$, we have $z=x \circ y$. Moreover, for any $a \in h(\sigma)$, we have $\chi_{z(a)}=\psi_{y(a)} \circ \varphi_{x(y(a))}$. Hence, for any $u \in \mathscr{L}_\sigma$ and $a \in h(\sigma)$, we obtain

$$\begin{aligned}(\chi_\sigma(u))(a) &= (\chi_{z(a)} \circ u \circ z)(a) \\ &= \psi_{y(a)}((\varphi_{x(y(a))} \circ u \circ x)(y(a))) \\ &= (\psi_{y(a)} \circ \varphi_\sigma(u) \circ y)(a) \\ &= ((\psi_{f(\sigma)} \circ \varphi_\sigma)(u))(a).\end{aligned}$$

This implies $\chi_\sigma = \psi_{f(\sigma)} \circ \varphi_\sigma$. Hence the lemma is established.

Definition 5.5.4. Let $\mathbf{SL}^{li}$ be the category defined as follows: An object of $\mathbf{SL}^{li}$ is a triple $(\sigma, K, \mathscr{L})$, where $(K, \mathscr{L})$ is an object of $\mathbf{L}^{li}$ and σ is a simplex of K, and a morphism $(\sigma, K, \mathscr{L}) \to (\tau, L, \mathscr{M})$ of $\mathbf{SL}^{li}$ is a morphism $(f, \varphi) : (K, \mathscr{L}) \to (L, \mathscr{M})$ of $\mathbf{L}^{li}$ such that $f(\sigma) = \tau$.

Definition 5.5.5. Lemma 5.5.3 implies that the correspondence $(\sigma, K, \mathscr{L}) \mapsto \mathscr{L}_\sigma$ defines a covariant functor from $\mathbf{SL}^{li}$ into the category of R-modules. This functor is called the *local section functor.*

Let σ and τ be two simplices such that $\tau \subseteq \sigma$. Since $u \in \mathscr{L}_\sigma$ is a map defined on σ, one can speak of the restriction $r_\tau^\sigma(u)$ of u into τ. By the definition of local system, it is easily seen that $r_\tau^\sigma(u) \in \mathscr{L}_\tau$ and $r_\tau^\sigma : \mathscr{L}_\sigma \to \mathscr{L}_\tau$ is an R-isomorphism. Definition 5.5.2 immediately implies the following:

Lemma 5.5.6. *For any face τ of σ, there is a commutative diagram of R-modules:*

$$\begin{tikzcd} \mathscr{L}_\sigma \arrow[r, "\varphi_\sigma"] \arrow[d, "r_\tau^\sigma"'] & \mathscr{M}_{f(\sigma)} \arrow[d, "r_{f(\tau)}^{f(\sigma)}"] \\ \mathscr{L}_\tau \arrow[r, "\varphi_\tau"'] & \mathscr{M}_{f(\tau)}. \end{tikzcd}$$

(5.6) The categories $\mathbf{L}_{ord}$ and $\mathbf{L}_{ord}^{li}$.

Definition 5.6.1. The categories $\mathbf{L}_{ord}$ and $\mathbf{L}_{ord}^{li}$ are defined obviously; an object of $\mathbf{L}_{ord}$ is a pair $(K, \mathscr{L})$, where K is an ordered simplicial complex and $\mathscr{L}$ is a local system on K, and $\mathbf{L}_{ord}^{li}$ is the subcategory of $\mathbf{L}_{ord}$ whose morphism is locally injective morphism of $\mathbf{L}_{ord}$.

Remark 5.6.2. There is a natural commutative diagram of functors:

$$\begin{tikzcd}
\mathbf{L}^{li}_{ord} \arrow[r, "\text{subcategory}"] \arrow[d, "\text{forgetful}"'] & \mathbf{L}_{ord} \arrow[d, "\text{forgetful}"] \\
\mathbf{L}^{li} \arrow[r, "\text{subcategory}"'] & \mathbf{L}
\end{tikzcd}$$

6. Subdivision functor

(6.1) The subdivision functor $\mathrm{Sd} : \mathbf{S} \to \mathbf{S}_{ord}$.

The subdivision Sd is a covariant functor from $\mathbf{S}$ into $\mathbf{S}_{ord}$. By composing it with the forgetful functor $\mathbf{S}_{ord} \to \mathbf{S}$ (see (4.3)), Sd is also considered as a functor from $\mathbf{S}$ into itself.

Definition 6.1.1. Let K be a simplicial complex. A vertex of $\mathrm{Sd}K$ is a simplex of K, i.e., $V_{\mathrm{Sd}K} = K$. The partial order on $V_{\mathrm{Sd}K}$ is defined by putting $\tau < \sigma$ if $\tau \subset \sigma$. A simplex of $\mathrm{Sd}K$ is a finite set $\{\sigma_0, \sigma_1, \cdots, \sigma_q\}$ of simplices of K such that $\sigma_0 < \sigma_1 < \cdots < \sigma_q$. If $f : K \to L$ is a simplicial map, then the simplicial map $\mathrm{Sd}f : \mathrm{Sd}K \to \mathrm{Sd}L$ is defined by $\mathrm{Sd}f : V_{\mathrm{Sd}K}(=K) \to V_{\mathrm{Sd}L}(=L)$, $\sigma \mapsto f(\sigma)$.

(6.2) The subdivision functor $\mathrm{Sd} : \mathbf{L}^{li} \to \mathbf{L}^{li}_{ord}$.

The functor $\mathrm{Sd} : \mathbf{S} \to \mathbf{S}_{ord}$ extends to a covariant functor from $\mathbf{L}^{li}$ into $\mathbf{L}^{li}_{ord}$.

Definition 6.2.1. Let $\mathscr{L} = \{\mathscr{L}_a, \xi_{ba}\}$ be a local system on K. Then the local system $\mathrm{Sd}\mathscr{L} = \{(\mathrm{Sd}\mathscr{L})_\sigma, (\mathrm{Sd}\xi)_{\tau\sigma}\}$ on $\mathrm{Sd}K$ is defined by

(1) for any $\sigma \in V_{\mathrm{Sd}K} = K$, $(\mathrm{Sd}\mathscr{L})_\sigma = \mathscr{L}_\sigma$, and

(2) for any $\{\sigma, \tau\} \in (\mathrm{Sd}K)^{(1)}$, $(\mathrm{Sd}\xi)_{\tau\sigma} : \mathscr{L}_\sigma \to \mathscr{L}_\tau$ is given by

$$(\mathrm{Sd}\xi)_{\tau\sigma} = \begin{cases} r^\sigma_\tau & \text{if } \sigma \supseteq \tau, \\ (r^\tau_\sigma)^{-1} & \text{if } \sigma \subseteq \tau. \end{cases}$$

Definition 6.2.2. If $(f, \varphi) : (K, \mathscr{L}) \to (L, \mathscr{M})$ is a morphism of $\mathbf{L}^{li}$, then we put $\mathrm{Sd}(f, \varphi) = (\mathrm{Sd}f, \mathrm{Sd}\varphi)$, where $\mathrm{Sd}\varphi = \{(\mathrm{Sd}\varphi)_\sigma\}$ is defined by

$$(\mathrm{Sd}\varphi)_\sigma = \varphi_\sigma : \mathscr{L}_\sigma \to \mathscr{M}_{f(\sigma)}, \qquad (\sigma \in V_K \in K).$$

where φ_σ is defined in Definition 5.5.2.

Lemma 5.5.6 and Definition 6.2.1 imply that $(\mathrm{Sd}f, \mathrm{Sd}\varphi)$ is a morphism of $\mathbf{L}^{li}_{ord}$.

7. External product functor

(7.1) The direct product functor $\mathbf{S}_{ord}\times\cdots\times\mathbf{S}_{ord}\to\mathbf{S}_{ord}$.

Let $K_1, K_2, \cdots, K_n$ be ordered simplicial complexes. Then there exists an ordered simplicial complex $K=K_1\times K_2\times\cdots\times K_n$, called the *direct product* of $K_1, K_2, \cdots, K_n$. Let us describe it explicitly, with the following:

Notation 7.1.1. The following notations are used in Definition 7.1.3.

$K_1, K_2, \cdots, K_n$: ordered simplicial complexes,

$K=K_1\times K_2\times\cdots\times K_n$: to be defined in Definition 7.1.3,

$q=(q_1, q_2, \cdots, q_n)$: an n-tuple of nonnegative integers,

$r=q_1+q_2+\cdots+q_n$,

$r_i=q_1+q_2+\cdots+q_i$, $(i=1, 2, \cdots, n)$,

$j_i=(j_{i1}, j_{i2}, \cdots, j_{iq_i})$: a q_i-tuple of integers such that

$$1\leq j_{i1}\leq j_{i2}\leq\cdots\leq j_{iq_i}\leq r,$$

$j=(j_1, j_2, \cdots, j_n)$: the map from $\{1, 2, \cdots, r\}$ into itself defined by

$$j=\begin{pmatrix} 1 & \cdots & r_1 & r_1+1 & \cdots & r_2 & \cdots & r_{n-1}+1 & \cdots & r_n \\ j_{11} & \cdots & j_{1q_1} & j_{21} & \cdots & j_{2q_2} & \cdots & j_{n1} & \cdots & j_{nq_n} \end{pmatrix},$$

$\mathfrak{S}_r$: the symmetric group on $\{1, 2, \cdots, r\}$,

$J(q)$: the set of all j's such that $j\in\mathfrak{S}_r$,

$\sigma_i=\{a_{i0}, a_{i1}, \cdots, a_{iq_i}\}$: *a* q_i-simplex of K_i such that $a_{i0}<a_{i1}<\cdots<a_{iq_i}$,

$\Sigma(q)=\{\sigma=(\sigma_1, \sigma_2, \cdots, \sigma_n)\,;\, \sigma_i$ is a q_i-simplex of $K_i\}$

Remark 7.1.2. Elementary combinatrics shows that the cardinality $\#J(q)$ of $J(q)$ is given by

$$\#J(q)=\frac{r!}{q_1!\,q_2!\cdots q_n!}.$$

Definition 7.1.3. A *vertex* of K is an n-tuple $a=(a_1, a_2, \cdots, a_n)$, where a_i is a vertex of K_i, $(i=1, 2, \cdots, n)$, i.e., $V_K=V_{K_1}\times V_{K_2}\times\cdots\times V_{K_n}$. The partial order on V_K is the *lexicographic order*, i.e., for $a=(a_1, a_2, \cdots, a_n)$ and $b=(b_1, b_2, \cdots, b_n)$, we put $a<b$ if and only if $a_j=b_j$ $(j<i)$ and $a_i<b_i$ for some $i\in\{1, 2, \cdots, n\}$. Sometimes $(a_1, a_2, \cdots, a_n)$ is denoted by $a_1\times a_2\times\cdots\times a_n$.

The simplices of K : For any $\sigma\in\Sigma(q)$ and $j\in J(q)$ in Notation 7.1.1, we shall define a simplex $\langle\sigma\,;j\rangle$ of K. For $i=1, 2, \cdots, n$, we put

$$(b_{i0}, b_{i1}, \cdots, b_{ir}) = (\underbrace{\overset{0}{\breve{a}_{i0}}, \cdots, a_{i0}}, \underbrace{\overset{j_{i1}}{\breve{a}_{i1}}, \cdots, a_{i1}}, \underbrace{\overset{j_{i2}}{\breve{a}_{i2}}, \cdots, a_{i2}}, \cdots, \underbrace{\overset{j_{iq_i}}{\breve{a}_{iq_i}}, \cdots, \overset{r}{\breve{a}_{iq_i}}}).$$

For $k=0, 1, 2, \cdots, r$, we define a vertex c_k of K by $c_k=b_{1k}\times b_{2k}\times\cdots\times b_{nk}$. Since $j\in J(q)$, we have $c_0<c_1<\cdots<c_r$ in the lexicographic order. Now we define $\langle\sigma; j\rangle=\langle\sigma_1, \cdots, \sigma_n; j_1, \cdots, j_n\rangle$ by $\langle\sigma; j\rangle=\{c_0, c_1, \cdots, c_r\}$. A simplex of K is, by definition, a nonempty subset of $\langle\sigma; j\rangle$ for some $\sigma\in\Sigma(q)$ and $j\in J(q)$. Clearly, the partial order on V_K induces a total order on every simplices of K.

If $f_i: K_i\to L_i$, $(i=1, 2, \cdots, n)$ are morphisms of $\mathbf{S}_{ord}$, then a morphism $f=f_1\times\cdots\times f_n\ K:\to L=L_1\times\cdots\times L_n$ is defined by

$$f(a_1\times a_2\times\cdots\times a_n)=f_1(a_1)\times f_2(a_2)\times\cdots\times f_n(a_n), \qquad (a_i\in V_{K_i}).$$

We have obtained a covariant functor $\mathbf{S}_{ord}\times\cdots\times\mathbf{S}_{ord}\to\mathbf{S}_{ord}$, called the *direct product functor.*

(7.2) The external product functor $\mathbf{L}_{ord}\times\cdots\times\mathbf{L}_{ord}\to\mathbf{L}_{ord}$.

Definition 7.2.1. Let $(K_i, \mathscr{L}_i)$, $(i=1, 2, \cdots, n)$ be objects of $\mathbf{L}_{ord}$ with $\mathscr{L}_i=\{(\mathscr{L}_i)_{a_i}, (\xi_i)_{b_i, a_i}\}$. Then we have a new object $(K, \mathscr{L})$ with $\mathscr{L}=\{\mathscr{L}_a, \xi_{ba}\}$ defined by

(1) $K=K_1\times K_2\times\cdots\times K_n$,

(2) $\mathscr{L}_a=(\mathscr{L}_1)_{a_1}\otimes(\mathscr{L}_2)_{a_2}\otimes\cdots\otimes(\mathscr{L}_n)_{a_n}$ for each $a=a_1\times\cdots\times a_n\in V_K$,

(3) for each $\{a, b\}\in K^{(1)}$, with $a=a_1\times a_2\times\cdots\times a_n$ and $b=b_1\times b_2\times\cdots\times b_n$, we define $\xi_{ba}:\mathscr{L}_a\to\mathscr{L}_b$ by $\xi_{ba}=(\xi_1)_{b_1, a_1}\otimes(\xi_2)_{b_2, a_2}\otimes\cdots\otimes(\xi_n)_{b_n, a_n}$.

Definition 7.1.3 implies that if $\{a, b\}\in K^{(1)}$, then $\{a_i, b_i\}\in K_i^{(1)}$ for each $i=1, 2, \cdots, n$. So ξ_{ba} is well-defined. We write

$$\begin{aligned}(K, \mathscr{L})&=(K_1, \mathscr{L}_1)\boxtimes(K_2, \mathscr{L}_2)\boxtimes\cdots\boxtimes(K_n, \mathscr{L}_n)\\&=\mathop{\boxtimes}_{i=1}^{n}(K_i, \mathscr{L}_i)\\&=(K_1\times K_2\times\cdots\times K_n, \mathscr{L}_1\boxtimes\mathscr{L}_2\boxtimes\cdots\boxtimes\mathscr{L}_n).\end{aligned}$$

$(K, \mathscr{L})$ is called the *external product* of $(K_1, \mathscr{L}_1), (K_2, \mathscr{L}_2), \cdots, (K_n, \mathscr{L}_n)$.

Definition 7.2.2. Let $(f_i, \varphi_i):(K_i, \mathscr{L}_i)\to(L_i, \mathscr{M}_i)$, $(i=1, 2, \cdots, n)$ be morphisms of $\mathbf{L}_{ord}$ with $\varphi_i=\{(\varphi_i)_{a_i}\}$. Put $(K, \mathscr{L})=\boxtimes_{i=1}^n(K_i, \mathscr{L}_i)$ and $(L, \mathscr{M})=\boxtimes_{i=1}^n(L_i, \mathscr{M}_i)$. Then we have a new morphism $(f, \varphi):(K, \mathscr{L})\to(L, \mathscr{M})$ with $\varphi=\{\varphi_a\}$ defined by

(1) $f=f_1\times f_2\times\cdots\times f_n$,

(2) for each vertex $a=a_1\times a_2\times\cdots\times a_n\in V_K$, $\varphi_a:\mathscr{L}_a\to\mathscr{M}_{f(a)}$ is defined by

$$\varphi_a=(\varphi_1)_{a_1}\otimes(\varphi_2)_{a_2}\otimes\cdots\otimes(\varphi)_{a_n}.$$

Definitions 7.2.1 and 7.2.2 give the *external product functor* $\mathbf{L}_{ord}\times\cdots\times\mathbf{L}_{ord}\to\mathbf{L}_{ord}$.

8. Group actions on a local system

(8.1) Group actions on a simplicial complex and quotients.

Let K be a simplicial complex, and G be a group. Let Aut$K$ be the group of all simplicial automorphisms of K.

Definition 8.1.1. A *group action* of G on K is a group homomorphism $\rho: G\to \mathrm{Aut}K$.

If no confusion might occur, we write $\rho(g)a=ga$ for $g\in G$ and $a\in V_K$. If G acts on K, then we can define the *quotient simplicial complex* K/G.

Definition 8.1.2. A *vertex* of K/G is a G-orbit through a vertex of K. We have a natural projection $\pi: V_K\to V_{K/G}$, $a\mapsto[a]=Ga$, where $[a]=Ga$ is the G-orbit through a. For a set $s=\{a_0, a_1, \cdots, a_q\}$ of vertices of K, we put $\pi(s)=\{[a_0], [a_1], \cdots, [a_q]\}$. A nonempty finite subset σ of $V_{K/G}$ is a simplex of K/G if there exists a simplex $\tilde{\sigma}$ of K such that $\pi(\tilde{\sigma})=\sigma$. Such a simplex $\tilde{\sigma}$ is called a simplex *over* σ.

The definition immediately implies that K/G is a simplicial complex and the projection $\pi: K\to K/G$ is a simplicial map.

(8.2) Regular and quasi-regular actions.

Definition 8.2.1. An action of G on K is said to be *quasi-regular* if the following condition holds: For any $a\in V_K$ and $g\in G$, if $\{a, ga\}$ is a simplex of K, then $ga=a$.

Lemma 8.2.2. *If G acts on K quasi-regularly, then the canonical projection $\pi: K\to K/G$ is locally injective.*

Proof. For any simplex σ of K, we shall show that π is injective on σ. Assume that $a, b\in\sigma$ and $\pi(a)=\pi(b)$. Then there exists a $g\in G$ such that $b=ga$. Since $\{a, b\}$ is a subset of σ, $\{a, b\}$ is also a simplex of K. By quasi-regularity, we have $b=ga=a$. This shows that π is injective on σ.

If G acts on K then G acts on the set of all simplices of K; $g\in G$ sends a simplex $\sigma=\{a_0, a_1, \cdots, a_q\}$ of K into another simplex $g\sigma=\{ga_0, ga_1, \cdots, ga_q\}$. Given a simplex σ of K/G, let $O(\sigma)$ be the set of all simplices of K over σ. Then G acts on $O(\sigma)$.

Definition 8.2.3. An action of G on K is said to be *regular* if for any simplex σ of K/G, G acts on $O(\sigma)$ transtively.

Remark 8.2.4. It is easy to see that a regular action is a quasi-regular action.

(8.3) Induced action on the subdivision of a simplicial complex.

Definition 8.3.1. An action of G on K induces an action of G on the subdivision $\mathrm{Sd}K$ in a natural manner: $G\times V_{\mathrm{Sd}K} \to V_{\mathrm{Sd}K}$, $(g, \sigma)\mapsto g\sigma$.

Theorem 8.3.2. ([5], Chap.III, Proposition 1.1)

(1) *If G acts on K, then G acts on* $\mathrm{Sd}K$ *quasi-regularly,*

(2) *If G acts on K quasi-regularly, then G acts on* $\mathrm{Sd}K$ *regularly,*

(3) *If G acts on K, then G acts on* Sd^2K *regularly.*

(8.4) Group action on a local system.

Let $(K, \mathscr{L})$ be an object of **L** with $\mathscr{L}=\{\mathscr{L}_a, \xi_{ba}\}$, and let $\mathrm{Aut}(K, \mathscr{L})$ be the group of all automorphisms of $(K, \mathscr{L})$, (see (5.3)).

Definition 8.4.1. A *group action* of G on $(K, \mathscr{L})$ is a group homomorphism $\rho: G \to \mathrm{Aut}(K, \mathscr{L})$.

For each $g\in G$, $\rho(g)=(f(g), \varphi(g))$ with $\varphi(g)=\{\varphi_a(g)\}_{a\in V_K}$ is an automorphism of $(K, \mathscr{L})$. For simplicity of notation, we write $f(g)=g$ and $\varphi_a(g)=g$. Then for each $\{a, b\}\in K^{(1)}$, there is the following commutative diagram, (see Definition 5.3.1):

$$\begin{tikzcd}
\mathscr{L}_a \arrow[r, "g"] \arrow[d, "\xi_{ba}"'] & \mathscr{L}_{ga} \arrow[d, "\xi_{gb,ga}"] \\
\mathscr{L}_b \arrow[r, "g"'] & \mathscr{L}_{gb}.
\end{tikzcd}$$

(8.5) Induced action on the local sections.

Let $(K, \mathscr{L})$ be an object of $\mathbf{L}$, σ a simplex of K. An action of G on $(K, \mathscr{L})$ induces an R-isomorphism $g: \mathscr{L}_\sigma \to \mathscr{L}_{g\sigma}$ for each $g \in G$.

Definition 8.5.1. For each $g \in G$, the R-isomorphism $g: \mathscr{L}_\sigma \to \mathscr{L}_{g\sigma}$, $u \mapsto gu$ is defined by $(gu)(a) = g \cdot u(g^{-1}a)$, $(a \in g\sigma)$.

It is easy to see that $gu \in \mathscr{L}_{g\sigma}$, (cf. Definition 5.5.1).

Remark 8.5.2. For each pair (σ, τ) of simplices such that τ is a face of σ, there is the following commutative diagram of R-modules:

$$\begin{array}{ccc} \mathscr{L}_\sigma & \xrightarrow{g} & \mathscr{L}_{g\sigma} \\ {\scriptstyle r_\tau^\sigma}\downarrow & & \downarrow{\scriptstyle r_{g\tau}^{g\sigma}} \\ \mathscr{L}_\tau & \xrightarrow[g]{} & \mathscr{L}_{g\tau}. \end{array}$$

(8.6) Induced action on the subdivision of a local system.

By (8.3) and (8.5), an action of G on $(K, \mathscr{L})$ induces an action of G on $(\mathrm{Sd}K, \mathrm{Sd}\mathscr{L})$ naturally. The commutative diagram in Remark 8.5.2 leads to the compatibility condition i.e., the commutative diagram in Definition 5.3.1.

(8.7) A lift of a group action.

Assume that a group G acts on a simplicial complex K and let $\pi: K \to K/G$ be the canonical projection. Moreover, let $\mathscr{L}$ be a local system on K/G. Then the pull-back functor (see (5. 2)) gives us the local system $\pi^*\mathscr{L}$ on K.

Definition8.7.1. An action of G on K lifts to an action of G on $(K, \pi^*\mathscr{L})$. The R-isomorphism $g: (\pi^*\mathscr{L})_a \to (\pi^*\mathscr{L})_{ga}$ associated with $g \in G$ is defined so as to make the diagram

$$\begin{array}{ccc} (\pi^*\mathscr{L})_a & \xrightarrow{g} & (\pi^*\mathscr{L})_{ga} \\ \| & & \| \\ \mathscr{L}_{\pi(a)} & = & \mathscr{L}_{\pi(ga)}, \end{array} \qquad (a \in V_K,\ g \in G)$$

commutative. Moreover, there is a morphism $\pi: (K, \pi^*\mathscr{L}) \to (K/G, \mathscr{L})$ of $\mathbf{L}$, called the *canonical projection.* This consists of a simplicial map $\pi: K \to K/G$ and a collection of R-homomorphisms $\pi: (\pi^*\mathscr{L})_a \to \mathscr{L}_{\pi(a)}$, $(a \in V_K)$ defined by

$$\pi=\mathrm{id}:(\pi^*\mathscr{L})_a:=\mathscr{L}_{\pi(a)}\to\mathscr{L}_{\pi(a)}.$$

Remark 8.7.2. (i) For any $g\in G$, there is a commutative diagram of morphisms of $\mathbf{L}$:

$$\begin{array}{ccc}(K,\pi^*\mathscr{L}) & \xrightarrow{g} & (K,\pi^*\mathscr{L})\\ \pi\downarrow & & \downarrow\pi\\ (K/G,\mathscr{L}) & = & (K/G,\mathscr{L}).\end{array}$$

(ii) If G acts on K *quasi-regularly*, then Lemma 8.2.2 implies that this diagram is a commutative diagram of morphisms of $\mathbf{L}^{li}$.

Applying the local section functor (see Definition 5.5.5) to the above commutative diagram, we obtain the following:

Remark 8.7.3. Assume that G acts on K quasi-regularly. Then for any simplex σ of K and $g\in G$, there is a commutative diagram of R-modules:

$$\begin{array}{ccc}(\pi^*\mathscr{L})_\sigma & \xrightarrow{g} & (\pi^*\mathscr{L})_{g\sigma}\\ \pi_\sigma\downarrow & & \downarrow\pi_{g\sigma}\\ \mathscr{L}_{\pi(\sigma)} & = & \mathscr{L}_{\pi(\sigma)}.\end{array}$$

(8.8) The symmetric group acts on the exterior product.

Let $(K_i,\mathscr{L}_i)$, $(i=1,2,\cdots,n)$ be objects of $\mathbf{L}$, and $\mathfrak{S}_n$ be the symmetric group on $\{1,2,\cdots,n\}$.

Notation 8.8.1. Let $\tau\in\mathfrak{S}_n$. We use the following notations.

$$K=K_1\times K_2\times\cdots\times K_n,$$
$$\mathscr{L}=\mathscr{L}_1\boxtimes\mathscr{L}_2\boxtimes\cdots\boxtimes\mathscr{L}_n,$$
$$\tau K=K_{\tau(1)}\times K_{\tau(2)}\times\cdots\times K_{\tau(n)},$$
$$\tau L=L_{\tau(1)}\boxtimes L_{\tau(2)}\boxtimes\cdots\boxtimes L_{\tau(n)}.$$

Let us define an isomorphism $\tau:(K,\mathscr{L})\to(\tau K,\tau\mathscr{L})$ of $\mathbf{L}$.

Definition 8.8.2. The simplicial map $\tau:K\to\tau K$ is defined by the set-theoretical map $\tau:V_K\to V_{\tau K}$, $a_1\times a_2\times\cdots\times a_n\mapsto a_{\tau(1)}\times a_{\tau(2)}\times\cdots\times a_{\tau(n)}$. For each vertex $a=a_1\times a_2\times\cdots\times a_n\in V_K$, taking into account that

$$\mathscr{L}_a=(\mathscr{L}_1)_{a_1}\otimes(\mathscr{L}_2)_{a_2}\otimes\cdots\otimes(\mathscr{L}_n)_{a_n},$$
$$(\tau\mathscr{L})_{\tau(a)}=(\mathscr{L}_{\tau(1)})_{a_{\tau(1)}}\otimes(\mathscr{L}_{\tau(2)})_{a_{\tau(2)}}\otimes\cdots\otimes(\mathscr{L}_{\tau(n)})_{a_{\tau(n)}},$$

we define an R-isomorphism $\tau:\mathscr{L}_a\to(\tau\mathscr{L})_{\tau(a)}$ by

$$\tau(u_1\otimes u_2\otimes\cdots\otimes u_n)=u_{\tau(1)}\otimes u_{\tau(2)}\otimes\cdots\otimes u_{\tau(n)}.$$

The following special case is of particular importance:

$$(K,\ \mathscr{L}):=(K_1,\ \mathscr{L}_1)=(K_2,\ \mathscr{L}_2)=\cdots=(K_n,\ \mathscr{L}_n).$$

Here we used the notation $(K,\ \mathscr{L})$ in a different sense as before. Applying Definition 8.8.1 to this case, we obtain the following:

Lemma 8.8.3. *Let $(K,\ \mathscr{L})$ be an object of* $\mathbf{L}$, *and n be a positive integer. Then there is a natural action of $\mathfrak{S}_n$ on $(K^n,\ \overset{n}{\boxtimes}\mathscr{L})$, where $\overset{n}{\boxtimes}\mathscr{L}=\overbrace{\mathscr{L}\boxtimes\cdots\boxtimes\mathscr{L}}^{n\text{-factors}}$.*

9. Chain complex functors

(9.1) Ordered and oriented simplices.

Definition 9.1.1. Let K be a simplicial complex, and σ be a q-simplex of K. An *ordered simplex over* σ is a bijective map $\phi:\{0,\ 1,\ 2,\ \cdots,\ q\}\to\sigma$. An ordered simplex of K is, by definition, an ordered simplex over a simplex of K. If ϕ is an ordered simplex of σ, then σ is said to be the simplex *under* ϕ. We put $\sigma=\langle\phi\rangle$.

The set of all ordered simplices of K is denoted by K_{ord}. There is a natural forgetful map: $K_{ord}\to K$, $\phi\mapsto\langle\phi\rangle$. Let $\mathfrak{S}_{q+1}$ be the symmetric group on $\{0,\ 1,\ 2,\ \cdots,\ q\}$. If ϕ is an ordered simplex over a q-simplex ϕ, then for any $s\in\mathfrak{S}_{q+1}$, $\phi\cdot s$ is also an ordered simplex over σ. All ordered simplices over σ are obtained in this manner.

Definition 9.1.3. Let ϕ and ψ be ordered simplex over a q-simplex σ. They are said to be equivalent if there exists an $s\in\mathfrak{A}_{q+1}$ such that $\psi=\phi\circ s$, where $\mathfrak{A}_{q+1}$ is the alternating subgroup of $\mathfrak{S}_{q+1}$. The equivalence class $[\phi]$ determined by ϕ is called the oriented simplex under ϕ. An oriented simplex over σ is, by definition, an oriented simplex under an ordered simplex over σ. An oriented simplex of K is an oriented simplex over a simplex of K.

Over any q-simplex with $q \geq 1$ there are exactly two oriented simplices. The set of all oriented simplices of K is denoted by K_{ori}. We have a sequence of forgetful maps: $K_{ord} \to K_{ori} \to K$, $\phi \mapsto [\phi] \mapsto \langle\phi\rangle$.

Remark 9.1.4. A group action on K lifts to K_{ord} and K_{ori}, so that the forgetful maps are equivariant with respect to these actions.

(9.2) Principal faces.

For $i=0, 1, \cdots, q$, let Δ_i^q be an element of $\mathfrak{S}_{q+1}$ defined by

$$\Delta_i^q(j)=\begin{cases} j & (0 \leq j < i) \\ j+1 & (i \leq j < q) \\ i & (j=q). \end{cases}$$

Note that $\operatorname{sgn} \Delta_i^q = (-1)^{q-i}$.

Definition 9.2.1. Let ϕ be an ordered q-simplex. For $i=0, 1, \cdots, q$ the ith *principal face* $\partial_i\phi$ of ϕ is the ordered $(q-1)$-simplex defined by

$$\partial_i\phi = \phi \circ \Delta_i^q \quad \text{restricted to} \quad \{0, 1, \cdots, q-1\}.$$

An ordered simplex ψ is said to be a principal face of an ordered simplex ϕ if $\psi = \partial_i\phi$ for some i. An oriented simplex τ is said to be a principal face of an oriented simplex σ if there exist ordered simplices ψ, ϕ such that $\tau=[\psi]$, $\sigma=[\phi]$ and ψ is a principal face of ϕ.

Notation 9.2.2. We write $\tau \ll \sigma$, if τ is a principal face of σ.

(9.3) The ordered chain complex.

Definition 9.3.1. Let $(K, \mathscr{L})$ be an object of **L**. An *ordered chain* c of $(K, \mathscr{L})$ is a formal sum:

$$c = \sum_{\phi \in K_{ord}} u_\phi \cdot \phi,$$

such that

(1) for each $\phi \in K_{ord}$, $u_\phi \in \mathscr{L}_{\langle\phi\rangle}$,

(2) $\operatorname{supp}(c)$ is a finite set, where $\operatorname{supp}(c) = \{\phi \in K_{ord}\,;\ u_\phi \neq 0\}$.

Let $C_\bullet(K, \mathscr{L})_{ord}$ be the R-module of all ordered chains of $(K, \mathscr{L})$.

We call supp(c) the *support* of c. If supp(c) contains ordered q-simplices only, then c is called an ordered q-chain. Let $C_q(K, \mathscr{L})_{ord}$ be the R-module of all ordered q-chains of $(K, \mathscr{L})$.

Definition 9.3.2. Let c be an ordered q-chain (see Definition 9.3.1). The boundary ∂c of c is the ordered $(q-1)$-chain defined by

$$\partial c = \sum_{\phi \in K_{ord}} \sum_{i=0}^{q} (-1)^i (u_\phi|_{\langle \partial_i \phi \rangle}) \cdot \partial_i \phi,$$

where $u_\phi|_{\langle \partial_i \phi \rangle}$ is the restriction of u_ϕ to $\langle \partial_i \phi \rangle$.

We have an R-homomorphim $\partial : C_\bullet(K, \mathscr{L})_{ord} \to C_\bullet(K, \mathscr{L})_{ord}$ of degree -1. It is not hard to show that $\partial^2 = 0$. Thus we have obtained a chain complex $C_\bullet(K, \mathscr{L})_{ord}$, called the *ordered chain complex* of $(K, \mathscr{L})$.

(9.4) The ordered chain complex functor $C_\bullet(\cdot)_{ord} : \mathbf{L}^{li} \to \mathbf{C}$.

Let $\mathbf{C}$ be the category of chain complexes. Throughout the paper, a chain complex is a chain complex of R-modules. The correspondence $(K, \mathscr{L}) \mapsto C_\bullet(K, \mathscr{L})_{ord}$ is set up as a covariant functor from $\mathbf{L}^{li}$ into $\mathbf{C}$. To see this we have to describe the correspondence of morphisms. Let $(f, \varphi) : (K, \mathscr{L}) \to (L, \mathscr{M})$ be a morphism of $\mathbf{L}^{li}$. By Definition 5.5.2, for any simplex σ of K there is a natural R-homomorphism $\varphi_\sigma : \mathscr{L}_\sigma \to \mathscr{M}_{f(\sigma)}$. We shall define the chain homomorphism:

$$C_\bullet(f, \varphi)_{ord} : C_\bullet(K, \mathscr{L})_{ord} \to C_\bullet(L, \mathscr{M})_{ord}.$$

Definition 9.4.1. For any oriented chain $c = \sum_{\phi \in K_{ord}} u_\phi \cdot \phi$ of $(K, \mathscr{L})$, we put

$$C_\bullet(f, \varphi)_{ord} c = \sum_{\phi \in K_{ord}} \varphi_{\langle \phi \rangle}(u_\phi) \cdot (f \circ \phi).$$

Note that if $\phi : \{0, 1, \cdots, q\} \to \langle \phi \rangle$ is an ordered q-simplex of K, then $f \circ \phi : \{0, 1, \cdots, q\} \to \langle f \circ \phi \rangle = f(\langle \phi \rangle)$ is an ordered simplex of L. Moreover note that $\varphi_{\langle \phi \rangle}(u_\phi) \in M_{f(\langle \phi \rangle)}$. Hence, $C_\bullet(f, \varphi)_{ord} c$ is well-defined.

Remark 9.4.2. (i) Let $(f, \varphi) : (K, \mathscr{L}) \to (L, \mathscr{M})$ and $(g, \psi) : (L, \mathscr{M}) \to (M, \mathscr{N})$ be morphisms of $\mathbf{L}^{li}$ and put $(h, \chi) = (g, \psi) \circ (f, \varphi)$. Then Lemma 5.5.3 yields

$$C_\bullet(h, \chi)_{ord} = C_\bullet(g, \psi)_{ord} \circ C_\bullet(f, \varphi)_{ord}.$$

This shows that $C_\bullet(\cdot)_{ord}$ is indeed a functor.

(ii) The functor $C_\bullet(\cdot)_{ord}$ is not necessarily well-defined on $\mathbf{L}$.

(9.5) The subcomplex $I_\bullet(K, \mathscr{L})$.

Definition 9.5.1. Let $I_\bullet(K, \mathscr{L})$ be the R-submodule of $C_\bullet(K, \mathscr{L})_{ord}$ generated by all elements of the form: $c=u\cdot\{\phi-(\operatorname{sgn} s)(\phi\circ s)\}$, where ϕ is an ordered q-simplex of K (for some q), $s\in \mathfrak{S}_{q+1}$ and $u\in L_{\langle\phi\rangle}$.

Lemma 9.5.2. *For any generater c in Definition 9.5.1, we have*

$$\partial c=\sum_{i=0}^{q}(-1)^i(u|_{\langle\partial_i\phi\rangle})\cdot\{\partial_i\phi-(\operatorname{sgn} s_i)((\partial_i\phi)\circ s_i)\},$$

where $s_i\in\mathfrak{S}_q$, $(i=0, 1, \cdots, q)$ *is defined by*

$$s_i=(\Delta_i^q)^{-1}\circ s\circ \Delta_{s^{-1}(i)}^q \quad \textit{restricted to} \quad \{0, 1, \cdots, q-1\}.$$

This lemma implies that the boundary operator ∂ preserves the submodule $I_\bullet(K, \mathscr{L})$. Hence $I_\bullet(K, \mathscr{L})$ is a subcomplex of $C_\bullet(K, \mathscr{L})_{ord}$.

(9.6) The (oriented) chain complex functor $C_\bullet: \mathbf{L}^{li}\to \mathbf{C}$.

Definition 9.6.1. The oriented chain complex of $(K, \mathscr{L})$ is defined by

$$C_\bullet(K, \mathscr{L})=\frac{C_\bullet(K, \mathscr{L})_{ord}}{I_\bullet(K, \mathscr{L})}.$$

Unless otherwise stated explicitly, a chain of $(K, \mathscr{L})$ is an *oriented* chain of $(K, \mathscr{L})$. If $(f, \varphi):(K, \mathscr{L})\to(L, \mathscr{M})$ is a morphism of $\mathbf{L}^{li}$, then it is easy to see that $C_\bullet(f, \varphi)_{ord}: C_\bullet(K, \mathscr{L})_{ord}\to C_\bullet(L, \mathscr{M})_{ord}$ sends $I_\bullet(K, \mathscr{L})$ into $I_\bullet(L, \mathscr{M})$. This observation gives the following:

Lemma 9.6.2. *The covariant functor* $C_\bullet(\cdot)_{ord}: \mathbf{L}^{li}\to\mathbf{C}$ *induces a covariant functor* $C_\bullet: \mathbf{L}^{li}\to\mathbf{C}$ *in a natural manner.*

The functor $C_\bullet: \mathbf{L}^{li}\to\mathbf{C}$ is called the *(oriented) chain complex functor*. The adjective "oriented" will be often omitted.

Definition 9.6.3. Let K be a simplicial complex. An *ordering* of K is a right-inverse $K\to K_{ord}$, $\sigma\mapsto\phi_\sigma$ of the forgetful map $K_{ord}\to K$. Similarly an *orientation* of K is a right-inverse $K\to K_{ori}$, $\sigma\mapsto\hat{\sigma}$ of the forgetful map $K_{ori}\to K$.

An ordering $\sigma \mapsto \phi_\sigma$ induces an orientation $K \to K_{ori}$, $\sigma \mapsto \hat{\sigma}=[\phi_\sigma]$, called the *associated* orientation.

Lemma 9.6.4. *Given an orientation $K \to K_{ori}$, $\sigma \mapsto \hat{\sigma}$, then any oriented chain $c \in C_\bullet(K, \mathcal{L})$ is uniquely expressed in the form*

$$c=\sum_{\sigma \in K} u_\sigma \hat{\sigma} \qquad \textit{with} \quad u_\sigma \in \mathcal{L}_\sigma.$$

Definition 9.6.5. An ordered simplicial complex $(K, <)$ admits a unique ordering $K \to K_{ord}$, $\sigma \mapsto \phi_\sigma$ such that for any simplex $\sigma \in K$,

$$\phi_\sigma(0) < \phi_\sigma(1) < \phi_\sigma(2) < \cdots.$$

The associated orientation $\sigma \mapsto \hat{\sigma}=[\phi_\sigma]$ is called the *natural* orientation of $(K, <)$.

Remark 9.6.6. Hereafter, choosing an orientation $\sigma \mapsto \hat{\sigma}$, we express any oriented chain of $(K, \mathcal{L})$ as in Lemma 9.6.4, though there is no canonical choice of it. If K is an ordered simplicial complex, then the natural orientation associated with the ordering is chosen without comment. Moreover, for simplicity of notation, we often express an oriented chain c as

$$c=\sum_{\sigma \in K} u_\sigma \cdot \sigma$$

instead of $c=\sum_{\sigma \in K} u_\sigma \cdot \hat{\sigma}$. This convention should cause no confusion.

(9.7) The incidence number.

Let σ and τ be oriented simplices. We define the *incidence number* $[\sigma : \tau]$.

Definition 9.7.1. If $\tau \ll \sigma$ then there exist ordered simplices ϕ and ψ such that $\sigma=[\phi]$, $\tau=[\psi]$ and ψ is the ith principal face of ϕ for some i (see Definition 9.2.1). We can show that $(-1)^i$ is independent of the choice of ϕ and ψ. We put $[\sigma : \tau]=(-1)^i$. If τ is not a principal face of σ, then we put $[\sigma : \tau]=0$.

Remark 9.7.2. Fix an orientation $\sigma \mapsto \hat{\sigma}$. The boundary of an oriented simplex $\hat{\sigma}$ is given by

$$\partial \hat{\sigma}=\sum_{\tau \in K} [\hat{\sigma} : \hat{\tau}](u_\sigma|_\tau) \cdot \hat{\tau}.$$

(9.8) A chain homotompy equivalence.

Lemma 9.8.1. *The canonical projection $C_\bullet(K, \mathcal{L})_{ord} \to C_\bullet(K, \mathcal{L})$ is a chain homotopy equiva-*

lence.

This lemma is proved in a similar manner as in the classical case where the local system $\mathscr{L}$ is trivial.

(9.9) The subdivision of a chain complex.

Let $(K, \mathscr{L})$ be an object of $\mathbf{L}$, σ a q-simplex of K, $\hat{\sigma}$ an oriented simplex over σ. We denote by $S(\sigma)$ the set of all simplices $\tau=\{\sigma_0, \sigma_1, \cdots, \sigma_q\}$ of $\mathrm{Sd}K$ such that

$$\sigma_0 \ll \sigma_1 \ll \cdots \sigma_{q-1} \ll \sigma_q = \sigma.$$

For $i=0, 1, \cdots, q-1$, take an oriented simplex $\hat{\sigma}_i$ over σ_i. By convention we put $\hat{\sigma}_q=\hat{\sigma}$. Noting that $\prod_{i=1}^{q}[\hat{\sigma}_i : \hat{\sigma}_{i-1}]$ depends only on τ and $\hat{\sigma}$, we introduce the following:

Notation 9.9.1.

$$\epsilon(\hat{\sigma}; \tau)=(-1)^{\frac{q(q+1)}{2}} \prod_{i=1}^{q}[\hat{\sigma}_i : \hat{\sigma}_{i-1}].$$

For any $u \in \mathscr{L}_\sigma$ we define a local section $\mathrm{Sd}_\tau^\sigma u \in (\mathrm{Sd}\mathscr{L})_\tau$ by

$$(\mathrm{Sd}_\tau^\sigma u)(\sigma_i)=r_{\sigma_i}^{\sigma}(u), \qquad (i=0, 1, \cdots, q).$$

Definition 9.9.2. A a natural chain map, called the *subdivision*

$$\mathrm{Sd} : C_\bullet(K, \mathscr{L}) \to C_\bullet(\mathrm{Sd}K, \mathrm{Sd}\mathscr{L}), \quad u \cdot \hat{\sigma} \mapsto \mathrm{Sd}(u \cdot \hat{\sigma}),$$

is defined by

$$\mathrm{Sd}(u \cdot \hat{\sigma})=\sum_{\tau \in S(\sigma)} \epsilon(\hat{\sigma}; \tau)(\mathrm{Sd}_\tau^\sigma u) \cdot \hat{\tau},$$

where $\hat{\tau}$ is the natural orientation of τ (see Definition 9.6.5).

Composing the subdivision functor $\mathrm{Sd} : \mathbf{L}^{li} \to \mathbf{L}^{li}_{ord}$ (see Definitions 6.2.1 and 6.2.2) with the chain complex functor $C_\bullet : \mathbf{L}^{li}_{ord} \to \mathbf{C}$, we obtain a covariant functor $C_\bullet \circ \mathrm{Sd} : \mathbf{L}^{li} \to \mathbf{C}$.

Lemma 9.9.3. *The subdivision* Sd *in Definition 9.9.2 is a natural transformation from the chain complex functor* $C_\bullet : \mathbf{L}^{li} \to \mathbf{C}$ *to the functor* $C_\bullet \circ \mathrm{Sd} : \mathbf{L}^{li} \to \mathbf{C}$.

Lemma 9.9.4. *The subdivision* $\mathrm{Sd}: C_\bullet(K, \mathscr{L}) \to C_\bullet(\mathrm{Sd}K, \mathrm{Sd}\mathscr{L})$ *is a chain homotopy equivalence.*

As in the classical case where the local system $\mathscr{L}$ is trivial, we can apply the method of acyclic models to prove this lemma. So the proof is omitted.

10. Homology functor

(10.1) The (oriented) homology functor $H_\bullet: \mathbf{L}^{li} \to \mathbf{M}$.

Let $\mathbf{M}$ be the category of graded R-modules.

Definition 10.1.1. Composing the homology functor with the oriented chain complex functor, we have a covariant functor $H_\bullet: \mathbf{L}^{li} \to \mathbf{M}$, called the *(oriented) homology functor.*

The adjective "oriented" will be often omitted in what follows.

Remark 10.1.2. The *ordered* homology functor $H_\bullet(\cdot)_{ord}: \mathbf{L}^{li} \to \mathbf{M}$ is defined in a similar manner by composing the homology functor with the ordered chain complex functor. However, Lemma 9.8.1 implies that the canonical projection $C_\bullet(K, \mathscr{L})_{ord} \to C_\bullet(K, \mathscr{L})$ induces an isomorphism $H_\bullet(K, \mathscr{L})_{ord} \to H_\bullet(K, \mathscr{L})$. So it is not necessary to consider the ordered homology functor.

Remark 10.1.3. Another kind of homology functor — the singular homology functor — will be considered in (18.3). In order to distinguish these two homology functors, the present one is called the *simplicial homology functor* and is denoted by $H_\bullet^{simp}: \mathbf{L}^{li} \to \mathbf{M}$.

(10.2) The subdivision isomorphism.

Lemmas 9.9.3 and 9.9.4 immediately imply the following:

Theorem 10.2.1. *For any object* $(K, \mathscr{L})$ *of* $\mathbf{L}^{li}$, *there exists an isomorphism of R-modules:* $\mathrm{Sd}: H_\bullet(K, \mathscr{L}) \to H_\bullet(\mathrm{Sd}K, \mathrm{Sd}\mathscr{L})$. *For any morphism* $(f, \varphi): (K, \mathscr{L}) \to (L, \mathscr{M})$ *of* $\mathbf{L}^{li}$, *there exists a commutative diagram of R-modules:*

$$\begin{array}{ccc} H_\bullet(K, \mathscr{L}) & \xrightarrow{\mathrm{Sd}} & H_\bullet(\mathrm{Sd}K, \mathrm{Sd}\mathscr{L}) \\ {\scriptstyle H_\bullet(f,\varphi)}\downarrow & & \downarrow{\scriptstyle H_\bullet(\mathrm{Sd}f, \mathrm{Sd}\varphi)} \\ H_\bullet(L, \mathscr{M}) & \xrightarrow[\mathrm{Sd}]{} & H_\bullet(\mathrm{Sd}L, \mathrm{Sd}\mathscr{M}). \end{array}$$

11. Group actions on a chain complex

(11.1) The induced action on a chain complex.

Let $(K, \mathscr{L})$ be an object of $\mathbf{L}^{li}$ and assume that a group G acts on $(K, \mathscr{L})$. Then each element g of G induces an automorphism of $(K, \mathscr{L})$. Applying the chain complex functor $C_\bullet$: $\mathbf{L}^{li} \to \mathbf{C}$, we obtain an action of G on the chain complex $C_\bullet(K, \mathscr{L})$. This is called the *induced action* on $C_\bullet(K, \mathscr{L})$. Explicitly the action of $g\in G$ on $C_\bullet(K, \mathscr{L})$ is given by $g\sum_{\sigma\in K} u_\sigma\cdot\sigma= \sum_{\sigma\in K}(gu_\sigma)\cdot(g\sigma)$, where gu_σ is defined in Definition 8.5.1.

Notation 11.1.1. $C_\bullet(K, \mathscr{L})^G=\{c\in C_\bullet(K, \mathscr{L})\,;\, gc=c\ (\forall g\in G)\}$.

(11.2) The canonical projection.

Assume that a group G acts on a simplicial complex K. Let $\mathscr{L}$ be a local system on K/G. Then there is the canonical projection $\pi:(K, \pi^*\mathscr{L}) \to (K/G, \mathscr{L})$, (see Definition 8.7.1).

Definition 11.2.1. If G acts on K *quasi-regularly*, then Lemma 8.2.2 implies that $\pi:(K, \pi^*\mathscr{L}) \to (K/G, \mathscr{L})$ is a locally injective morphism, i.e. a morphism of $\mathbf{L}^{li}$. Hence, applying the chain complex functor $C_\bullet: \mathbf{L}^{li} \to \mathbf{C}$, we have a chain homomorphism $\pi: C_\bullet(K, \pi^*\mathscr{L}) \to C_\bullet(K/G, \mathscr{L})$. This is also called the *canonical projection.*

Remark 11.2.2. The canonical projection is not necessarily well-defined, unless the action of G on K is quasi-regular.

(11.3) The pull-back of a local section.

In the situation of (11.2), let σ be a simplex of K/G, and let $u\in\mathscr{L}_\sigma$. Consider the pull-back π^*u of u by π, which is defined by

$$\pi^*u=u\circ\pi:\pi^{-1}(\sigma)\ni a\mapsto u(\pi(a))\in\mathscr{L}_{\pi(a)}=(\pi^*\mathscr{L})_a.$$

Let $S(\sigma)$ be the set of all simplices $\tilde{\sigma}\in K$ such that $\pi(\tilde{\sigma})=\sigma$. Given $\tilde{\sigma}\in S(\sigma)$, let $\pi^*u\,|_{\tilde{\sigma}}$ be the restriction of π^*u to $\tilde{\sigma}$.

Lemma 11.3.1. *For any* $\tilde{\sigma}\in S(\sigma)$, $\pi^*u\,|_{\tilde{\sigma}}\in(\pi^*\mathscr{L})_{\tilde{\sigma}}$.

Proof. For any a, $b\in\tilde{\sigma}$, we have

$$(\pi^*\xi)_{ba}(\pi^*u\,|_{\tilde{\sigma}})(a)=\xi_{\pi(b)\pi(a)}u(\pi(a))=u(\pi(b))=(\pi^*u\,|_{\tilde{\sigma}})(b).$$

This shows that $\pi^*u\,|_{\tilde{\sigma}}\in(\pi^*\mathscr{L})_{\tilde{\sigma}}$.

Lemma 11.3.2. *For any* $\tilde{\sigma} \in S(\sigma)$ *and* $h \in G$, $h \cdot (\pi^* u|_{\tilde{\sigma}}) = \pi^* u|_{h\tilde{\sigma}}$.

Proof. For any $a \in h\tilde{\sigma}$, we have

$$(h \cdot (\pi^* u|_{\tilde{\sigma}}))(a) = (\pi^* u|_{\tilde{\sigma}})(h^{-1}a) = u(\pi(h^{-1}a)) = u(\pi(a)) = (\pi^* u|_{h\tilde{\sigma}})(a).$$

This establishes the lemma.

(11.4) The transfer.

Assume that a group G acts on a simplicial complex K. Let $\mathscr{L}$ be a local system of R-modules on X/G, R being a commutative ring with unity. We make the following:

Assumption 11.4.1.

(1) G is a *finite group*,

(2) G acts on K *regularly*, and

(3) the order $\#G$ of G is a *unit element* of R.

Composing the canonical projection $\pi: C_\bullet(K, \pi^*\mathscr{L}) \to C_\bullet(K/G, \mathscr{L})$ with the inclusion map $C_\bullet(K, \pi^*\mathscr{L})^G \hookrightarrow C_\bullet(K, \pi^*\mathscr{L})$, we obtain a natural chain map:

$$\pi: C_\bullet(K, \pi^*\mathscr{L})^G \to C_\bullet(K/G, \mathscr{L}).$$

This is an R-isomorphism. To see this we construct its inverse chain map, called the transfer.

Definition 11.4.2. Under Assumption 11.3.1, the *transter*

$$\mathrm{tf}: C_\bullet(K/G, \mathscr{L}) \to C_\bullet(K, \pi^*\mathscr{L})^G$$

is defined by

$$\mathrm{tf}(u \cdot \sigma) = \frac{1}{\#G} \sum_{g \in G} (\pi^* u|_{g\tilde{\sigma}}) \cdot (g\tilde{\sigma}),$$

where σ is a simplex of K/G, $u \in \mathscr{L}_\sigma$ and $\tilde{\sigma}$ is an element of $O(\sigma)$.

By assumption, G acts on K regularly, i.e., G acts on $O(\sigma)$ transtively (see Definition 8.2.3). So this definition is well-defined, i.e., independent of the choice of $\tilde{\sigma} \in O(\sigma)$. Moreover we have to show that tf is a map into $C_\bullet(K, \pi^*\mathscr{L})^G$. Indeed we have the following:

Lemma 11.4.3. $\mathrm{tf}(u \cdot \sigma) \in C_\bullet(K, \pi^*\mathscr{L})^G$.

Proof. For any $h \in G$, Lemma 11.3.2 implies

$$h\cdot\mathrm{tf}(u\cdot\sigma)=\frac{1}{\# G}\sum_{g\in G}h(\pi^* u\,|_{g\tilde{\sigma}})\cdot(hg\tilde{\sigma})=\frac{1}{\# G}\sum_{g\in G}(\pi^* u\,|_{hg\tilde{\sigma}})\cdot(hg\tilde{\sigma})=\mathrm{tf}(u\cdot\sigma).$$

This establishes the lemma.

Theorem 11.4.4. *Under Assumption* 11.4.1, *the chain map*

$$\pi: C_\bullet(K,\ \pi^*\mathscr{L})^G \to C_\bullet(K/G,\ \mathscr{L})$$

is an R-isomorphism and its inverse is given by the transfer

$$\mathrm{tf}: C_\bullet(K/G,\ \mathscr{L}) \to C_\bullet(K,\ \pi^*\mathscr{L})^G.$$

Proof. It is easy to show $\pi\circ\mathrm{tf}=\mathrm{id}$. We shall show $\mathrm{tf}\circ\pi=\mathrm{id}$. Let $c=\sum_{\sigma\in K}u_\sigma\cdot\sigma$ be any element of $C_\bullet(K,\ \pi^*\mathscr{L})^G$, where $u_\sigma\in(\pi^*\mathscr{L})_\sigma$. Note that $\pi^*\pi_\sigma(u_\sigma)|_\sigma=u_\sigma$, where $\pi_\sigma:(\pi^*\mathscr{L})_\sigma \to \mathscr{L}_{\pi(\sigma)}$ is defined in Lemma 5.5.2 (see also Remark 8.7.3). We have $\pi(c)=\sum_{\sigma\in K}\pi_\sigma(u_\sigma)\cdot\pi(\sigma)$ and hence

$$\begin{aligned}(\mathrm{tf}\circ\pi)(c)&=\sum_{\sigma\in K}\frac{1}{\# G}\sum_{g\in G}(\pi^*\pi_\sigma(u_\sigma)\,|_{g\sigma})\cdot(g\sigma)\\&=\frac{1}{\# G}\sum_{g\in G}\sum_{\sigma\in K}g(\pi^*\pi_\sigma(u_\sigma)\,|_{\sigma})\cdot(g\sigma)\qquad\text{(by Lemma 11.3.2)}\\&=\frac{1}{\# G}\sum_{g\in G}\sum_{\sigma\in K}(gu_\sigma)\cdot(g\sigma)\\&=\frac{1}{\# G}\sum_{g\in G}g\sum_{\sigma\in K}u_\sigma\cdot\sigma\\&=\frac{1}{\# G}\sum_{g\in G}gc\\&=c.\end{aligned}$$

This shows $\mathrm{tf}\circ\pi=\mathrm{id}$ and establishes the lemma.

Corollary 11.4.5. Under Assumption 11.4.1, the transfer induces an isomorphism of R-modules: $\mathrm{tf}: H_\bullet(K/G,\ \mathscr{L}) \to H_\bullet(C_\bullet(K,\ \pi^*\mathscr{L})^G)$.

12. Invariant functor

(12.1) The invariant functor.

Definition 12.1.1. We define the category $\mathbf{C}^G$: An object of $\mathbf{C}^G$ is an object of $\mathbf{C}$ together with an action of G on it. A morphism of $\mathbf{C}^G$ is a morphism of $\mathbf{C}$ which commutes with the actions of G. The category $\mathbf{M}^G$ is defined exactly in the same manner.

Definition 12.1.2. We define the *G-invariant functor* $I^G: \mathbf{C}^G \to \mathbf{C}$: For an object $C_\bullet$ of $\mathbf{C}^G$, $I^G(C_\bullet)=C_\bullet^G$ is given by $C_\bullet^G=\{c\in C_\bullet; gc=c \quad (\forall g\in G)\}$. For a morphism $f: C_\bullet \to D_\bullet$ of $\mathbf{C}^G$, $I^G(f)=f^G$ is the restriction of f into $C_\bullet^G$, i.e., $f^G=f|_{C_\bullet^G}$. The G-invariant functor $I^G: \mathbf{M}^G \to \mathbf{M}$ is defined exactly in the same manner.

(12.2) The first cohomology of a group.

The first cohomology of the group G will be used later. So we shall recall it briefly. For details of the group cohomology, see [15]. Let $R[G]$ be the group ring of G over R, A an $R[G]$-module.

Definition 12.2.1.

$$C^1(G, A)=\{f: G \to A\,; \text{maps}\},$$
$$Z^1(G, A)=\{f\in C^1(G, A)\,; \delta f=0\},$$
$$B^1(G, A)=\{f\in C^1(G, A)\,; \exists\, a\in A \text{ such that } f=\delta a\},$$

where $\delta f: G\times G \to A$ is defined by $(\delta f)(g_1, g_2)=g_1f(g_2)-f(g_1g_2)+f(g_1)$, and δa is an element of $C^1(G, A)$ defined by $(\delta f)(g)=(g-1)a$.

$$H^1(G, A)=\frac{Z^1(G, A)}{B^1(G, A)}.$$

The following cohomology vanishing is a key to the main result (Theorem 12.3.2) in this section.

Lemma 12.2.2. *If the order $\#G$ of G is a unit of R, then $H^1(G, A)=0$.*

(12.3) Commutativity of the homology functor and the invariant functor.

Let $C_\bullet$ be an object of $\mathbf{C}^G$. The inclusion map $i: C_\bullet^G \hookrightarrow C_\bullet$ induces an R-homomorphism $i_*=H_\bullet(i): H_\bullet(C_\bullet^G) \to H_\bullet(C_\bullet)$. The image of i_* is clearly contained in the G-invariant component $H_\bullet(C_\bullet)^G$ of $H_\bullet(C_\bullet)$. Hence we have an R-homomorphism: $i_*: H_\bullet(C_\bullet^G) \to H_\bullet(C_\bullet)^G$. Note that i_* is thought of as a natural transformation $i_*: H_\bullet \circ I^G \to I^G \circ H_\bullet$, where $H_\bullet$ is the homology functor.

Lemma 12.3.1. *If $\#G$ is a unit of R, then $i_*=(i_*)_{C_\bullet}: H_\bullet(C_\bullet^G) \to H_\bullet(C_\bullet)^G$ is injective.*

Proof. We define an R-homomorphism $\mu : C_\bullet \to C_\bullet^G$ by

$$\mu(c)=\frac{1}{\# G}\sum_{g\in G} g\cdot c, \qquad (c\in C_\bullet).$$

Clearly we have $\mu\circ i=\mathrm{id}$. Applying the homology functor, we obtain $\mu_*\circ i_*=\mathrm{id}: H_\bullet(C_\bullet^G)\to H_\bullet(C_\bullet)^G$. This implies that i_* is injective and μ_* is the left-inverse of i_*.

Theorem 12.3.2. *If $\# G$ is a unit of R, then $i_*=(i_*)_{C_\bullet}: H_\bullet(C_\bullet^G)\to H_\bullet(C_\bullet)^G$ is an R-isomorphism, i.e., $i_*: H_\bullet\circ I^G\to I^G\circ H_\bullet$ is a natural equivalence of functors.*

Proof. By Lemma 12.3.1, it is sufficient to prove that i_* is surjective. Let Z_q and B_q be the q-cycles of $C_\bullet$ and the q-boundaries of $C_\bullet$, respectively. They are $R[G]$-submodules of C_q. For $z\in Z_q$, the homology class in $H_q(C_\bullet)$ represented by z is denoted by $[z]$. For $z\in Z_q^G$, the homology class in $H_q(C_\bullet^G)$ represented by z is denoted by $[[z]]$. For any $h\in H_q(C_\bullet)^G\subset H_q(C_\bullet)$, take a representative $z\in Z_q$ such that $h=[z]$. Since h is G-invariant, we have $[(g-1)z]=(g-1)[z]=(g-1)h=0$, i.e., $(g-1)z\in B_q$ for any $g\in G$.

The boundary operator $\partial : C_{q+1}\to B_q$ induces an R-isomorphism $\partial : A\to B_q$, where $A=C_{q+1}/Z_{q+1}$. Note that A is an $R[G]$-module. Since $(g-1)z\in B_q$ for any $g\in G$ and $\partial : A\to B_q$ is isomorphism, there exists a unique $f\in C^1(G, A)$ such that $(g-1)z=\partial f(g)$ for any $g\in G$. We shall show $f\in Z^1(G, A)$. For any $g_1, g_2\in G$, we have $(\delta f)(g_1, g_2)=g_1 f(g_2)-f(g_1 g_2)+f(g_1)\in A$. Applying the boundary operator ∂ to this equation, we obtain

$$\begin{aligned}\partial(\delta f)(g_1, g_2)&=g_1\partial f(g_2)-\partial f(g_1 g_2)+\partial f(g_1)\\&=g_1(g_2-1)z-(g_1 g_2-1)z+(g_1-1)z\\&=\{(g_1 g_2-g_1)-(g_1 g_2-1)+(g_1-1)\}z=0.\end{aligned}$$

Since $\partial : A\to B_q$ is injective, we obtain $(\delta f)(g_1, g_2)=0$, i.e., $f\in Z^1(G, A)$. Since $\# G$ is assumed to be a unit of R, Lemma 12.2.2 implies $H^1(G, A)=0$, i.e., $Z^1(G, A)=B^1(G, A)$. Hence there exists an $a\in A$ such that $f=\delta a$, i.e., $f(g)=(\delta a)(g)=(g-1)a$ for any $g\in G$. If we put $w=z-\partial a$, then $z\in Z_q^G$. Indeed, we have $(g-1)w=(g-1)(z-\partial a)=\partial f(g)-\partial(g-1)a=0$. If we put $h'=[[w]]\in H_q(C_\bullet^G)$, then $i_*(h')=[w]=h$. This implies that $i_*: H_q(C_\bullet^G)\to H_q(C_\bullet)^G$ is surjective.

From Corollary 11.4.5 and Theorem 12.3.2, we immediately obtain the following:

Corollary 12.3.3. *Under Assumption 11.4.1, there exists an isomorphism of R-modules:* tf : $H_\bullet(K/G, \mathscr{L})\to H_\bullet(K, \pi^*\mathscr{L})^G$.

This isomorphism is also called the *transfer.*

13. The chain complex of external products

Throughout this section, we use Notations 7.1.1 and 8.8.1.

(13.1) The cross product.

Let $(K_i, \mathscr{L}_i)$, $(i=1, 2, \cdots, n)$ be objects of $\mathbf{L}_{ord}$. We put

$$K=K_1\times K_2\times\cdots\times K_n,$$
$$\mathscr{L}=\mathscr{L}_1\boxtimes\mathscr{L}_2\boxtimes\cdots\boxtimes\mathscr{L}_n, \qquad \text{(see Notation 8.8.1).}$$

Let $q=(q_1, \cdots, q_n)$ be a multi-index, and let $\sigma=(\sigma_1, \sigma_2, \cdots, \sigma_n)\in\Sigma(q)$, and $j=(j_1, j_2, \cdots, j_n)\in J(q)$. Recall that $\langle\sigma; j\rangle$ is a simplex of K (see Notation 7.1.1 and Definition 7.1.3).

Definition 13.1.1. For $u_i\in\mathscr{L}_{\sigma_i}$, $(i=1, 2, \cdots, n)$, the local section $u_{\langle\sigma;\phi\rangle}\in\mathscr{L}_{\langle\sigma;j\rangle}$ is defined by

$$\begin{aligned} u_{\langle\sigma;j\rangle}(a)&=u_1(a_1)\otimes u_2(a_2)\otimes\cdots\otimes u_n(a_n)\\ &\in(\mathscr{L}_1)_{a_1}\otimes(\mathscr{L}_2)_{a_2}\otimes\cdots\otimes(\mathscr{L}_n)_{a_n}=\mathscr{L}_a. \end{aligned}$$

where $a=a_1\times\cdots\times a_n\in\langle\sigma; j\rangle$.

Definition 13.1.2. The *cross product*

$$\begin{array}{ccc} C_\bullet(K_1, \mathscr{L}_1)\otimes\cdots\otimes C_\bullet(K_n, \mathscr{L}_n) & \longrightarrow & C_\bullet(K, \mathscr{L}) \\ u_1\cdot\sigma_1\otimes\cdots\otimes u_n\cdot\sigma_n & \mapsto & u_1\cdot\sigma_1\times\cdots\times u_n\cdot\sigma_n \end{array}$$

is a chain map defined by $u_1\cdot\sigma_1\times\cdots\times u_n\cdot\sigma_n=\sum_{j\in J(q)}(\operatorname{sgn} j)u_{\langle\sigma;j\rangle}\cdot\langle\sigma; j\rangle$.

Lemma 13.1.3. *The cross product* $C_\bullet(K_1, \mathscr{L}_1)\otimes\cdots\otimes C_\bullet(K_n, \mathscr{L}_n)\to C_\bullet(K, \mathscr{L})$ *is a chain homotopy equivalence.*

We apply the method of acyclic models to prove this lemma, just as in the classical case where the local systems are trivial.

(13.2) $\mathfrak{S}_n$-equivariance of the cross product.

Definition 13.2.1. (i) For a weight $q=(q_1, \cdots, q_n)$, put

$$\Delta(x_1, \cdots, x_n; q_1 \cdots, q_n) = \prod_{i<j} (x_i - x_j)^{q_i q_j}.$$

(ii) For $\tau \in \mathfrak{S}_n$, the *weighted signature* of τ with weight q is the number $\mathrm{sgn}_q \tau \in \{\pm 1\}$ defined by

$$\Delta(x_{\tau(1)}, x_{\tau(2)}, \cdots, x_{\tau(n)}; q_{\tau(1)}, q_{\tau(2)}, \cdots, q_{\tau(n)})$$
$$= (\mathrm{sgn}_q \tau) \Delta(x_1, x_2, \cdots, x_n; q_1, q_2, \cdots, q_n).$$

Definition 13.2.2. For any $\tau \in \mathfrak{S}_n$, we define a chain isomorphism:

$$\tau : C_\bullet(K_1, \mathscr{L}_1) \otimes \cdots \otimes C_\bullet(K_n, \mathscr{L}_n)$$
$$\to C_\bullet(K_{\tau(1)}, \mathscr{L}_{\tau(n)}) \otimes \cdots \otimes C_\bullet(K_{\tau(n)}, \mathscr{L}_{\tau(n)})$$

by $\tau(u_1 \cdot \sigma_1 \otimes \cdots \otimes u_n \cdot \sigma_n) = (\mathrm{sgn}_q \tau)(u_{\tau(1)} \cdot \sigma_{\tau(1)} \otimes \cdots \otimes u_{\tau(n)} \cdot \sigma_{\tau(n)})$, where $\sigma = (\sigma_1, \cdots, \sigma_n) \in S(q)$ and $u_i \in \mathscr{L}_{\sigma i}$, $(i=1, \cdots, n)$.

Given $\tau \in \mathfrak{S}_n$, we put

$$\tau K = K_{\tau(1)} \times K_{\tau(2)} \times \cdots \times K_{\tau(n)},$$
$$\tau \mathscr{L} = \mathscr{L}_{\tau(1)} \boxtimes \mathscr{L}_{\tau(2)} \boxtimes \cdots \boxtimes \mathscr{L}_{\tau(n)}, \quad \text{(see Notation 8.8.1).}$$

Recall that there is an isomorphism $\tau : (K, \mathscr{L}) \to (\tau K, \tau \mathscr{L})$ of $\mathbf{L}_{ord}$ (see Definition 8.8.2). By Definitions 13.1.2 and 13.2.2, we can easily show the following:

Lemma 13.2.3. *For any $\tau \in \mathfrak{S}_n$, there is a commutative diagram of chain complexes:*

$$\begin{array}{ccc}
C_\bullet(K_1, \mathscr{L}_1) \otimes \cdots \otimes C_\bullet(K_n, \mathscr{L}_n) & \xrightarrow{\text{cross product}} & C_\bullet(K, \mathscr{L}) \\
\tau \downarrow & & \downarrow \tau \\
C_\bullet(K_{\tau(1)}, \mathscr{L}_{\tau(1)}) \otimes \cdots \otimes C_\bullet(K_{\tau(n)}, \mathscr{L}_{\tau(n)}) & \xrightarrow[\text{cross product}]{} & C_\bullet(\tau K, \tau \mathscr{L}).
\end{array}$$

This shows that the cross product is equivariant with respect to the "action" of $\mathfrak{S}_n$ on the exterior product of $(K_1, \mathscr{L}_1), \cdots, (K_n, \mathscr{L}_n)$.

(13.3) An $\mathfrak{S}_n$-equivariant isomorphism for homology groups.

Applying the homology functor to the diagram in Lemma 13.2.3 and using Lemma 13.1.3, we obtain the following:

Theorem 13.3.1. *There is a commutative diagram of R-isomorphisms:*

$$\begin{array}{ccc} H_\bullet(C_\bullet(K_1, \mathscr{L}_1)\otimes\cdots\otimes C_\bullet(K_n, \mathscr{L}_n)) & \xrightarrow{\text{cross product}} & H_\bullet(K, \mathscr{L}) \\ \tau\downarrow & & \downarrow\tau \\ H_\bullet(C_\bullet(K_{\tau(1)}, \mathscr{L}_{\tau(1)})\otimes\cdots\otimes C_\bullet(K_{\tau(n)}, \mathscr{L}_{\tau(n)})) & \xrightarrow[\text{cross product}]{} & H_\bullet(\tau K, \tau\mathscr{L}) \end{array}$$

The special case where $(K, \mathscr{L}):=(K_1, \mathscr{L}_1)=\cdots=(K, \mathscr{L}_n)$, (cf. Lemma 8.8.3) is of particular importance in what follows. The group $\mathfrak{S}_n$ acts on the chain complexes $\overset{n}{\otimes} C_\bullet(K, \mathscr{L})$, $C_\bullet(K^n, \overset{n}{\boxtimes}\mathscr{L})$, and therefore on the homology groups $H_\bullet(\overset{n}{\otimes} C_\bullet(K, \mathscr{L}))$, $H_\bullet(K^n, \overset{n}{\boxtimes}\mathscr{L})$. Theorem 13.3. 1 immediately implies the following:

Corollary 13.3.2. Let $(K, \mathscr{L})$ be an object of L. Then for any positive integer n, the cross product $H_\bullet(\overset{n}{\otimes} C_\bullet(K, \mathscr{L})) \to H_\bullet(K^n, \overset{n}{\boxtimes}\mathscr{L})$ is an $\mathfrak{S}_n$-equivariant R-isomorphism.

14. Application of the Künneth formula

(14.1) The Künneth formula.

In standard texts on algebraic topology, the Künneth formula is formulated over a principal ideal domain (PID). In this paper, however, since we are concerned with local systems over a commutative ring R which is not necessarily a PID, we need a slight modification of the Künneth formula. What we need here is the following:

Theorem 14.1.1. *Let $C_\bullet$ and $D_\bullet$ be chain complexes of R-modules. Assume that the cycles $Z_\bullet(C_\bullet)$ and the boundaries $B_\bullet(C_\bullet)$ of $C_\bullet$ are flat R-modules. Then there exists a natural short exact sequence of R-modules:*

$$\begin{aligned} 0 \to \bigoplus_{p+q=r} H_p(C_\bullet)\otimes H_q(D_\bullet) &\xrightarrow{\zeta} H_r(C_\bullet\otimes D_\bullet) \\ \to \bigoplus_{p+q=r-1} \mathrm{Tor}_1^R(H_p(C_\bullet), H_q(D_\bullet)) &\to 0, \end{aligned}$$

where ζ is induced from the natural map $Z_p(C_\bullet)\otimes Z_q(D_\bullet) \to Z_{p+q}(C_\bullet\otimes D_\bullet)$.

Proof. Essentially this theorem was already established in the proof of Theorem V.2.1 in [15]. Consider the following two conditions:

(A) $Z_\bullet(C_\bullet)$ and $B_\bullet(C_\bullet)$ are flat,

(B) R is a PID and $C_\bullet$ is flat.

Hilton and Stammbach [15] proved Theorem 14.1.1 under assumption (B). By a careful reading of their proof, however, one observes that condition (A) is sufficient to establish the theorem. Indeed, since any submodule of a flat module over a PID is again flat, conditon (B) implies condition (A), and they needed only the latter condition in their proof.

The naturality of the isomorphism ζ plays a crucial role when one considers group actions. As a corollary of Theorem 14.1.1, we obtain

Corollary 14.1.2. *If $H_\bullet(C_\bullet)$ and $Z_\bullet(C_\bullet)$ are flat R-modules, then there exists a natural isomorphism of R-modules:*

$$\bigoplus_{p+q=r} H_p(C_\bullet)\otimes H_q(D_\bullet) \xrightarrow{\zeta} H_r(C_\bullet\otimes D_\bullet),$$

where ζ is induced from the natural map $Z_p(C_\bullet)\otimes Z_q(D_\bullet)\to Z_{p+q}(C_\bullet\otimes D_\bullet)$.

Proof. By Lemma 14.1.3 below, the assumption of the corollary implies that $B_\bullet(C_\bullet)$ is also flat. Since $H_\bullet(C_\bullet)$ is flat, we have $\mathrm{Tor}_1^R(H_p(C_\bullet), H_q(D_\bullet))=0$. So Theorem 14.1.1 implies Corollary 14.1.2 immediately.

Lemma 14.1.3. *Let A, B and C be R-modules and assume that C is flat. If there exists an exact sequence $0\to A\to B\to C\to 0$ of R-modules, then A is flat if and only if B is flat.*

(14.2) An application of the Künneth formula.

As an application of the Künneth formula, we obtain the following:

Theorem 14.2.1. *Let $C_\bullet$ be a chain complex of R-modules such that*

(1) *$Z_\bullet(C_\bullet)$ is a flat R-module,*

(2) *$H_q(C_\bullet)=0$ if $q\neq r$, and*

(3) *$H_r(C_\bullet)$ is a flat R-module.*

Then we have

(i) *$H_q(\overset{n}{\otimes} C_\bullet)=0$ if $q\neq nr$, and*

(ii) *There exists a canonical isomorphism of R-modules:*

$$\overset{n}{\otimes} H_r(C_\bullet) \xrightarrow{\zeta} H_{nr}(\overset{n}{\otimes} C_\bullet),$$

where ζ is induced from the natural map $\overset{n}{\otimes} Z_r(C_\bullet)\to Z_{nr}(\overset{n}{\otimes} C_\bullet)$.

(iii) *This isomorphism is natural, i.e., if $C'_\bullet$ is another chain complex satisfying the conditions* (1)−(3) *and $f: C_\bullet \to C'_\bullet$ is a chain homomorphism, then there exists a commutative diagram of R-modules:*

$$\begin{array}{ccc} \overset{n}{\otimes} H_r(C_\bullet) & \xrightarrow{\zeta} & H_{nr}(\overset{n}{\otimes} C_\bullet) \\ {\scriptstyle \overset{n}{\otimes} H_r(f)}\downarrow & & \downarrow{\scriptstyle H_{nr}(\overset{n}{\otimes} f)} \\ \overset{n}{\otimes} H_r(C'_\bullet) & \xrightarrow[\zeta]{} & H_{nr}(\overset{n}{\otimes} C'_\bullet). \end{array}$$

Remark 14.2.2. Since the natural map $\overset{n}{\otimes} Z_r(C_\bullet) \to Z_{nr}(\overset{n}{\otimes} C_\bullet)$ is $\mathfrak{S}_n$-equivariant, the isomorphism ζ is also $\mathfrak{S}_n$-equivariant.

Proof of Theorem 14.2.1. We show this theorem by induction on n. If $n=1$, there is nothing to show. Assume that the theorem holds for $n-1$ with $n \geq 2$. Putting $D_\bullet = \overset{n-1}{\otimes} C_\bullet$, we apply Corollary 14.1.2 to obtain an R-isomorphism

$$\bigoplus_{p+q=m} H_p(C_\bullet) \otimes H_q(\overset{n-1}{\otimes} C_\bullet) \xrightarrow{\zeta} H_m(\overset{n}{\otimes} C_\bullet).$$

Since, by assumption, $H_p(C_\bullet)=0$ $(p \neq r)$, we have an R-isomorphism

$$(*)_{m,n} \qquad H_r(C_\bullet) \otimes H_{m-r}(\overset{n-1}{\otimes} C_\bullet) \xrightarrow{\eta_{m,n}} H_m(\overset{n}{\otimes} C_\bullet),$$

where $\eta_{m,n}$ is induced from the natural map $Z_r(C_\bullet) \otimes Z_{m-r}(\overset{n-1}{\otimes} C_\bullet) \to Z_m(\overset{n}{\otimes} C_\bullet)$. If $m \neq nr$, then by induction hypothesis, $H_{m-r}(\overset{n-1}{\otimes} C_\bullet)=0$. So $(*)_{m,n}$ implies $H_m(\overset{n}{\otimes} C_\bullet)=0$. Next consider the R-homomorphism $\overset{n}{\otimes} H_r(C_\bullet) \xrightarrow{\zeta_n} H_{nr}(\overset{n}{\otimes} C_\bullet)$, where ζ_n is induced from the natural map $\overset{n}{\otimes} Z_r(C_\bullet) \to Z_{nr}(\overset{n}{\otimes} C_\bullet)$. Then there exists a commutative diagram of R-modules:

$$\begin{array}{ccc} \overset{n}{\otimes} H_r(C_\bullet) & \xrightarrow{1 \otimes \zeta_{n-1}} & H_r(C_\bullet) \otimes H_{(n-1)r}(\overset{n-1}{\otimes} C_\bullet) \\ \| & & \downarrow{\scriptstyle \eta_{nr,n}} \\ \overset{n}{\otimes} H_r(C_\bullet) & \xrightarrow[\zeta_n]{} & H_{nr}(\overset{n}{\otimes} C_\bullet). \end{array}$$

By induction hypothesis, ζ_{n-1} is an R-isomorphism. Since $H_r(C_\bullet)$ is assumed to be flat, $1 \otimes \zeta_{n-1}$ is also an R-isomorphism. Moreover, by $(*)_{nr,n}$, $\eta_{nr,n}$ is an R-isomorphism. Hence the commutative diagram implies that ζ_n is also an R-isomorphism. This completes the induction argument.

15. Twisted homology of the configuration space

(15.1) The configuration space of n-points.

Let K be an ordered simplicial complex, and $\mathscr{L}$ be a local system of R-modules on K. Let n be a natural number fixed throughout. By Theorem 8.3.2,(3), the natural action of $\mathfrak{S}_n$ on K^n induces a *regular* action of $\mathfrak{S}_n$ on $\mathrm{Sd}^2K^n := \mathrm{Sd}^2(K^n)$.

Definition 15.1.1. The quotient simplicial complex $K_n := \mathrm{Sd}^2K^n/\mathfrak{S}_n$ is called the *configuration space of n-points* in K. Let $\pi : \mathrm{Sd}^2K^n \to K_n$ denote the canonical projection.

Remark 15.1.2. By Lemma 8.2.2, π is a morphism of $\mathbf{L}^{li}_{ord}$. The regularity of the $\mathfrak{S}_n$-action will be essential in what follows. So we shall consider $\mathrm{Sd}^2K^n/\mathfrak{S}_n$ instead of $K^n/\mathfrak{S}_n$. The reason for this will be clear in § 21.

(15.2) Twisted homology groups of the configuration space of n-points.

Let $\mathscr{L}$ and $\mathscr{M}$ be local systems (of R-modules) on K and on K_n, respectively. Applying the external product functor (7.2) and the subdivision functor (6.2), we obtain a local system $\mathrm{Sd}^2 \overset{n}{\boxtimes} L$ on Sd^2K^n. We make the following:

Assumption 15.2.1.

(1) $n!$ is a unit element of R, i.e., $n! \in R^{\times}$.

(2) $\pi^*\mathscr{M} = \mathrm{Sd}^2 \overset{n}{\boxtimes} \mathscr{L}$.

Theorem 15.2.2. *Under Assumption* 15.2.1, *there exists a natural isomorphism of R-modules:*

$$H_\bullet(K_n, \mathscr{M}) \to H_\bullet(\overset{n}{\otimes} C_\bullet(K, \mathscr{L}))^{\mathfrak{S}_n}.$$

Proof. If we put $G = \mathfrak{S}_n$ and we replace K by Sd^2K^n, then Assumption 15.2.1,(1) implies Assumption 11.4.1. So we can apply Corollary 12.3.3 to obtain an isomorphism:

$$\begin{aligned} H_\bullet(K_n, \mathscr{M}) &\underset{\mathrm{tf}}{\simeq} H_\bullet(\mathrm{Sd}^2K^n, \pi^*\mathscr{M})^{\mathfrak{S}_n} \\ &= H_\bullet(\mathrm{Sd}^2K^n, \mathrm{Sd}^2 \overset{n}{\boxtimes} \mathscr{L})^{\mathfrak{S}_n}. \quad \text{(Assumption 15.2.1,(2))} \end{aligned}$$

By Theorem 10.2.1, we have an $\mathfrak{S}_n$-equivariant isomorphism

$$\mathrm{Sd}^2 : H_\bullet(K^n, \overset{n}{\boxtimes} \mathscr{L}) \to H_\bullet(\mathrm{Sd}^2K^n, \mathrm{Sd}^2 \overset{n}{\boxtimes} \mathscr{L}),$$

which induces an isomorphism

$$(\mathrm{Sd}^2)^{-1}: H_\bullet(\mathrm{Sd}^2 K^n, \mathrm{Sd}^2 \overset{n}{\boxtimes} \mathscr{L})^{\mathfrak{S}_n} \to H_\bullet(K^n, \overset{n}{\boxtimes} \mathscr{L})^{\mathfrak{S}_n}.$$

Hence we obtain an isomorphism $H_\bullet(K_n, \mathscr{M}) \simeq H_\bullet(K^n, \overset{n}{\boxtimes} \mathscr{L})^{\mathfrak{S}_n}$. Moreover, by Corollary 13. 3.2, we have an isomorphism

$$H_\bullet(\overset{n}{\otimes} C_\bullet(K, \mathscr{L}))^{\mathfrak{S}_n} \xrightarrow{\text{cross product}} H_\bullet(K^n, \overset{n}{\boxtimes} \mathscr{L})^{\mathfrak{S}_n}.$$

Combining these isomorphisms, we obtain the desired isomorphism.

(15.3) The case of pure homology.

If the homology group $H_\bullet(K, \mathscr{L})$ is pure and some additional conditions are satisfied, then we can get more detailed result for the homology group $H_\bullet(K_n, \mathscr{M})$, which we shall state in what follows. In addition to Assumption 15.2.1, we make the following:

Assumption 15.3.1. (cf. Assumption of Theorem 14.2.1)

(1) $Z_\bullet(K, \mathscr{L})$ is a flat R-module,

(2) $H_q(K, \mathscr{L})=0$ if $q \neq r$, and

(3) $H_r(K, \mathscr{L})$ is a flat R-module.

Notation 15.3.2. Given an R-module A, let $\overset{n}{\wedge} A$ and $\overset{n}{\odot} A$ denote the nth *exterior* and *symmetric power* module of A, respectively. They are well-defined, since $n!$ is assumed to be a unit element of R.

Theorem 15.3.3. *Under Assumptions* 15.2.1 *and* 15.3.1, *we have*

$$H_q(K_n, \mathscr{M})=0 \qquad (q \neq rn)$$

and a canonical isomorphism of R-modules:

$$H_{nr}(K_n, \mathscr{M}) \simeq \begin{cases} \overset{n}{\wedge} H_r(K, \mathscr{L}) & (r: \mathit{odd}), \\ \overset{n}{\odot} H_r(K, \mathscr{L}) & (r: \mathit{even}). \end{cases}$$

Proof. If $q \neq nr$, Theorem 14.2.1,(i) implies that $H_q(\overset{n}{\otimes} C_\bullet(K, \mathscr{L}))=0$, and hence $H_q(\overset{n}{\otimes} C_\bullet(K, \mathscr{L}))^{\mathfrak{S}_n}=0$. By using Theorem 15.2.2, we obtain $H_q(K_n, \mathscr{M})=0$. Next consider the case $q=nr$. By Theorem 14.2.1,(ii) and Remark 14.2.2, we have an $\mathfrak{S}_n$-equivariant isomorphism $\overset{n}{\otimes} H_r(K, \mathscr{L})$

$\simeq H_{nr}(\overset{n}{\otimes} C_\bullet(K, \mathscr{L}))$, which induces an isomorphism $\{\overset{n}{\otimes} H_r(K, \mathscr{L})\}^{\mathfrak{S}_n} \simeq H_{nr}(\overset{n}{\otimes} C_\bullet(K, \mathscr{L}))^{\mathfrak{S}_n}$. By Theorem 15.2.2, we obtain an isomorphism $H_{nr}(K_n, \mathscr{M}) \simeq \{\overset{n}{\otimes} H_r(K, \mathscr{L})\}^{\mathfrak{S}_n}$. Noting that

$$\{\overset{n}{\otimes} H_r(K, \mathscr{L})\}^{\mathfrak{S}_n} = \begin{cases} \overset{n}{\wedge} H_r(K, \mathscr{L}) & (r : \text{odd}), \\ \overset{n}{\odot} H_r(K, \mathscr{L}) & (r : \text{even}), \end{cases}$$

we have obtained the desired isomorphism.

16. Local systems on a bouquet

We consider local systems on a bouquet. They are simple but quite important examples to which our twisted simplicial theory applies. These examples will play an important role in application to hypergeometric functions (cf. § 22).

(16.1) Bouquets.

Definition 16.1.1. The m-bouquet B_m is a 1-dimensional ordered simplicial complex whose vertices are $a_1, \cdots, a_m, b_1, \cdots, b_m, c$, and whose ordered 1-simplices are (c, a_i), (a_i, b_i), (b_i, c), $(i=1, 2, \cdots, m)$.

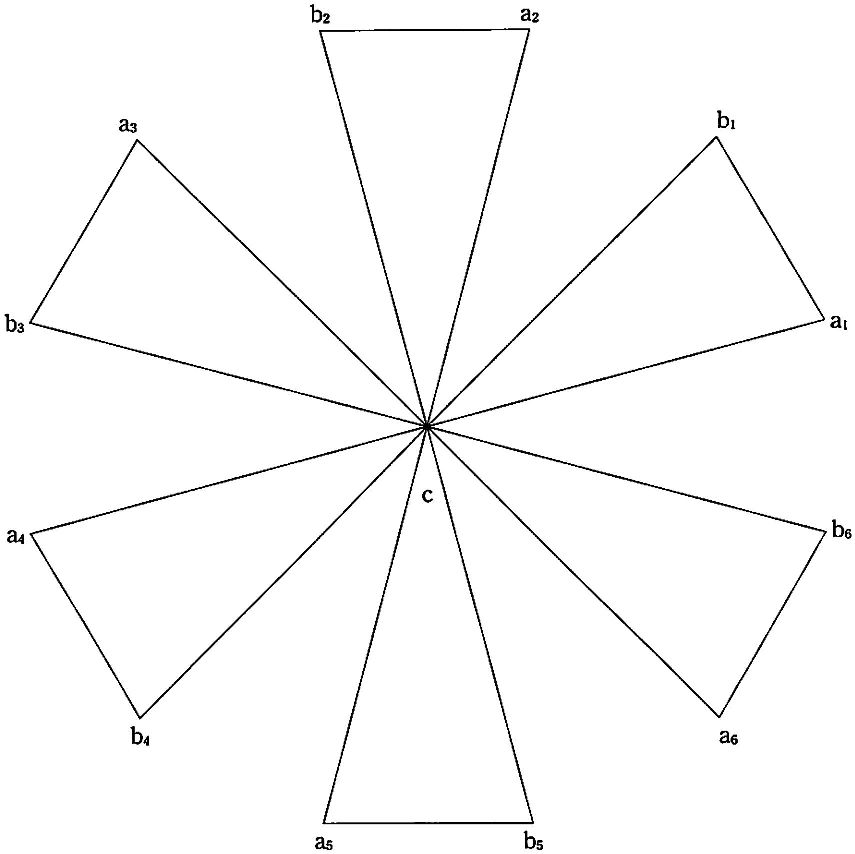

Figure 1. Bouquet $B_m(m=6)$

Remark 16.1.2. The topological realization $|B_m|$ of B_m is homotopic to $\mathbf{C}-\{w_1, w_2, \cdots, w_m\}$, where $w_1, w_2, \cdots, w_m$, are distinct m-points in $\mathbf{C}$. This simple fact is essential in application to hypergeometric functions (cf.(22.2)).

(16.2) Local systems on a bouquet.

Let R be a commutative ring with unity, and let $e_1, e_2, \cdots, e_m \in R^\times$. We put $e=(e_1, e_2, \cdots, e_m)$. Let $K=B_m$ be the m-bouquet.

Definition 16.2.1. The local sytem $\mathscr{L}=\mathscr{L}_e$ on K is defined by $\mathscr{L}_{a_i}=\mathscr{L}_{b_i}=\mathscr{L}_c=R$, and

$$\begin{cases} \zeta_{a_i,c} : \mathscr{L}_c \to \mathscr{L}_{a_i}, & \zeta_{a_i,c} = \mathrm{id}_R, \\ \zeta_{b_i,a_i} : \mathscr{L}_{a_i} \to \mathscr{L}_{b_i}, & \zeta_{b_i,c_i} = \mathrm{id}_R, \\ \zeta_{c,b_i} : \mathscr{L}_{b_i} \to \mathscr{L}_c, & \zeta_{c,b_i} = e_i \cdot \mathrm{id}_R. \end{cases}$$

(16.3) The chain complex of $(K, \mathscr{L})$.

Let us describe the chain complex $C_\bullet(K, \mathscr{L})$. First,

$$\begin{cases} C_q(K, \mathscr{L}) = 0 \qquad (q \neq 0, 1), \\ C_0(K, \mathscr{L}) = \oplus_{i=1}^m (Ra_i \oplus Rb_i) \oplus Rc, \\ C_1(K, \mathscr{L}) = \oplus_{i=1}^m \{R(c, a_i) \oplus R(a_i, b_i) \oplus R(b_i, c)\}. \end{cases}$$

The boundary operator $\partial : C_1(K, \mathscr{L}) \to C_0(K, \mathscr{L})$ is given by

$$\begin{aligned} &\partial \sum_{i=1}^m \{u_i(c, a_i) + v_i(a_i, b_i) + w_i(b_i, c)\} \\ &= \sum_{i=1}^m \{(u_i - v_i)a_i + (v_i - w_i)b_i\} + \sum_{i=1}^m (e_i w_i - u_i)c, \end{aligned}$$

where $u_i, v_i, w_i \in R$. To describe $B_0(K, \mathscr{L})$ and $Z_1(K, \mathscr{L})$, we put

$$\sigma_i := (c, a_i) + (a_i, b_i) + (b_i, c), \qquad (i = 1, 2, \cdots, m).$$

Definition 16.3.1. Let $\phi : R^m \to R$ be an R-homomorphism defined by

$$\phi : u = {}^t(u_1, u_2, \cdots, u_m) \mapsto \sum_{i=1}^m (1 - e_i) u_i.$$

A straightforward computation gives the following:

Lemma 16.3.2. (i) *Any element of $B_0(K, \mathscr{L})$ is of the form:*

$$\sum_{i=1}^m (v_i a_i + w_i b_i) + \sum_{i=1}^m \{(e_i - 1)u_i - e_i(v_i + w_i)\}c \qquad \textit{with} \quad u_i, v_i, w_i \in R.$$

(ii) *Any element of $Z_1(K, \mathscr{L})$ is of the form:*

$$\sum_{i=1}^m u_i a_i \qquad \textit{with} \quad u = {}^t(u_1, u_2, \cdots, u_m) \in \operatorname{Ker} \phi.$$

(16.4) The homology groups of $(K, \mathscr{L})$.

Lemma 16.4.1. *$H_0(K, \mathscr{L})=0$ if and only if $\phi: R^m \to R$ is surjective.*

Proof. $H_0(K, \mathscr{L})=0$ if and only if $B_0(K, \mathscr{L})=C_0(K, \mathscr{L})$. By Lemma 16.3.2,(i) this is the case if and only if for any $v_1, w_i, u \in R$ $(i=1, 2, \cdots, m)$, there exist $u_i \in R$ $(i=1, 2, \cdots, m)$ such that $\sum_{i=1}^m\{(e_i-1)u_i-c_i(v_i+w_i)\}=u$. This is equivalent to say that ϕ is surjective.

By Lemma 16.3.2,(ii), we have an R-isomorphism $H_1(K, \mathscr{L})=Z_1(K, \mathscr{L})\simeq \operatorname{Ker}\phi$.

Proposition 16.4.2. *If $\phi: \mathrm{R}^m \to \mathrm{R}$ is surjective, then*

(1) *$Z_\bullet(K, \mathscr{L})$ is a flat R-module,*

(2) *$H_q(K, \mathscr{L})=0$ if $q\neq 1$, and*

(3) *$H_1(K, \mathscr{L})$ is a flat R-module.*

Proof. $Z_q(K, \mathscr{L})=H_q(K, \mathscr{L})=0$ if $q\neq 0, 1$. By Lemma 16.4.1, $H_0(K, \mathscr{L})=0$. Moreover, $Z_0(K, \mathscr{L})=C_0(K, \mathscr{L})$ is free, and hence flat. So it suffices to show that $H_1(K, \mathscr{L})=Z_1(K, \mathscr{L})\simeq \operatorname{Ker}\phi$ is flat. Since ϕ is assumed to be surjective, there exists a $u^0 \in R^m$ such that $\phi(u^0)=1$. There exists a direct sum decomposition of R-modules: $R^m=\operatorname{Ker}\phi \oplus Ru^0$, where the projection $\psi: R^m \to \operatorname{Ker}\phi$ is given by $\psi(u)=u-\phi(u)u^0$. Hence $\operatorname{Ker}\phi$ is a direct summand of the free module R^m. So $\operatorname{Ker}\phi$ is projective, and hence flat.

(16.5) The configuration space of n-points in a bouquet.

As before, consider $K=B_m$ and $\mathscr{L}=\mathscr{L}_e$ with $e=(e_1, e_2, \cdots, e_m)$, where the e_i's are unit elements of R. Let K_n be the configuration space of n-points in K, $\pi: \mathrm{Sd}^2K^n \to K^n$ the canonical projection, and $\mathscr{M}$ a local system on K_n. We shall compute the homology groups of $(K_n, \mathscr{M})$ under certain conditions. To state the result we introduce the following:

Notation 16.5.1. (cf.(3.3)) Let I_e be the ideal of R generated by $1-e_1, 1-e_2, \cdots, 1-e_m$, i.e., $I_e := \sum_{i=1}^m R(1-e_i)$. Let V_e be the R-module defined by

$$V_e := \{u=(u_1, u_2, \cdots, u_m)\in R^m\,;\, \sum_{i=1}^m u_i(1-e_i)=0\}.$$

Theorem 16.5.2. *Assume that*

(1) $n! \in R^\times$,

(2) $I_e=R$, and

(3) $\pi^*\mathscr{M}=\mathrm{Sd}^2 \overset{n}{\boxtimes} \mathscr{L}$.

Then there exists a canonical isomorphism of R-modules:

$$H_q(K_n, \mathscr{M}) \simeq \begin{cases} 0 & (q \neq n), \\ \overset{n}{\wedge} H_1(K, \mathscr{M}) & (q=n), \end{cases}$$

where $H_1(K, \mathscr{L})$ is R-isomorphic to V_e.

Proof. The assumption of the theorem and Proposition 16.4.2 imply Assumptions 15.2.1 and 15.3.1 (with $r=1$). So we can apply Theorem 15.3.3 to establish the theorem.

III. Singular Local Systems

In Chapter III we consider a twisted *singular* homology group and establish a comparison theorem between it and a twisted *simplicial* homology group discussed in Chapters I and II. The singular theory fits into the de Rham theory (see [16]), while the simplicial theory is convenient in explicit computation.

17. Category of singular local systems

(17.1) The category $\mathbf{L}(X)$ of local systems on a topological space X.

As to singular local systems, we follow the exposition of [12] (§ 6.1). Let $\Pi : \mathbf{T} \to \mathbf{G}$ be the fundamental groupoid functor, where $\mathbf{T}$ is the category of topological spaces and $\mathbf{G}$ is the category of groupoids (see e.g. [14]). Let $\mathbf{M}$ be the category of R-modules. For any topological space X, we define the category $\mathbf{L}(X)$ of *singular local systems* of R-modules on X.

Definition 17.1.1. An object of $\mathbf{L}(X)$, called a *singular local system* on X, is a covariant functor from ΠX into $\mathbf{M}$. For any objects $\mathscr{L}$, $\mathscr{M}$ of $\mathbf{L}(X)$, a morphism from $\mathscr{L}$ into $\mathscr{M}$ is a natural transformation from $\mathscr{L}$ into $\mathscr{M}$.

Remark 17.1.2. An object $\mathscr{L}$ of $\mathbf{L}(X)$ is a collection of R-modules $\mathscr{L}_p$ $(p \in X)$ together with R-isomorphisms $\zeta(\gamma) : \mathscr{L}_p \to \mathscr{L}_q$, where γ is a homotopy class of curves in X with initial point p and terminal point q. In this situation, we often write $\mathscr{L}=(\mathscr{L}, \zeta)$. For any objects $\mathscr{L}=(\mathscr{L}, \xi)$, $\mathscr{M}=(\mathscr{M}, \eta)$ of $\mathbf{L}(X)$, a morphism $\varphi=\{\varphi_a\} : \mathscr{L} \to \mathscr{M}$ is a collection of R-homomorphisms $\varphi_p : \mathscr{L}_p \to \mathscr{M}_p$ $(p \in X)$ such that for any curve γ in X with initial point p and terminal point q, the following diagram is commutative:

$$\begin{tikzcd} \mathscr{L}_p \arrow[r, "\varphi_p"] \arrow[d, "\xi(\gamma)"'] & \mathscr{M}_p \arrow[d, "\eta(\gamma)"] \\ \mathscr{L}_q \arrow[r, "\varphi_q"'] & \mathscr{M}_q \end{tikzcd}$$

Hereafter, unless otherwise stated explicitly, a local system is a singular local system.

(17.2) The pull-back functor.

Definition 17.2.1. A continuous map $f: X \to Y$ induces a covariant functor $f^*: \mathbf{L}(X) \to \mathbf{L}(Y)$, called the *pull-back functor*. For any object $\mathscr{L}$ of $\mathbf{L}(Y)$, we put

$$f^*\mathscr{L} := \mathscr{L} \circ \Pi f,$$

where $\Pi f: \Pi X \to \Pi Y$ is the groupoid homomorphism associated with f. Then $f^*\mathscr{L}$ is an object of $\mathbf{L}(X)$.

(17.3) The category of singular local systems.

Definition 17.3.1. The category $\mathbf{L}^{sing}$ of singular local systems is defined as follows: An object of $\mathbf{L}^{sing}$ is a pair $(X, \mathscr{L})$ of a topological space X and a singular local system $\mathscr{L}$ on X. A morphism $(f, \varphi): (X, \mathscr{L}) \to (Y, \mathscr{M})$ of $\mathbf{L}^{sing}$ is a pair of a continuous map $f: X \to Y$ and a morphism $\varphi: \mathscr{L} \to f^*\mathscr{M}$ of $\mathbf{L}(X)$.

(17.4) The topological realization functor $|\cdot|$.

The topological realization functor $|\cdot|: \mathbf{S} \to \mathbf{T}$ from the category of simplicial complexes into that of topological spaces is formulated in [28](Chapter 3, § 1). Let us recall the minimal ingredient of it.

Definition 17.4.1. For a simplicial complex K, let $|K|$ be the set of all functions $\alpha: V_K \to [0, 1]$ such that

(1) $\operatorname{supp} \alpha := \{a \in V_K\,;\, \alpha(a) \neq 0\}$ is a simplex of K, and

(2) $\sum_{a \in V_K} \alpha(a) = 1$.

Definition 17.4.2. For a vertex $a \in V_K$, let $\langle a \rangle \in |K|$ be defined by

$$\langle a \rangle(b) := \begin{cases} 1 & (b = a), \\ 0 & (b \neq a). \end{cases}$$

Definition 17.4.3. For a simplicial map $f: K \to L$, let $|f|: |K| \to |L|$ be a map defined by

$$|f|(\alpha) := \sum_{a \in V_K} \alpha(a) \langle f(a) \rangle.$$

We provide $|K|$ with the *coherent topology*. Then $|f|:|K| \to |L|$ becomes a continuous map ([28], Chapter 3, § 2), and hence $|\cdot|: \mathbf{S} \to \mathbf{T}$ is a covariant functor.

(17.5) Simplicial local systems induced from singular local systems.

For any simplicial complex K, there exists a covariant functor $\theta=\theta_K : \mathbf{L}(|K|) \to \mathbf{L}(K)$ from the category of singular local systems on $|K|$ into that of simplicial local systems on K. We define this functor in what follows.

Definition 17.5.1. Let $\mathscr{L}=(\mathscr{L}, \xi)$ be an object of $\mathbf{L}(|K|)$. For any vertex $a \in V_K$, we put $(\theta_K \mathscr{L})_a := \mathscr{L}_{\langle a\rangle}$ $(a \in V_K)$, where $\langle a\rangle$ is defined in Definition 17.4.2. For any $\{a, b\} \in K^{(1)}$, we put $(\theta_K \xi)_{ba} := \xi(\gamma_{ba}) : (\theta_K \mathscr{L})_a \to (\theta_K \mathscr{L})_b$, where $\gamma_{ba} : [0, 1] \to |K|$ is the curve in $|K|$ defined by $\gamma_{ba}(t) := t\langle a\rangle + (1-t)\langle b\rangle$ $(0 \le t \le 1)$.

For simplicity, $\theta_K \mathscr{L}$ will often be denoted by $\mathscr{L}$. This abuse of notation should cause no confusion. The following lemmas are easily proved.

Lemma 17.5.2. *The functor θ is natural in the sense that for any simplicial map $f: K \to L$, there exists a commutative diagram of functors:*

$$\begin{array}{ccc} \mathbf{L}(|L|) & \xrightarrow{\theta_L} & \mathbf{L}(L) \\ {\scriptstyle |f|^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathbf{L}(|K|) & \xrightarrow[\theta_K]{} & \mathbf{L}(K). \end{array}$$

Lemma 17.5.3. *Assume that a group G acts on K regularly. Let $\pi: K \to K/G$ be the canonical simplicial projection. Then there exists a commutative diagram of functors:*

$$\begin{array}{ccc} \mathbf{L}(|K/G|) & \xrightarrow{\theta_{K/G}} & \mathbf{L}(K/G) \\ {\scriptstyle |\pi|^*}\downarrow & & \downarrow{\scriptstyle \pi^*} \\ \mathbf{L}(|K|) & \xrightarrow[\theta_K]{} & \mathbf{L}(K). \end{array}$$

(17.6) Base change functors on local systems.

We have constructed the following *base change functors* on local systems:

(1) for simplicial complexes K, L and a simplicial map $f: K \to L$,

$$f^* : \mathbf{L}(\mathrm{L}) \to \mathbf{L}(\mathrm{K}) : \text{the pull-back functor (5.2),}$$

(2) for a simplicial complex K,

$$\mathrm{Sd}:\mathbf{L}(K)\to\mathbf{L}(\mathrm{Sd}K):\text{the subdivision functor (6.2)},$$

(3) for topological spaces X, Y and a continuous map $f:X\to Y$,

$$f^*:\mathbf{L}(Y)\to\mathbf{L}(X):\text{the pull-back functor (17.2)},$$

(4) for a simplicial complex K,

$$\theta_K:\mathbf{L}(|K|)\to\mathbf{L}(K):\text{defined in (17.5)},$$

18. Comparison theorem

(18.1) The singular chain complex functor $S_\bullet:\mathbf{L}^{sing}\to\mathbf{C}$.

The singular chain complex functor $S_\bullet:\mathbf{L}^{sing}\to\mathbf{C}$ is defined in [12](§ 6.2), where $\mathbf{C}$ is the category of chain complexes of R-modules. For later use, we shall recall it briefly. Let Δ^q be the standard q-simplex with vertices $v_0, v_1, \cdots, v_q$. For any singular q-simplex $\sigma:\Delta^q\to X$, let γ_σ be the curve in X defined by

$$\gamma_\sigma(t):=\sigma((1-t)v_0+tv_1)\qquad(0\leq t\leq 1).$$

Definition 18.1.1. Let $(K, \mathscr{L})$ be an object of $\mathbf{L}^{sing}$ with $\mathscr{L}=(\mathscr{L}, \xi)$. A q-chain $c\in S_q(X, \mathscr{L})$ is a formal sum $c=\sum_\sigma u_\sigma\cdot\sigma$, where the sum is taken over all singular q-simplices σ in X, $u_\sigma\in\mathscr{L}_{\sigma(v_0)}$ and $u_\sigma=0$ except for a finite number of σ's. The boundary operator $\partial:S_q(X, \mathscr{L})\to S_{q-1}(X, \mathscr{L})$ is defined by

$$\partial c:=\sum_\sigma\{\xi(\gamma_\sigma)(u_\sigma)\cdot\partial_0\sigma+\sum_{i=1}^q(-1)^i u_\sigma\cdot\partial_i\sigma\}.$$

(18.2) The natural chain map $C_\bullet(K, \theta_K\mathscr{L})\to S_\bullet(|K|, \mathscr{L})$.

Let K be a simplicial complex, and $\mathscr{L}$ be a singular local system on $|K|$.

Definition 18.2.1. For any q-simplex $\sigma=\{a_0, a_1, \cdots, a_q\}$ of K, we define the singular q-simplex $\tilde{\sigma}:\Delta^q\to|K|$ by

$$\tilde{\sigma}\Big(\sum_{i=0}^q t_i v_i\Big):=\sum_{i=0}^q t_i\langle a_i\rangle,$$

for $t_i\geq 0$ and $\sum t_i=1$. For any local section $u\in(\theta_K\mathscr{L})_\sigma$, we define $\tilde{u}\in\mathscr{L}_{\tilde{\sigma}(v_0)}=\mathscr{L}_{\langle a_0\rangle}$ by $\tilde{u}:=u(a_0)$. Since $u\in(\theta_K\mathscr{L})_\sigma$ is a map defined on σ and $u(a_0)\in(\theta_K\mathscr{L})_{a_0}=\mathscr{L}_{\langle a_0\rangle}$, this is well-defined.

Definition 18.2.2. The chain map $\theta_K : C_\bullet(K, \theta_K\mathscr{L}) \to S_\bullet(|K|, \mathscr{L})$ is defined by

$$\theta_K(\sum u_\sigma \cdot \sigma) := \sum \tilde{u}_\sigma \cdot \tilde{\sigma},$$

where $\tilde{u}_\sigma$ and $\tilde{\sigma}$ are defined in Definition 18.2.1.

Lemma 18.2.3. *The chain map $\theta = \theta_K$ is natural in the following sense: If L is another chain complex, $\mathscr{M}$ is a singular local system on $|L|$ and $(f, \varphi) : (K, \theta_K\mathscr{L}) \to (L, \theta_L\mathscr{M})$ is a morphism of $\mathbf{L}^{li}$, then there exists a commutative diagram of chain maps*

$$\begin{array}{ccc} C_\bullet(K, \theta_K\mathscr{L}) & \xrightarrow{\theta_K} & S_\bullet(|K|, \mathscr{L}) \\ {\scriptstyle C_\bullet(f,\varphi)}\downarrow & & \downarrow{\scriptstyle S_\bullet(|f|, |\varphi|)} \\ C_\bullet(L, \theta_L\mathscr{M}) & \xrightarrow[\theta_L]{} & S_\bullet(|L|, \mathscr{M}). \end{array}$$

(18.3) The singular homology functor.

The *singular homology functor* $H_\bullet^{sing} : \mathbf{L}^{sing} \to \mathbf{M}$ is defined by composing the homology functor with the singular chain complex functor, where $\mathbf{M}$ is the category of graded R-modules. Recall that the simplicial homology functor $H_\bullet^{simp} : \mathbf{L}^{li} \to \mathbf{M}$ was defined in § 10 (see Remark 10.1.3). Let us compare these two functors.

(18.4) The comparison theorem.

Let K be a simplicial complex, $\mathscr{L}$ a singular local system on $|K|$. Then we have a chain map $\theta_K : C_\bullet(K, \theta_K\mathscr{L}) \to S_\bullet(|K|, \mathscr{L})$, which induces an R-homomorphism

$$\theta_K : H_\bullet^{simp}(K, \theta_K\mathscr{L}) \to H_\bullet^{sing}(|K|, \mathscr{L}).$$

The following theorem is fundamental:

Theorem 18.4.1. *θ_K is an isomorphism which is natural in the following sense: If L is another chain complex, $\mathscr{M}$ is another singular local system on $|L|$ and $(f, \varphi) : (K, \theta_K\mathscr{L}) \to (L, \theta_L\mathscr{M})$ is a morphism of $\mathbf{L}^{li}$, then there exists a commutative diagram of R-homomorphisms*

$$\begin{array}{ccc} H_\bullet^{simp}(K, \theta_K\mathscr{L}) & \xrightarrow{\theta_K} & H_\bullet^{sing}(|K|, \mathscr{L}) \\ {\scriptstyle H_\bullet^{simp}(f,\varphi)}\downarrow & & \downarrow{\scriptstyle H_\bullet^{sing}(|f|, |\varphi|)} \\ H_\bullet^{simp}(L, \theta_L\mathscr{M}) & \xrightarrow[\theta_L]{} & H_\bullet^{sing}(|L|, \mathscr{M}). \end{array}$$

Proof. This theorem is proved in an almost similar manner as in the classical case where the local system is trivial. The naturality follows from Lemma 18.2.3.

(18.5) Polyhedra.

Definition 18.5.1. A topological space X is said to be an *polyhedron* if there exists a simplicial complex K and a continuous map $f : |K| \to X$ such that f is a homotopy equivalence between $|K|$ and X. We call (K, f) an *underlying simplicial structure* of X.

Remark 18.5.2. In the usual definition of polyhedra, one requires that $f : |K| \to X$ is a homeomorphism.

Let X be a polyhedron with underlying simplicial structure (K, f), $\mathscr{L}$ a singular local system on X. Applying the pull-back functor f^*, we obtain a singular local system $f^*\mathscr{L}$ on $|K|$ and a morphism $f_* : (|K|, f^*\mathscr{L}) \to (X, \mathscr{L})$ of $\mathbf{L}^{sing}$. Applying the singular homology functor, we obtain an R-homomorphism

$$f_* : H_\bullet^{sing}(|K|, f^*\mathscr{L}) \to H_\bullet^{simp}(X, \mathscr{L}).$$

This is an R-isomorphism due to the homotopy invariance of the singular homology functor. By the comparison theorem (Theorem 18.4.1), we have an R-isomorphism $\theta_K : H_\bullet^{simp}(K, \theta_K f^*\mathscr{L}) \to H_\bullet^{sing}(X, \mathscr{L})$. Composing these isomorphisms, we obtain the following:

Proposition 18.5.3. *There exists an R-isomorphism:*

$$H_\bullet^{simp}(K, \theta_K f^*\mathscr{L}) \simeq H_\bullet^{sing}(X, \mathscr{L}).$$

This shows that if X is a polyhedron with underlying simplicial structure (K, f), then the singular homology group of $(X, \mathscr{L})$ is computed as the simplicial homology group of $(K, \theta_K f^* \mathscr{L})$.

19. Naturality of the topological realization functor

Let K be a simplicial complex. In the simplicial category, the configuration space of n-points in K is the simplicial complex $K_n = \mathrm{Sd}^2 K^n/\mathfrak{S}_n$(15.1), whereas in the topological category, the corresponding configuration space is the topological space $|K|_n := |K|^n/\mathfrak{S}_n$. We have to establish a *canonical* homeomorphism $|K_n| \cong |K|_n$. To do so, we shall establish the naturality of the topological realization functor $|\cdot|$ with respect to the subdivision and the direct product functor. Since this is one of the essential points in this paper and no standard text has treated it explicitly, some explanation should be included here. Moreover, we recall the naturality of $|\cdot|$ with respect to a group action (see [5]).

(19.1) Naturality of $|\cdot|$ with respect to Sd.

Lemma 19.1.1. *For any simplicial complex K, $|K|$ and $|\mathrm{Sd}K|$ are canonically homeomorphic. If a group G acts on K, then this homeomorphism is G-equivariant.*

Proof. We explicitly give a bijection $|K| \leftrightarrow |\mathrm{Sd}K|$, $\alpha \leftrightarrow \beta$. First, if β is given then α is defined by

$$\alpha(a):=\sum_{\sigma\ni a}\frac{\beta(\sigma)}{\#\sigma} \qquad (a\in V_K),$$

where the sum is taken over all simplices σ such that $a\in\sigma$. Next, if α is given then β is defined as follows: Let $t_0>t_1>\cdots t_q$ be the positive range of α arranged decreasingly and, by convention, put $t_{q+1}=0$. For $0\le i\le q$, let $\sigma_i:=\{a\in V_K\,;\,\alpha(a)\ge t_i\}$. Then the σ_i's are simplices of K such that $\sigma_0<\sigma_1<\cdots<\sigma_q$. Hence $\{\sigma_0, \sigma_1, \cdots, \sigma_q\}$ is a q-simplex of $\mathrm{Sd}K$. For any $\sigma=V_{\mathrm{Sd}K}=K$, we put

$$\beta(\sigma):=\begin{cases}(t_i-t_{i+1})\#\sigma_i & \text{if } \sigma=\sigma_i,\ 0\le i\le q,\\ 0 & \text{otherwise.}\end{cases}$$

Note that supp $\beta=\{\sigma_0, \sigma_1, \cdots, \sigma_q\}$ and $\sum_\sigma\beta(\sigma)=1$, and hence $\beta\in|\mathrm{Sd}K|$.

The bijection mentioned above gives a homeomorphism between $|K|$ and $|\mathrm{Sd}K|$. The equivariance of a group action easily follows from the construction of the bijection.

(19.2) Naturality of $|\cdot|$ with respect to the direct product functor.

Definition 19.2.1. Let K be an ordered simplicial complex. Then $\mathfrak{S}_n$ acts on $|K^n|$ by $(\tau\alpha)(a_1, a_2, \cdots, a_n)=\alpha(a_{\tau(1)}, a_{\tau(2)}, \cdots, a_{\tau(n)})$, where $\tau\in\mathfrak{S}_n$, $\alpha\in|K|^n$ and $a_1, a_2, \cdots, a_n$ are vertices of K. Moreover $\mathfrak{S}_n$ acts on $|K|^n$ by $\tau(a_1, a_2, \cdots, a_n)=(a_{\tau(1)}, a_{\tau(2)}, \cdots, a_{\tau(n)})$, where $\tau\in\mathfrak{S}_n$ and $a_1, a_2, \cdots, a_n\in|K|$.

Lemma 19.2.2. (i) *Let $K_1, K_2, \cdots, K_n$ be ordered simplicial complexes. Then there exists a canonical homeomorphism*:

$$|K_1\times K_2\times\cdots\times K_n|\cong|K_1|\times|K_2|\times\cdots\times|K_n|.$$

(ii) *Let K be an ordered simplicial complex. Then the canonical homeomorphism $|K^n|\cong|K|^n$ is $\mathfrak{S}_n$-equivariant.*

Proof. (i) Put $V_i=V_{K_i}$, $(i=1, 2, \cdots, n)$. We explicitly give a bijection:

$$\begin{array}{ccc} |K_1\times K_2\times\cdots\times K_n| & \leftrightarrow & |K_1|\times|K_2|\times\cdots\times|K_n| \\ \alpha & \leftrightarrow & (\alpha_1, \alpha_2, \cdots, \alpha_n) \end{array}$$

If α is given then $\alpha_i(i=1, 2, \cdots, n)$ are defined by

$$\alpha_i(a_i):=\sum_{a_1\in V_1}\cdots\sum_{a_{i-1}\in V_{i-1}}\sum_{a_{i+1}\in V_{i+1}}\cdots\sum_{a_n\in V_n}\alpha(a_1, \cdots, a_i, \cdots, a_n)$$

for $a_i\in V_i$. We shall show that $\alpha_i\in|K_i|$. Indeed, it is easy to show $\alpha_i\geq 0$ and $\sum_{a_i\in V_i}\alpha_i(a_i)=1$. So it suffices to show that supp α_i is a simplex of K_i. Since supp α is a simplex of $K_1\times K_2\times\cdots\times K_n$, there exist $\sigma=(\sigma_1, \sigma_2, \cdots, \sigma_n)\in\Sigma(q)$ and $j=(j_1, j_2, \cdots, j_n)\in J(q)$ such that supp $\alpha\subseteq\langle\sigma; j\rangle$ (see Definition 7.1.3). In particular supp $\alpha\subseteq\sigma_1\times\sigma_2\times\cdots\times\sigma_n$. From this it is easy to deduce supp $\alpha_i\subseteq\sigma_i$. Hence supp α_i is a simplex of K_i.

Conversely, if $\alpha_i\in|K_i|$, $(i=1, 2, \cdots, n)$ are given, then $\alpha\in|K_1\times K_2\times\cdots\times K_n|$ is defined as follows: Put

$$\sigma_i=\{a_{i0}, a_{i1}, \cdots, a_{iq_i}\}:=\text{supp } \alpha_i, \qquad \sigma:=(\sigma_i, \sigma_2, \cdots, \sigma_n),$$

where $a_{i0}<a_{i1}<\cdots<a_{iq_i}$. Moreover, put

$$\alpha_{ik}:=\alpha_i(a_{ik}), \qquad \tilde{\alpha}_{ik}:=\alpha_{i0}+\alpha_{i1}+\cdots+\alpha_{ik},$$

for $K=0, 1, \cdots, q_i$. Note that $\tilde{\alpha}_{iq_i}=1$. We introduce a total order in the set $\Lambda:=\{(i, k); 1\leq i\leq n, 1\leq k\leq q_i\}$ by putting $(i, k)<(i', k')$ if one of the following conditions holds:

(1) $\tilde{\alpha}_{i,k-1}<\tilde{\alpha}_{i',k'-1}$,

(2) $\tilde{\alpha}_{i,k-1}=\tilde{\alpha}_{i',k'-1}$ and $i<i'$,

(3) $\tilde{\alpha}_{i,k-1}=\tilde{\alpha}_{i',k'-1}$, $i=i'$ and $k<k'$.

For each $(i, k)\in\Lambda$, let j_{ik} be the natural number such that (i, k) is the j_{ik}-th smallest element in Λ. Then the map $J:\Lambda\to\{1, 2, \cdots, r\}$, $(i, k)\mapsto j_{ik}$ is a bijection, where $r:=\sum_{i=1}^{n}q_i$. Since $j_{ik}<j_{ik'}\Leftrightarrow(i, k)<(i, k')\Leftrightarrow k<k'$, we have $1\leq j_{i1}<j_{i2}<\cdots<j_{iq_i}\leq r$ for each i. Put $j_i=(j_{i1}, j_{i2}, \cdots, j_{iq_i})$ and $j=(j_1, j_2, \cdots, j_n)$. Then one can speak of an r-simplex $\langle\sigma; j\rangle$ of $K_1\times K_2\times\cdots\times K_n$ (see Definition 7.1.3). Put $\langle\sigma; j\rangle=\{c_0, c_1, \cdots, c_r\}$, where $c_0<c_1<\cdots c_r$ is arranged in the lexicographic order. Now we define $\beta\in|K_1\times K_2\times\cdots\times K_n|$ as follows: supp $\beta\subseteq\{c_0, c_1, \cdots, c_r\}$ and $\beta(c_l)$, $(l=0, 1, \cdots, r)$ are defined by

$$\beta(c_l):=\begin{cases}\beta_l & (l=0, 1, \cdots, r-1),\\ 1-\sum_{l=0}^{r-1}\beta_l & (i=r).\end{cases}$$

Here $\beta_0, \beta_1, \cdots, \beta_{l-1}$ are defined inductively by

$$\beta_0+\beta_1+\cdots+\beta_{l-1}=\tilde{\alpha}_{i(l),k(l)-1} \qquad (l=1, 2, \cdots, r),$$

where $(i(l), k(l))=J^{-1}(l)\in\Lambda$.

The maps $\alpha\mapsto(\alpha_0, \alpha_1, \cdots, \alpha_n)$ and $(\alpha_0, \alpha_1, \cdots, \alpha_n)\mapsto\alpha$ defined above are the inverse of each other. It is easy to see that this bijection is a homeomorphism.

(ii) The $\mathfrak{S}_n$-equivariance of the canonical homeomorphism $|K^n|\cong|K|^n$ follows from Definition 16.2.1 and the construction of the homeomorphism in (i).

(19.3) Naturality of $|\cdot|$ with respect to a group action.

Let K be a simplicial complex and assume that a finite group G acts on K. Let $\pi: K\to K/G$ be the canonical simplicial projection, $\pi:|K|\to|K|/G$ the canonical topological projetion, respectively. Then we have the following:

Lemma 19.3.1. *If G acts on K regularly, then there exists a homeomorphism $\chi:|K|/G\to|K/G|$ such that the following diagram is commutative:*

$$\begin{tikzcd} {|K|} \arrow[r, equal] \arrow[d, "|\pi|"'] & {|K|} \arrow[d, "p"] \\ {|K/G|} \arrow[r, "\chi"'] & {|K|/G.} \end{tikzcd}$$

Proof. See [5] page 117.

20. External product functor for singular local systems

(20.1) The external product functor $\mathbf{L}^{sing}\times\cdots\times\mathbf{L}^{sing}\to\mathbf{L}^{sing}$.

In (7.2) we defined the external product functor in the simplicial category. We can define its counterpart in the singular category. Here we only define the correspondence of objects.

Definition 20.1.1. Let $(X_i, \mathscr{L}_i)$, $(i=1, 2, \cdots, n)$ be objects of $\mathbf{L}^{sing}$ with $\mathscr{L}_i=(\mathscr{L}, \xi)$. Then a new object $(X, \mathscr{L})$ of $\mathbf{L}^{sing}$ with $\mathscr{L}=(\mathscr{L}, \xi)$ is defined as follows:

(1) $X:=X_1\times X_2\times\cdots\times X_n$,

(2) for each point $p=(p_1, p_2, \cdots, p_n)\in X$, put $\mathscr{L}_p:=\mathscr{L}_{p1}\otimes\mathscr{L}_{p2}\otimes\cdots\otimes\mathscr{L}_{pn}$,

(3) for each curve $\gamma:[0, 1]\to X$ with $\gamma(0)=p=(p_1, p_2, \cdots, p_n)$ and $\gamma(1)=q=(q_1, q_2, \cdots q_n)$, put $\gamma(t)=(\gamma_1(t), \gamma_2(t), \cdots\gamma_n(t))$. Then γ_i is a curve in X_i such that $\gamma_i(0)=p_i$ and $\gamma_i(1)=q_i$. We difine $\xi(\gamma):\mathscr{L}_p\to\mathscr{L}_q$ by

$$\xi(\gamma):=\xi_1(\gamma_1)\otimes\xi_2(\gamma_2)\otimes\cdots\otimes\xi_n(\gamma_n).$$

We write

$$(X, \mathscr{L})=(X_1, \mathscr{L}_1)\boxtimes(X_2, \mathscr{L}_2)\boxtimes\cdots\boxtimes(X_n, \mathscr{L}_n)$$
$$=(X_1\times X_2\times\cdots\times X_n, \mathscr{L}_1\boxtimes\mathscr{L}_2\boxtimes\cdots\boxtimes\mathscr{L}_n).$$

and call it the external product of $(X_1, \mathscr{L}_1)$, $(X_2, \mathscr{L}_2)$, $\cdots$, $(X_n, \mathscr{L}_n)$.

(20.2) Naturality of the external products.

Let X_i and Y_i be topological spaces, $f_i: X_i \to Y_i$ continuous maps $(i=1, 2, \cdots, n)$. We define the map $f_1\times f_2\times\cdots\times f_n$ by

$$f_1\times f_2\times\cdots\times f_n: X_1\times X_2\times\cdots\times X_n \to Y_1\times Y_2\times\cdots\times Y_n$$
$$(p_1, p_2, \cdots, p_n)\mapsto (f_1(p_1), f_2(p_2), \cdots, f_n(p_n)).$$

The following lemma is easily established.

Lemma 20.2.1. *If $\mathscr{L}_i$ is a singular local system on $Y_i(i=1, 2, \cdots, n)$, then*

$$(f_1\times f_2\times\cdots\times f_n)^*(\mathscr{L}_1\boxtimes\mathscr{L}_2\boxtimes\cdots\boxtimes\mathscr{L}_n)=f_1^*\mathscr{L}_1\boxtimes f_2^*\mathscr{L}_2\boxtimes\cdots\boxtimes f_n^*\mathscr{L}_n.$$

Let $K_i(i=1, 2, \cdots, n)$ be ordered simplicial complexs and put $K=K_1\times K_2\times\cdots\times K_n$. Recall that there exist covariant functors

$$\theta_{K_i}: \mathbf{L}(|K_i|) \to \mathbf{L}(\mathrm{K}_i) \quad\text{and}\quad \theta_K: \mathbf{L}(|K|) \to \mathbf{L}(K).$$

(see Definition 17.5.1). Moreover, by Lemma 19.2.2, there exists a canonical homeomorphism $\psi: |K| \to |K_1|\times|K_2|\times\cdots\times|K_n|$. Consider the sequence of covariant functors

$$\mathbf{L}(|K_1|\times|K_2|\times\cdots\times|K_n|)\xrightarrow{\psi^*}\mathbf{L}(|K|)\xrightarrow{\theta_K}\mathbf{L}(K).$$

From the above definitions we can easily obtain the following:

Lemma 20.2.2. *If $\mathscr{L}_i\in\mathbf{L}(|K_i|)$, $(i=1, 2, \cdots, n)$, then*

$$\theta_K\circ\psi^*(\mathscr{L}_1\boxtimes\mathscr{L}_2\boxtimes\cdots\boxtimes\mathscr{L}_n)=\theta_{K_1}\mathscr{L}_1\boxtimes\theta_{K_2}\mathscr{L}_2\boxtimes\cdots\boxtimes\theta_{K_n}\mathscr{L}_n.$$

21. Twisted singular homology groups of the configuration space

(21.1) Local systems on the topological configuration space of n-points.

In this section we are concerned with various projections and local systems. In order to avoid notational confusion, we employ the following:

Notation 21.1.1. (i) If a group G acts on a simplicial complex K, then we denote by $\pi_K : K \to K/G$ the canonical simplicial projection; if a group G acts on a topological space X, then we denote by $\pi_X : X \to X/G$ the canonical topological projection. If there exists a G-equivariant simplicial map $f : K \to L$, then we denote by $f/G : K/G \to L/G$ the simplicial map induced from f by passing to the quotient; if there exists a G-equivariant continuous map $f : X \to Y$, then we denote by $f/G : X/G \to Y/G$ the continuous map induced from f by passing to the quotient.

(ii) A simplicial local system on a simplicial complex K is denoted by $\mathscr{L}_K$, and a singular local system on a topological space X is denoted by $\mathscr{L}_X$.

Definition 21.1.2. (cf. Definition 15.1.1) Let X be a topological space. The group $\mathfrak{S}_n$ acts on the product space X^n by permutation of n-points. The quotient space $X_n := X^n/\mathfrak{S}_n$ is called the (*topological*) *configuration space of n-points* in X. We denote by $\pi_{X^n} : X^n \to X_n$ the canonical topological projection (cf. Notation 21.1.1). For any continuous map $f : X \to Y$, we define $f^n : X^n \to Y^n$ by

$$f^n(p_1, p_2, \cdots, p_n) := (f(p_1), f(p_2), \cdots, f(p_n)).$$

This map is $\mathfrak{S}_n$-equivariant and hence induces a continuous map

$$f_n := f^n/\mathfrak{S}_n : X_n \to Y_n.$$

Let X be a polyhedron with underlying simplicial structure (K, f). Moreover, let $\mathscr{L}_X$ and $\mathscr{L}_{X_n}$ be singular local systems on X and on X_n. We make the following:

Assumption 21.1.3. $\pi^*{}_{X^n}\, \mathscr{L}_{X_n} = \overset{n}{\boxtimes}\, \mathscr{L}_X$.

The purpose of this section is to express the singular homology of $(X_n, \mathscr{L}_{X_n})$ in terms of the simplicial homology of $(K_n, \mathscr{L}_{K_n})$, where $\mathscr{L}_{K_n}$ is a certain simplicial local system on K_n to be defined in Definition 21.3.1.

(21.2) Commutative diagrams of functors.

Since $f : |X| \to X$ is a homotopy equivalence, $f^n : |X|^n \to X^n$ is an $\mathfrak{S}_n$-equivariant homotopy equivalence. Passing to the quotient, we obtain a homotopy equivalence $f_n : |K|_n \to X_n$. There exists a sequence of $\mathfrak{S}_n$-equivariant continuous maps:

$$|\mathrm{Sd}^2K^n| = |\mathrm{Sd}^2K^n| \xrightarrow{\varphi} |K^n| \xrightarrow{\psi} |K|^n \xrightarrow{f^n} X^n,$$

where φ and ψ are homeomorphisms obtained by Lemmas 19.1.1 and 19.2.2, respectively. Passing to the quotient, we obtain the following commutative diagram of continuous maps:

Diagram 21.2.1.

$$\begin{tikzcd}
{|\mathrm{Sd}^2K^n|} \arrow[r, equal] \arrow[d, "|\pi_{\mathrm{Sd}^2K^n}|"] & {|\mathrm{Sd}^2K^n|} \arrow[r, "\varphi"] \arrow[d, "\pi_{|\mathrm{Sd}^2K^n|}"] & {|K^n|} \arrow[r, "\psi"] \arrow[d, "\pi_{|K^n|}"] & {|K|^n} \arrow[r, "f^n"] \arrow[d, "\pi_{|K|^n}"] & X^n \arrow[d, "\pi_{X^n}"] \\
{|K_n|} \arrow[r, "\chi"'] & {|\mathrm{Sd}^2K^n|/\mathfrak{S}_n} \arrow[r, "\varphi/\mathfrak{S}_n"'] & {|K^n|/\mathfrak{S}_n} \arrow[r, "\psi/\mathfrak{S}_n"'] & {|K|_n} \arrow[r, "f_n"'] & X_n,
\end{tikzcd}$$

where χ is a homeomorphism obtained by Lemma 19.3.1.

Applying the $\mathbf{L}$-functor to Diagram 21.2.1, using the base change functors (17.6) and Lemma 17.5.3, we obtain the following commutative diagram of functors:

Diagram 21.2.2.

$$\begin{tikzcd}
\mathbf{L}(X_n) \arrow[r, "\pi^*_{X^n}"] \arrow[d, "f_n^*"'] & \mathbf{L}(X^n) \arrow[d, "(f^n)^*"] & \\
\mathbf{L}(|K|_n) \arrow[r, "\pi^*_{|K|^n}"] \arrow[d, "(\psi/\mathfrak{S}_n)^*"'] & \mathbf{L}(|K|^n) \arrow[d, "\psi^*"] & \\
\mathbf{L}(|K^n|/\mathfrak{S}_n) \arrow[r, "\pi^*_{|K^n|}"] \arrow[d, "(\varphi/\mathfrak{S}_n)^*"'] & \mathbf{L}(|K^n|) \arrow[r, "\theta_{K^n}"] \arrow[d, "\varphi^*"'] & \mathbf{L}(K^n) \arrow[d, "\mathrm{Sd}^2"] \\
\mathbf{L}(|\mathrm{Sd}^2K^n|/\mathfrak{S}_n) \arrow[r, "\pi^*_{|\mathrm{Sd}^2K^n|}"] \arrow[d, "\chi^*"'] & \mathbf{L}(|\mathrm{Sd}^2K^n|) \arrow[r, "\theta_{\mathrm{Sd}^2K^n}"] \arrow[d, equal] & \mathbf{L}(\mathrm{Sd}^2K^n) \arrow[d, equal] \\
\mathbf{L}(|K_n|) \arrow[r, "|\pi^*_{\mathrm{Sd}^2K^n}|"] \arrow[d, equal] & \mathbf{L}(|\mathrm{Sd}^2K^n|) \arrow[r, "\theta_{\mathrm{Sd}^2K^n}"] & \mathbf{L}(\mathrm{Sd}^2K^n) \arrow[d, equal] \\
\mathbf{L}(|K_n|) \arrow[r, "\theta_{K_n}"'] & \mathbf{L}(K_n) \arrow[r, "\pi^*_{\mathrm{Sd}^2K^n}"'] & \mathbf{L}(\mathrm{Sd}^2K^n)
\end{tikzcd}$$

Let $g : |K_n| \to X_n$ be a continuous map defined by

$$g := f_n \circ (\psi/\mathfrak{S}_n) \circ (\varphi/\mathfrak{S}_n) \circ \chi.$$

Since f_n is a homotopy equivalence and $\psi/\mathfrak{S}_n$, $\varphi/\mathfrak{S}_n$ are homeomorphisms, g is a homotopy equivalence between $|K_n|$ and X_n. Hence we obtain the following:

Lemma 21.2.3. *X_n is a polyhedron with underlying simplicial structure (K_n, g).*

From Diagram 21.2.2, we obtain the following commutative diagram.

Diagram 21.2.4.

$$\begin{tikzcd}
\mathrm{L}(X_n) \arrow[r, "g^*"] \arrow[d, "\pi^*_{X^n}"'] & \mathrm{L}(|K_n|) \arrow[r, "\theta_{K_n}"] & \mathrm{L}(K_n) \arrow[r, "\pi^*_{\mathrm{Sd}^2K^n}"] & \mathrm{L}(\mathrm{Sd}^2K^n) \arrow[d, equal] \\
\mathrm{L}(X^n) \arrow[r, "(f^n \circ \psi)^*"'] & \mathrm{L}(|K^n|) \arrow[r, "\theta_{K^n}"'] & \mathrm{L}(K^n) \arrow[r, "\mathrm{Sd}^2"'] & \mathrm{L}(\mathrm{Sd}^2K^n).
\end{tikzcd}$$

(21.3) Twisted singular homology groups of the configuration space.

We keep the notation of (21.1) and (21.2).

Definition 21.3.1. Let $\mathscr{L}_K$ and $\mathscr{L}_{K_n}$ be simplicial local systems on K and on K_n defined by $\mathscr{L}_K := \theta_K \circ f^* \mathscr{L}_K$ and $\mathscr{L}_{K_n} := \theta_{K_n} \circ g^* \mathscr{L}_{X_n}$, where $g : |K_n| \to X_n$ is defined in (21.2).

Lemma 21.3.2. *Under Assumption* 21.1.3, *we have*

$$\pi^*_{\mathrm{Sd}^2K^n} \mathscr{L}_{K_n} = \mathrm{Sd}^2 \overset{n}{\boxtimes} \mathscr{L}_K,$$

where $\pi^*_{\mathrm{Sd}^2K^n} : \mathrm{Sd}^2K^n \to K_n$ *is the canonical projection.*

Proof.

$$\begin{aligned}
\pi^*_{\mathrm{Sd}^2K^n} \mathscr{L}_{K_n} &= \pi_{\mathrm{Sd}^2K^n} \circ \theta_{K_n} \circ g^* \mathscr{L}_{X_n} && \text{(Definition 21.3.1)} \\
&= \mathrm{Sd}^2 \circ \theta_{K^n} \circ (f^n \circ \psi)^* \circ \pi^*_{X^n} \mathscr{L}_{X_n} && \text{(Diagram 21.2.4)} \\
&= \mathrm{Sd}^2 \circ \theta_{K^n} \circ (f^n \circ \psi)^* \overset{n}{\boxtimes} \mathscr{L}_X && \text{(Assumption 21.1.3)} \\
&= \mathrm{Sd}^2 \circ \theta_{K^n} \circ \psi^* \circ (f^n)^* \overset{n}{\boxtimes} \mathscr{L}_X && \\
&= \mathrm{Sd}^2 \circ \theta_{K^n} \circ \psi^* \overset{n}{\boxtimes} (f^* \mathscr{L}_X) && \text{(Lemma 20.2.1)} \\
&= \mathrm{Sd}^2 \overset{n}{\boxtimes} (\theta_K \circ f^* \mathscr{L}_X) && \text{(Lemma 20.2.2)} \\
&= \mathrm{Sd}^2 \overset{n}{\boxtimes} \mathscr{L}_X && \text{(Definition 21.3.1).}
\end{aligned}$$

Now we are in a position to state the main theorem.

Theorem 21.3.3 *Let X be a polyhedron having (K, f) as its underlying simplicial structure. Let $\mathscr{L}_X$ and $\mathscr{L}_{X_n}$ be simplicial local systems of R-modules on X and on X_n. Assume that*

(1) *$n! \in R^\times$, and*

(2) $\pi^*{}_{X^n}\mathscr{L}_{X_n}=\overset{n}{\boxtimes}\mathscr{L}$, *where* $\pi_{X^n}: X^n \to X_n$ *is the canonical projection.*
Then,
(i) *There exists an R-isomorphism:*

$$H_\bullet^{sing}(X_n, \mathscr{L}_{X_n})\simeq H_\bullet(\overset{n}{\otimes} C_\bullet(K, \mathscr{L}_K))^{\mathfrak{S}_n},$$

where $\mathscr{L}_K$ *is the simplicial local system on* K *defined by* $\mathscr{L}_K: =\theta_K \circ f^* \mathscr{L}_X$.
(ii) *Assume further that*
(3) $Z_\bullet(K, \mathscr{L}_K)$ *is a flat R-module,*
(4) $H_q(K, \mathscr{L}_K)=0$ *if* $q\neq r$, *and*
(5) $H_r(K, \mathscr{L}_K)$ *is a flat R-module.*
Then,

$$H_q^{sing}(X_n, \mathscr{L}_{X_n})=0 \qquad (q\neq nr),$$

and there exists an R-isomorphism:

$$H_{nr}^{sing}(X_n, \mathscr{L}_{X_n})\simeq\begin{cases}\overset{n}{\bigwedge} H_r(K, \mathscr{L}_K) & (r: \text{odd}),\\ \overset{n}{\odot} H_r(K, \mathscr{L}_K) & (r: \text{even}).\end{cases}$$

Proof. Since X_n is a polyhedron having (K_n, g) as its underlying simplicial structure (Lemma 21.2.3), Proposition 18.5.3 implies that

$$H_\bullet^{sing}(X_n, \mathscr{L}_{X_n})\simeq H_\bullet^{simp}(K_n, \mathscr{L}_{K_n}),$$

where $\mathscr{L}_{K_n}$ is defined in Definition 21.3.1. By assumption (2) and Lemma 21.3.2, we have $\pi^*{}_{\mathrm{Sd}^2K^n}\mathscr{L}_{K_n}=\mathrm{Sd}^2\overset{n}{\boxtimes}\mathscr{L}_X$. In view of this and assumption (1), we apply Theorem 15.2.2 to obtain $H_\bullet^{simp}(K_n, \mathscr{L}_{K_n})\simeq H_\bullet(\overset{n}{\otimes} C_\bullet(K, \mathscr{L}_K))^{\mathfrak{S}_n}$. Combining the above isomorphisms, we obtain the first assertion (i). The second assertion (ii) is an immediate consequence of Theorem 15.3.3. This establishes the theorem.

22. Application to hypergeometric functions

(22.1) The complement of a Veronese arrangement.
We use the notation of Part I. For any $z\in M(m+1, n+1)$, we put

$$H_z:=\bigcup_{i=0}^{m} H_i(z),$$

where $H_i(z)$ is the hyperplane in $\mathbf{P}^n$ defined by $\{u\in\mathbf{P}^n;\ x_i(zu)=0\}$; H_z is an arrangement of $(m+1)$-hyperplanes in $\mathbf{P}^n$. Note that $E_z=\mathbf{P}^n\backslash H_z$, i.e., E_z is the complement of the arrangement E_z in $\mathbf{P}^n$.

Definition 22.1.1. An arrangement H_z is called a *Veronese arrangement* if z belongs to the Veronese image (see Definition 3.1.1).

Assume that $z=\mathrm{Vero}(w)$ is the Veronese image of

$$w=\begin{pmatrix}\cdots & \cdots\\ w_{i0} & w_{i1}\\ \cdots & \cdots\end{pmatrix}\in M(m+1,2).$$

We use the following:

Notation 22.1.2. (cf. Definitions 2.1.1, 2.1.2, 2.2.1 and 2.2.2.)

$$\alpha=(\alpha_0,\alpha_1,\cdots,\alpha_m)\in A(m+1,n+1),$$
$$\tilde{\alpha}=(\tilde{\alpha}_0,\tilde{\alpha}_1,\cdots,\tilde{\alpha}_m)\in A(m+1,2)$$

with $\beta_i:=\alpha_i-\tilde{\alpha}_i\in\mathbf{Z}$. Note that $\sum_{i=0}^m\beta_i=1-n$.

$$Y:=E_z(m+1,n+1),\quad \mathscr{L}_Y:=\mathscr{L}_z^\vee(m+1,n+1;\alpha),$$
$$X:=E_w(m+1,2),\quad \mathscr{L}_X:=\mathscr{L}_w^\vee(m+1,2;\tilde{\alpha}),$$

We consider $\mathscr{L}_Y$ and $\mathscr{L}_X$ as singular local systems of R-modules, where R is the ring mentioned in (3.2).

Remark 22.1.3. Note that Y is the complement of the Veronese arrangement H_z in $\mathbf{P}^n$ and $X=\mathbf{P}_1\backslash\{w_0^*,w_1^*,\cdots,w_m^*\}$, where $w_i^*=[w_{i0}:-w_{i1}]\in\mathbf{P}^1$.

We shall show that Y is the configuration space of n-points in X. Let $t={}^t(t_1,t_2,\cdots,t_n)$ be any point in $(\mathbf{P}^1)^n$, where $t_i=[t_{i0}:t_{i1}]$ is the homogeneous coordinate of the ith component $\mathbf{P}^1$ of $(\mathbf{P}^1)^n$. For $j=0,1,\cdots,n$, we put

$$u_j=u_j(t):=\sum_{\#J=j}u_J.$$

Here the sum is taken over all subset J of $\{1,2,\cdots,n\}$ such that $\#J=j$ and u_J is defined by

$$u_J=u_J(t):=\prod_{i\in J}t_{i0}\prod_{i\in J^c}t_{i1},$$

where $J^c=\{1, 2, \cdots, n\}\backslash J$. Moreover, we put $u=u(t)={}^t(u_0, u_1, \cdots, u_n)\in\mathbf{C}^{n+1}$. Since z is the Veronese image of w, we have $z_{ij}=w_{i0}^{n-j}w_{i1}^j$ (cf. Definition 3.1.1). Using this relation, we can easily show the following:

Lemma 22.1.4.

$$\sum_{j=0}^n z_{ij}u_j=\prod_{k=1}^n(w_{i0}t_{k1}+w_{i1}t_{k0}).$$

Corollary 22.1.5. *For any $t\in(\mathbf{P}^1)^n$, $u=u(t)\in\mathbf{C}^{n+1}$ is a nonzero vector.*

Proof. We can take a $w\in M(m+1, 2)$ so that

$$w_0^*\in\mathbf{P}_1\backslash\{t_1, t_2, \cdots, t_n\}, \text{ i.e. } \prod_{k=1}^n(w_{00}t_{k1}+w_{01}t_{k0})\neq 0.$$

By Lemma 22.1.4, we have $\sum_{j=0}^n z_{0j}u_j\neq 0$ and hence $u\neq 0$.

In view of Corollary 22.1.5, the map $\pi:(\mathbf{P}^1)^n\to\mathbf{P}^n$, $t\mapsto u(t)$ is well-defined, where the nonzero vector $u(t)$ is regarded as the homogeneous coordinate of $\mathbf{P}^n$. By using Lemma 22.1.4, we can show that π maps X^n onto Y. Moreover we have the following:

Lemma 22.1.6. *Y is the configuration space of n-points in X and $\pi: X^n\to Y$ is the canonical projection (cf. Definition* 21.1.2).

(22.2) Twisted singular homology groups of $(Y, \mathscr{L}_Y)$.

We consider the twisted singular homology groups of $(Y, \mathscr{L}_Y)$.

Lemma 22.2.1. *Let π: $X^n\to Y$ be the canonical projection (cf. Lemma* 22.1.6). *Then we have* $\pi^*\mathscr{L}_Y=\overset{n}{\boxtimes}\mathscr{L}_X$.

Proof. For any $k\in\mathbf{Z}$, we put $\mathcal{O}_Y(k):=\mathcal{O}_{\mathbf{P}^n}(k)|_Y$ and $\mathcal{O}_X(k):=\mathcal{O}_{\mathbf{P}^1}(k)|_X$. Recall that $f(z,\cdot;m+1, n+1, \alpha)$ is a multi-valued section of $\mathcal{O}_Y(-n-1)$ and determines the local system $\mathscr{L}_Y\subset\mathcal{O}_Y(-n-1)$ (cf. Definitions 2.2.1, 2.2.2 and Notation 22.1.2). Similarly, $f(w,\cdot;m+1, 2, \tilde{\alpha})$ is a multi-valued section of $\mathcal{O}_X(-2)$ and determines the local system $\mathscr{L}_X\subset\mathcal{O}_X(-2)$. We put

$$g(w, t; m+1, \beta):=\prod_{i=0}^n(w_{i0}t_0+w_{i1}t_0)^{\beta_i},$$

where $t={}^t(t_0, t_1)$. Since the β_i's are integers and $\sum\beta_i=1-n$ (cf. Notation 22.1.2), $g(w,\cdot;m+1, \beta)$ is a *single-valued* nowhere vanishing section of $\mathcal{O}_X(1-n)$. Hence $g:\mathcal{O}_X(-2)\to\mathcal{O}_X(-n-1)$, $s\mapsto g(w,\cdot;m+1, \beta)s$ gives an isomorphism between $\mathcal{O}_X(-2)$ and $\mathcal{O}_X(-n-1)$. By using this isomorphism g, we may consider that the local system $\mathscr{L}_X$ is realized in $\mathcal{O}_X(-n-1)$. Then $\mathscr{L}_X$

is determined by the multi-valued section $g(w,\cdot;m+1,\beta)f(w,\cdot;m+1,2,\tilde{\alpha})$ of $\mathcal{O}_X(-n-1)$. By Definition 2.2.1 and Lemma 22.1.4 we obtain

$$\begin{aligned} f(z,u(t);m+1,n+1,\alpha) &= \prod_{i=0}^{m}\Big(\sum_{j=0}^{n} z_{ij}u_j\Big)^{\alpha_i} \\ &= \prod_{i=0}^{m}\prod_{k=1}^{n}(w_{i0}t_{k1}+w_{i1}t_{k0})^{\alpha_i} \\ &= \prod_{k=1}^{n}\prod_{i=0}^{m}(w_{i0}t_{k1}+w_{i1}t_{k0})^{\tilde{\alpha}_i} \\ &= \prod_{k=1}^{n} g(w,t_k;m+1,\beta)f(w,t_k;m+1,2,\tilde{\alpha}). \end{aligned}$$

This implies $\pi^*\mathscr{L}_Y=\overset{n}{\boxtimes}\mathscr{L}_X$.

By Remark 22.1.3, $X=\mathbf{P}^1\backslash\{w_0^*, w_1^*, \cdots, w_m^*\}$. Without loss of generality we may assume $w_0^*=\infty$ and hence $X=\mathbf{C}\backslash\{w_1^*, w_2^*, \cdots, w_m^*\}$. Take $(2m+1)$-points a_i, $b_i(i=1,2,\cdots,m)$ and c in X so that w_i^* is in the interior of $\triangle_i$ and $w_j^*(j\neq i)$ is outside $\triangle_i$, where $\triangle_i=\triangle(c,a_i,b_i)$ is the triangle in $\mathbf{C}$ with vertices c, a_i and b_i. The orientation of $\triangle_i$ is given by $\overrightarrow{ca_ib_i}$. We may assume that this orientation coincides with the anti-clockwise orientation of $\mathbf{C}$. Now we consider the bouquet $K=B_m$ with vertices a_i, $b_i(i=1,2,\cdots,m)$ and c(see Definition 16.1.1). The topological realization $|K|$ of K is the union of m-triangles $\triangle_i$:

$$|K|=\bigcup_{i=1}^{m}\triangle_i.$$

The inclusion map $f:|K|\hookrightarrow X$ is a homotopy equivalence between $|K|$ and X. So X is a polyhedron with underlying simplicial structure (K,f) (cf. Definition 18.5.1). Let $\mathscr{L}_K=\mathscr{L}_e^{\vee}$, where $\mathscr{L}_e$ is the simplicial local system defined in Definition 16.2.1, with $e_i=\exp(2\pi\sqrt{-1}\alpha_i)(i=1,2,\cdots,m)$. Clearly we have the following:

Lemma 22.2.2. $\mathscr{L}_K=\theta_K\circ f^*\mathscr{L}_X$.

The simplicial homology groups of $(K,\mathscr{L}_K)$ were discussed in § 16. Combining Theorems 16.5.2 and 21.3.3, Lemmas 22.2.1 and 22.2.2, we obtain the following:

Lemma 22.2.3. *For any* $\alpha=(\alpha_0,\alpha_1,\cdots,\alpha_m)\in A(m+1,n+1)$, *put* $e_i=\exp(2\pi\sqrt{-1}\alpha_i)$. *Let R be a ring such that*

(1) e_i, $n!\in R^\times(i=1,2,\cdots,m)$,

(2) $I_e=R$, *where I_e is the ideal of R generated by* $1-e_1, 1-e_2,\cdots,1-e_m$. *Then there exists a canonical isomorphism of R-modules:*

$$H_q^{sing}(Y, \mathscr{L}_Y) \simeq \begin{cases} \bigwedge^n H_1^{simp}(K, \mathscr{L}_K) & (q=n) \\ 0 & (q \neq n), \end{cases}$$

where $H_1^{simp}(K, \mathscr{L}_K)$ *is (non-canonically) isomorphic to the R-module*

$$V_e := \{r=(r_1, r_2, \cdots, r_m) \in R^m ; \sum_{i=1}^m r_i(1-e_i)=0\}.$$

Theorem 3.3.3 in Part I is a special case of this theorem.

Remark 22.2.4. In the above situation all the R-modules involved e.g., chain complexes, cycles, boundaries and homologies are projective, and the Hom-functor is exact on the projective modules. Applying the Hom-functor to chain complexes and passing to cochain complexes, we can make a parallel argument for the twisted simplicial (and singular) cohomology. This leads to the result on the twisted cohomology in Theorem 3.3.3.

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