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(2, p, q)-Generations of the Conway group Co_1

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Abstract

A group G is (l, m, n)-generated if it is a quotient group of the triangle group $T(l, m, n)=\langle x, y, z \mid x^l=y^m=z^n=xyz=1\rangle$. In some research papers the problem is posed to find all possible (l, m, n)-generations for the non-abelian finite simple groups. In this paper we partially answer this question for the Conway group Co_1. We find all $(2, p, q)$-generations, p and q are distinct primes, for Co_1.
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1. Introduction

A group G is said to be (l, m, n)-generated if it can be generated by two elements x and y such that $o(x)=l$, $o(y)=m$ and $o(xy)=n$. In this case G is the quotient of the triangle group $T(l, m, n)$ and for any permutation π of S_3, the group G is also $((l)\pi, (m)\pi, (n)\pi)$-generated. Therefore we may assume that $l\leq m\leq n$. By [1] if the simple group G is (l, m, n)-generated, then $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}<1$. Hence for a non-abelian finite simple group G and divisors l, m, n of the order of G such that $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}<1$, it is natural to ask if G is a (l, m, n)-generated group. The motivation for this question came from the calculation of the genus of finite simple groups [17]. It can be shown that the problem of finding the genus of a finite simple group can be reduced to one of generations (for details see [14]).

In a series of papers, [5], [6], [8] and [9] Moori and Ganief established all possible (p, q, r)-generations, where p, q, r are distinct primes, of the sporadic groups J_1, J_2, J_3, HS, McL, Co_3, Co_2, and F_{22}. The motivation for this study is outlined in these papers and the reader is encouraged to consult these papers for background material as well as basic computational techniques.

Throughout this paper we use the same notation as in [7]. In particular, $\Delta(G)=\Delta(lX, mY, nZ)$ denotes the structure constant of G for the conjugacy classes lX, mY, nZ, whose value is the cardinality of the set $\Lambda=\{(x, y) \mid xy=z\}$, where $x\in lX$, $y\in mY$ and z is a fixed element of the conjugacy class nZ. In Table I, we list the values $\Delta(2X, pY, pZ)$, p and q distinct prime divisors of $|Co_1|$, using the character table Co_1. Also, $\Delta^*(G)=\Delta^*_G(lX, mY, nZ)$ and $\Sigma(H_1\cup H_2\cup\cdots\cup H_r)$ denote the number of pairs $(x, y)\in\Lambda$ such that $G=\langle x, y\rangle$ and $\langle x, y\rangle$

Table I : Structure Constants of Co_1

pX	$\Delta(2A,3A,pX)$	$\Delta(2A,3B,pX)$	$\Delta(2A,3C,pX)$	$\Delta(2A,3D,pX)$
$7A$	0	0	0	497
$7B$	0	147	0	0
$11A$	0	11	0	88
$13A$	0	0	0	117
$23A$	0	0	0	92
$23B$	0	0	0	92
pX	$\Delta(2A,5A,pX)$	$\Delta(2A,5B,pX)$	$\Delta(2A,5C,pX)$	$\Delta(2A,7A,pX)$
$7A$	294	7350	0	-
$7B$	147	38220	1764	-
$11A$	44	7810	2178	6204
$13A$	26	754	3900	5837
$23A$	0	1127	3082	2093
$23B$	0	1127	3082	2093
pX	$\Delta(2A,7B,pX)$	$\Delta(2A,11A,pX)$	$\Delta(2A,13A,pX)$	
$11A$	137940	-	-	
$13A$	40716	795652	-	
$23A$	39399	728295	299552	
$23B$	39399	728295	299552	
pX	$\Delta(2B,3A,pX)$	$\Delta(2B,3B,pX)$	$\Delta(2B,3C,pX)$	$\Delta(2B,3D,pX)$
$7A$	0	0	0	17640
$7B$	0	14	0	2352
$11A$	0	11	66	4444
$13A$	0	156	104	6396
$23A$	0	23	138	3841
$23B$	0	23	138	3841

Table I (Continued)

pX	$\Delta(2B,5A,pX)$	$\Delta(2B,5B,pX)$	$\Delta(2B,5C,pX)$	$\Delta(2B,7A,pX)$
$7A$	27048	57624	17640	-
$7B$	14	18816	189336	-
$11A$	220	36300	157938	100342
$13A$	6084	109252	105612	437372
$23A$	345	54119	136988	107778
$23B$	345	54119	136988	107778

pX	$\Delta(2B,7B,pX)$	$\Delta(2B,11A,pX)$	$\Delta(2B,13A,pX)$
$11A$	1357092	-	-
$13A$	2416284	35187100	-
$23A$	1757591	31654003	13257200
$23B$	1757591	31654003	13257200

pX	$\Delta(2C,3A,pX)$	$\Delta(2C,3B,pX)$	$\Delta(2C,3C,pX)$	$\Delta(2C,3D,pX)$
$7A$	0	0	0	0
$7B$	0	1274	392	16464
$11A$	0	462	330	18546
$13A$	0	0	156	19344
$23A$	0	138	828	19182
$23B$	0	138	828	19182

pX	$\Delta(2C,5A,pX)$	$\Delta(2C,5B,pX)$	$\Delta(2C,5C,pX)$	$\Delta(2C,7A,pX)$
$7A$	0	17640	423360	-
$7B$	1274	775572	522732	-
$11A$	1848	467214	634392	490512
$13A$	0	93288	615264	118872
$23A$	2070	282210	711804	563592
$23B$	2070	282210	711804	563592

pX	$\Delta(2C,7B,pX)$	$\Delta(2C,11A,pX)$	$\Delta(2C,13A,pX)$
$11A$	12004608	-	-
$13A$	5577624	139089600	-
$23A$	9083160	163389240	68441376
$23B$	9083160	163389240	68441376

$\subseteq H_i$ (for some $1 \leq i \leq r$), respectively. The number of pairs $(x, y) \in \Lambda$ generating a subgroup H of G will be given by $\Sigma^*(H)$ and the centralizer of a representative of lX will be denoted by $C_G(lX)$. A general conjugacy class of a subgroup H of G with elements of order n will be denoted by nx. Clearly, if $\Delta^*(G) > 0$, then G is (lX, mY, nZ)-generated and (lX, mY, nZ) is called a generating triple for G. The number of conjugates of a given subgroup H of G containing a fix element z is given by $\chi_{N_G(H)}(z)$, where $\chi_{N_G(H)}$ is the permutation character of G with action on the conjugates of H (cf. [15]). In most cases we will calculate this value from the fusion map from $N_G(H)$ into G stored in GAP, [10].

Now we discuss techniques that are useful in resolving generation type questions for finite groups. We begin with a theorem of Scott that, in certain situations, is very effective at establishing non-generations (see [11]).

Theorem 1.1. [*11*] *Let $x_1, x_2, \cdots, x_m$ be elements generating a group G with $x_1 x_2 \cdots x_m = 1$, and let V be an irreducible module for G of dimension n. Let $C_V(x_i)$ denote the fixed point space of $\langle x_i \rangle$ on V, and let d_i be the dimension of $V/C_V(x_i)$. Then $d_1 + \cdots d_m \geq 2n$.*

Further useful results that we shall use are:

Lemma 1.2. [*2*] *Let G be a finite centerless group and suppose lX, mY and nZ are G-conjugacy classes for which $\Delta^*(G) = \Delta^*_G(lX, mY, nZ) < |C_G(z)|$, $z \in nZ$. Then $\Delta^*(G) = 0$ and therefore G is not (lX, mY, nZ)-generated.*

Lemma 1.3. [*16*] *Let G be a finite simple group and H a maximal subgroup of G containing a fixed element x. Then the number h of conjugates of H containing x is $\chi_H(x)$, where χ_H is the permutation character of G with action on the conjugates of H. In particular,*

$$h = \sum_{i=1}^{m} \frac{|C_G(x)|}{|C_H(x_i)|}$$

where $x_1, x_2, \cdots, x_m$ are representatives of the H-conjugacy classes that fuse to the G-conjugacy class of x.

In the present paper we investigate the $(2, p, q)$-generations, p, q distinct primes, for the Conway's group Co_1. We prove the following result:

Theorem. *The Conway's group Co_1 is $(2, p, q)$-generated for all $p, q \in \{3, 5, 7, 11, 13, 23\}$ with $p < q$, except when $(p, q) = (3, 5)$ or $(3, 7)$.*

2. $(2, 3, p)$-Generations for Co_1

We consider triples $(2, 3, p)$, in which $p \geq 5$ is prime. If the group Co_1 is $(2, 3, p)$-generated, then $\frac{1}{2}+\frac{1}{3}+\frac{1}{p}<1$. Thus we only need to consider the cases $p=7$, 11, 13, 23. We deal separately with each case in the following four subsections. For basic properties of the group Co_1 and information on its maximal subgroups the reader is referred to [3], [12] and [13]. It is a well known fact that Co_1 has exactly 22 conjugacy classes of maximal subgroups, as listed in Table II.

2.1. (2, 3, 7)-generation of Co_1. The group Co_1 acts on a 24-dimensional vector space $\overline{\Lambda}$ over $GF(2)$ and has three orbits on the set of non-zero vectors. The stabilizers are the groups Co_2, Co_3 and $2^{11}:M_{24}$, and the permutation character of Co_1 on $\overline{\Lambda}-\{0\}$ is $\chi=1_{Co_2}\uparrow^{Co_1}+1_{Co_3}\uparrow^{Co_1}+1_{2^{11}:M_{24}}\uparrow^{Co_1}$, and using GAP [10], we find that

$$\chi=3\cdot 1a+2\cdot 299a+3\cdot 17250a+3\cdot 80730a+376740a$$
$$+644644a+2055625a+2417415a+2\cdot 5494125a,$$

in which, na denotes the first irreducible character with degree n, in the library of GAP. Now for $g \in Co_1$, the value of $\chi(g)$ is the number of non-zero vectors of $\overline{\Lambda}$ fixed by g from which we can find the dimension δ_{nX} of the fixed space of $g \in nX$. Using the character table of Co_1 we list in Table III the values of $d_{pX}=24-\delta_{pX}$, for all conjugacy classes with prime order representatives. Finally, in Table IV, we list the partial fusion maps of the maximal subgroups into Co_1

Table II : The maximal subgroups of Co_1

Group	*Order*	*Group*	*Order*
Co_2	$2^{18}.3^6.5^3.7.11.23$	$3.Suz.2$	$2^{14}.3^8.5^2.7.11.13$
$2^{11}:M_{24}$	$2^{21}.3^3.5.7.11.23$	Co_3	$2^{10}.3^7.5^3.7.11.23$
$2_+^{1+8}.O_8^+(2)$	$2^{21}.3^5.5^2.7$	$U_6(2).S_3$	$2^{16}.3^7.5.7.11$
$(A_4\times G_2(4)):2$	$2^{15}.3^4.5^2.7.13$	$2^{2+12}:(A_8\times S_3)$	$2^{21}.3^3.5.7$
$2^{4+12}.(S_3\times 3S_6)$	$2^{21}.3^4.5$	$3^2.U_4(3).D_8$	$2^{10}.3^8.5.7$
$3^6:2M_{12}$	$2^7.3^9.5.11$	$(A_5\times J_2):2$	$2^{10}.3^4.5^3.7$
$3^{1+4}.2U_4(2).2$	$2^8.3^9.5$	$(A_6\times U_3(3)):2$	$2^9.3^5.5.7$
$3^{3+4}:2(S_4\times S_4)$	$2^7.3^9$	$A_9\times S_3$	$2^7.3^5.5.7$
$(A_7\times L_2(7)):2$	$2^7.3^3.5.7^2$	$(D_{10}\times(A_5\times A_5).2).2$	$2^7.3^2.5^3$
$5^{1+2}:GL_2(5)$	$2^5.3.5^4$	$5^3:(4\times A_5).2$	$2^5.3.5^4$
$5^2:2A_5$	$2^3.3.5^3$	$7^2:(3\times 2A_4)$	$2^3.3^2.7^2$

Table III : The codimensions $d_{nX}=\mathbf{dim}(V/C_V(nX))$

d_{2A}	d_{2B}	d_{2C}	d_{3A}	d_{3B}	d_{3C}	d_{3D}	d_{5A}
8	12	12	24	12	18	16	24
d_{5B}	d_{5C}	d_{7A}	d_{7B}	d_{11A}	d_{13A}	d_{23A}	d_{23B}
16	20	24	18	20	24	22	22

that we will use later. In this table h denotes the number of conjugates of the maximal subgroup H containing a fixed element z (see Lemma 1.3).

It was first proved in [4] that the group Co_1 is not a Hurwitz group. The following is an alternative proof.

Lemma 2.1. *The group Co_1 is not $(2,3,7)$-generated.*

Proof. We must consider triples $(2X,3Y,7Z)$ for which $\Delta_{Co_1}(2X,3Y,7Z)\neq 0$ and these are given in Table I. If $(2X,3Y,7Z)\neq(2B,3D,7A)$, $(2B,3D,7B)$, $(2C,3B,7B)$ or $(2C,3D,7B)$, then $\Delta_{Co_1}(2X,3Y,7Z)<|C_{Co_1}(7Z)|$. Therefore, by Lemma 1.2, $\Delta^*(G)=0$ and therefore Co_1 is not $(2X,3Y,7Z)$-generated, where $(2X,3Y,7Z)=(2A,3B,7B)$, $(2A,3D,7A)$, $(2B,3B,7B)$ or $(2C,3C,7B)$. Next we consider $(2X,3Y,7Z)$ to be one of the triples $(2B,3D,7B)$, $(2C,3B,7B)$ or $(2C,3D,7B)$. Using Table III, we have,

$$d_{2B}+d_{3D}+d_{7B}=12+16+18=46<48,$$
$$d_{2C}+d_{3B}+d_{7B}=12+12+18=42<48,$$
$$d_{2C}+d_{3D}+d_{7B}=12+16+18=46<48.$$

and hence, by Scott's theorem $(2X,3Y,7Z)$ is a non-generating triple of Co_1.

Finally, we show that Co_1 is not $(2B,3D,7A)$-generated. It is easy to see that, in this case, $\Delta(G)=|C_G(7A)|=17640$. By Table II, $3.Suz.2$ is a maximal subgroup with non-empty intersection with the classes $2B$, $3D$, $7A$ and $\Sigma(3.Suz.2)=1260$. Now we have, $\Delta^*(G)\leq\Delta(G)-1260<17640=|C_G(7A)|$ and non-generating of Co_1 by this triple follows from Lemma 1.2. This completes the proof. □

2.2. (2, 3, 11)-generation of Co_1. We will use the maximal subgroups of Co_1 listed in the ATLAS extensively, especially those with order divisible by 11. We listed in Table IV, fusion maps of these maximal subgroups into Co_1 (obtained from GAP) that will enable us to evaluate $\Delta^*_G(2X,3Y,11Z)$.

Table IV: Partial Fusion Maps into Co_1

Co_2-classes	2a	2b	2c	3a	3b	5a	5b	
$\to Co_1$	2A	2A	2C	3C	3B	5C	5B	
Co_2-classes	7a	11a	23a	23b				
$\to Co_1$	7B	11A	23A	23B				
h	21	6	1	1				
$3.Suz.2$-classes	2a	2b	2c	2d	3a	3b	3c	3d
$\to Co_1$	2A	2B	2B	2C	3A	3A	3B	3B
$3.Suz.2$-classes	3e	3f	5a	5b	7a	11a	13a	
$\to Co_1$	3C	3D	5A	5B	7A	11A	13A	
h					35	1	4	
$2^{11}:M_{24}$-classes	2a	2b	2c	2d	2e	2f	3a	3b
$\to Co_1$	2A	2C	2A	2C	2C	2B	3B	3D
$2^{11}:M_{24}$-classes	5a	7a	7b	11a	23a	23b		
$\to Co_1$	5B	7B	7B	11A	23A	23B		
h		14	14	3	1	1		
Co_3-classes	2a	2b	3a	3b	3c	5a	5b	
$\to Co_1$	2A	2C	3C	3B	3D	5C	5B	
Co_3-classes	7a	11a	11b	23a	23b			
$\to Co_1$	7B	11A	11A	23A	23B			
h	28	6	6	1	1			
$U_6(2).S_3$-classes	2a	2b	2c	2d	2e	3a	3b	3c
$\to Co_1$	2A	2A	2C	2A	2C	3B	3C	3B
$U_6(2).S_3$-classes	3d	3e	3f	3g	5a	7a	11a	
$\to Co_1$	3A	3B	3C	3D	5B	7B	11A	
h						28	2	
$(A_4 \times G_2(4)):2$-classes	2a	2b	2c	2d	2e	2f	3a	3b
$\to Co_1$	2A	2B	2B	2B	2B	2C	3A	3D
$(A_4 \times G_2(4)):2$-classes	3c	3d	3e	5a	5b	7a	13a	
$\to Co_1$	3A	3B	3D	5B	5A	7A	13A	
h						35	1	

Table IV (Continued)

$2_{+}^{1+8}.O_8^+(2)$-classes	2a	2b	2c	2d	2e	2f	2g	2h	3a	3b	3c
$\rightarrow Co_1$	2A	2A	2B	2A	2C	2C	2C	2A	3B	3C	3A
$2_{+}^{1+8}.O_8^+(2)$-classes	3d	3e	5a	5b	5c	7a					
$\rightarrow Co_1$	3B	3C	5A	5B	5C	7B					
h						21					
$2^{2+12}:(A_8 \times S_3)$-classes	2a	2b	2c	2d	2e	2f	2g	2h	2i	2j	2k
$\rightarrow Co_1$	2A	2A	2B	2C	2A	2C	2C	2A	2C	2C	2B
$2^{2+12}:(A_8 \times S_3)$-classes	2l	2m	3a	3b	3c	3d	3e	5a	7a	7b	
$\rightarrow Co_1$	2C	2B	3D	3A	3D	3B	3D	5A	7B	7B	
h										420	
$A_9 \times S_3$-classes	2a	2b	2c	2d	2e	3a	3b	3c	3d	3e	3f
$\rightarrow Co_1$	2A	2B	2B	2C	2C	3D	3A	3D	3C	3D	3B
$A_9 \times S_3$-classes	3g	5a	7a								
$\rightarrow Co_1$	3D	5A	7B								
h			420								
$(A_7 \times L_2(7)):2$-classes	2a	2b	2c	2d	2e	3a	3b	3c	3d	3e	5a
$\rightarrow Co_1$	2A	2B	2B	2B	2C	3D	3A	3D	3B	3D	5A
$(A_7 \times L_2(7)):2$-classes	7a	7b	7c	7d							
$\rightarrow Co_1$	7A	7B	7A	7B							
h			722	750							
$(A_5 \times J_2):2$-classes	2a	2b	2c	2d	2e	2f	3a	3b	3c	3d	3e
$\rightarrow Co_1$	2A	2B	2B	2B	2C	2B	3A	3D	3A	3B	3D
$(A_5 \times J_2):2$-classes	5a	5b	5c	5d	5e	5f	5g	7a			
$\rightarrow Co_1$	5B	5A	5A	5A	5B	5C	5B	7A			
h								21			
$3^2.U_4(3).D_8$-classes	2a	2b	2c	2d	2e	2f	3a	3b	3c	3d	3e
$\rightarrow Co_1$	2A	2B	2C	2A	2C	2B	3B	3A	3B	3C	3A
$3^2.U_4(3).D_8$-classes	3f	3g	3h	5a	7a						
$\rightarrow Co_1$	3C	3B	3D	5B	7A						
h					70						

Lemma 2.2. *The group Co_1 is $(2X, 3Y, 11Z)$-generated if and only if $(Y, Z)=(D, A)$ and $X \in \{B, C\}$.*

Proof. We first prove that Co_1 is $(2X, 3D, 11A)$-generated, where $X \in \{B, C\}$. To do this, assume that $X=B$, then by Table IV, the maximal subgroups of Co_1, up to isomorphisms, that contain $(2B, 3D, 11A)$-generated subgroups are $3.Suz.2$ and $2^{11}:M_{24}$. Furthermore, we calculate $\Delta(Co_1)=4444$, $\Sigma(3.Suz.2)=715$, and $\Sigma(2^{11}:M_{24})=154$. Therefore, $\Delta^*(Co_1) \geq 4444-715-3.154>0$ and so Co_1 is $(2B, 3D, 11A)$-generated. Next, we assume that $X=C$, then by Table IV, the maximal subgroups of Co_1, up to isomorphisms, that contain $(2C, 3D, 11A)$-generated subgroups are $3.Suz.2$, $2^{11}:M_{24}$, Co_3 and $U_6(2).S_3$. We calculate $\Delta(Co_1)=18546$, $\Sigma(3.Suz.2)=\Sigma(U_6(2).S_3)=0$, $\Sigma(2^{11}:M_{24})=968$ and $\Sigma(Co_3)=1342$. Using Table IV, we have, $\Delta^*(Co_1) \geq 18546-6 \cdot 1342-3 \cdot 968>0$, and so Co_1 is $(2C, 3D, 11A)$-generated.

Now we prove the non-generation of Co_1 by other $(2X, 3Y, 11Z)$ triples. First of all, if $(X, Y) \notin \{(2A, 3D), (2B, 3C), (2C, 3B), (2C, 3C)\}$, then $\Delta^*_{Co_1}(2X, 3Y, 11Z) \leq \Delta_{Co_1}(2X, 3Y, 11Z) < |C_{Co_1}(11Z)|$, and by Lemma 1.2, Co_1 is not $(2X, 3Y, 11Z)$-generated. Next, suppose that $(X, Y)=(A, D)$ or (C, B). We apply the Scott's theorem and Table III, have,

$$d_{2A}+d_{3D}+d_{11A}=8+16+20=44<48,$$
$$d_{2C}+d_{3B}+d_{11A}=12+12+20=44<48.$$

Therefore, Co_1 is not $(2A, 3D, 11A)$-and $(2C, 3B, 11A)$-generated. Also, if $(X, Y)=(B, C)$, then the only maximal subgroups of Co_1, up to isomorphisms, that contain $(2B, 3C, 11A)$-generated subgroups are $3.Suz.2$. Furthermore, we calculate $\Delta(Co_1)=66$ and $\Sigma(3.Suz.2)=66$. Thus, $\Delta^*(Co_1) \leq 66-66=0$ and so $\Delta^*(Co_1)=0$. Finally, we show that Co_1 is not $(2C, 3C, 11A)$-generated. The maximal subgroups, up to isomorphisms, that contain $(2C, 3C, 11A)$-generated subgroups are Co_2, $3.Suz.2$, Co_3 and $U_6(2).S_3$. Furthermore, we have $\Delta(Co_1)=330$, $\Sigma(Co_2)=55$ and $\Sigma(3.Suz.2)=\Sigma(Co_3)=\Sigma(U_6(2).S_3)=0$. Using similar argument as in above, the only maximal subgroups of Co_2, up to isomorphisms, that contain $(2C, 3C, 11A)$-generated subgroups are $U_6(2).2$ and that $\Sigma_{Co_2}(U_6(2).2)=0$. Therefore, $\Delta^*(Co_1)=330-6 \cdot 55=0$, proving the result. □

2.3. (2, 3, 13)-generation of Co_1. We will use the maximal subgroups of Co_1 with order divisible by 13. We listed in Table IV, fusion maps of these maximal subgroups into Co_1, stored in GAP. We have,

Lemma 2.3. *The group Co_1 is $(2X, 3Y, 13A)$-generated if and only if $(X, Y) \in \{(B, D), (C, C), (C, D)\}$.*

Proof. Suppose $T=\{(B,B),(B,D),(C,C),(C,D)\}$ and $(X,Y)\not\in T$. Since $\Delta(Co_1)<|C_G(13A)|$, it follows from Lemma 1.2, that Co_1 is not $(2X,3Y,13A)$-generated. Thus, it is enough to investigate four cases that $(X,Y)\in T$ which we treat separately.

Case $(2B,3B,13A)$. We can see that $\Delta(Co_1)=156$ and that $|C_{Co_1}(13A)|=156$. By Table IV, $3.Suz.2$ is only maximal subgroup of Co_1 with non-empty intersection with all the conjugacy classes in this triple and $\Sigma(3.Suz.2)=39$. But, $\Delta^*(Co_1)\leq 156-39=117<156=|C_{Co_1}(13A)|$. Thus by Lemma 1.2, we conclude that Co_1 is not $(2B,3B,13A)$-generated.

Case $(2B,3D,13A)$. In this case, $\Delta(Co_1)=6396$ and maximal subgroups of Co_1 with non-empty intersection with all the conjugacy classes in this triple are, up to isomorphisms, $3.Suz.2$ and $(A_4\times G_2(4)):2$. Our calculations give, $\Sigma(3.Suz.2)=858$ and $\Sigma((A_4\times G_2(4)):2)=0$. This implies that,

$$\begin{aligned}\Delta^*(Co_1)&\geq\Delta(Co_1)-4\Sigma(Suz)-\Sigma((A_4\times G_2(4)):2)\\&=6396-4.858=2964.\end{aligned}$$

Thus $\Delta^*(Co_1)>0$ and whence $(2B,3D,13A)$ is a generating triple for Co_1.

Case $(2C,3C,13A)$. By Table IV, $3.Suz.2$ is only maximal subgroups of Co_1, up to isomorphisms, with non-empty intersection with all the classes in this triple. Since $\Sigma(3.Suz.2)=0$, hence Co_1 is $(2C,3C,13A)$-generated.

Case $(2C,3D,13A)$. The only maximal subgroups of Co_1 that may contain $(2C,3D,13A)$-generated proper subgroups are isomorphic to $3.Suz.2$ and $(A_4\times G_2(4)):2$. Moreover, $\Sigma(3.Suz.2)=\Sigma((A_4\times G_2(4)):2)=0$ and therefore Co_1 is $(2C,3D,13A)$-generated. This completes the proof. □

2.4. (2, 3, 23)-generation of Co_1. Using character table of Co_1, stored in GAP, we can see that Co_1 has exactly two conjugacy classes of elements of order 23, i.e., $23A$ and $23B$, and that $23A^{-1}=23B$. Therefore, for investigation of $(2X,3Y,23Z)$-generation of Co_1, we can assume that $Z=A$.

Lemma 2.4. *The group Co_1 is $(2X,3Y,23Z)$-generated if and only if $X\in\{B,C\}$ and $Y\in\{C,D\}$.*

Proof. Set, $T=\{(A,D),(B,B),(B,C),(B,D),(C,B),(C,C),(C,D)\}$. If $(X,Y)\not\in T$ then $\Delta(Co_1)<|C_{Co_1}(23Z)|$, which implies the non-generation of Co_1 by triples $(2X,3Y,23Z)$. Using Table III, we have,

$$d_{2A}+d_{3D}+d_{23Z}=8+16+22=46<48,$$
$$d_{2B}+d_{3B}+d_{23Z}=12+12+22=46<48,$$
$$d_{2C}+d_{3B}+d_{23Z}=12+12+22=46<48.$$

hence, if $(X, Y)=(A, D)$, (B, B) or (C, B), then by Scott's theorem $(2X, 3Y, 23Z)$ is a non-generating triple of Co_1. Our main proof will consider a number of cases,

Case $(2B, 3C, 23A)$. Since, in this case, there is no maximal subgroups of Co_1 which intersect this triple, hence the group Co_1 is $(2B, 3C, 23A)$ generated.

Case $(2B, 3D, 23A)$. The only maximal subgroups of Co_1 that may contain $(2B, 3D, 23A)$-generated proper subgroups are isomorphic to $2^{11}:M_{24}$. Moreover, $\Delta(Co_1)=3841$ and $\Sigma(2^{11}:M_{24})=322$. On the other hand, $\Delta^*(Co_1)\geq\Delta(Co_1)-322>0$ and so Co_1 is $(2B, 3D, 23A)$-generated.

Case $(2C, 3C, 23A)$. The maximal subgroups of Co_1 that may contain $(2C, 3C, 23A)$-generated proper subgroups are isomorphic to Co_2 and Co_3. We calculate that $\Delta(Co_1)=828$, $\Sigma(Co_2)=69$ and $\Sigma(Co_3)=0$. Thus, $\Delta^*(Co_1)\geq\Delta(Co_1)-\Sigma(Co_2)>0$, and so Co_1 is $(2C, 3C, 23A)$-generated.

Case $(2C, 3D, 23A)$. In this case, $\Delta(Co_1)=19182$ and the maximal subgroups of Co_1 with non-empty intersection with all the conjugacy classes in this triple are, up to isomorphisms, $2^{11}:M_{24}$ and Co_3. We calculate that $\Sigma(2^{11}:M_{24})=414$ and $\Sigma(Co_3)=736$. Our calculations give, $\Delta^*(Co_1)\geq\Delta(Co_1)-736-414>0$, and therefore Co_1 is $(2C, 3D, 23A)$-generated. □

We now summarize the above results in the following theorem.

Theorem 2.5. *The Conway's group Co_1 is $(2, 3, p)$-generated for all $p\in\{11, 13, 23\}$.*

Proof. The proof follows from the lemmas proved in sections 2.1-2.4 and the fact that the triangular group $T(2, 3, 5)\cong A_5$. □

3. (2, 5, p)-Generations for Co_1

We consider triples $(2, 5, p)$, in which $p\geq7$ is prime. We deal separately with each case in the following four subsections.

3.1. (2, 5, 7)-generation of Co_1. In this section we obtain all the $(2, 5, 7)$-generations of the Conway's group Co_1, with the exception of $(2B, 5A, 7A)$ and $(2B, 5B, 7A)$. In Table IV, we list the partial fusion maps of maximal subgroups into Co_1 that we will use later.

Lemma 3.1. *Suppose $(X, Y, Z)\neq(B, A, A)$, (B, B, A). The group Co_1 is $(2X, 5Y, 7Z)$-*

generated if and only if $Y=C$, $X\in\{B, C\}$ *and* $Z\in\{A, B\}$ *or* $(X, Y, Z)=(C, B, A)$.

Proof. Set, $S=\{(A, B, B), (A, C, B), (B, B, B), (C, A, B), (C, B, A), (C, B, B)\}$ and $T=\{(B, C, A), (B, C, B), (C, C, A), (C, C, B)\}$. We calculate that if $(X, Y, Z)\notin S\cup T$ then $\Delta(Co_1)<|C_{Co_1}(7Z)|$. Thus by Lemma 1.2 we conclude that Co_1 is not $(2X, 5Y, 7Z)$-generated. On the other hand, by Table III, we have,

$$d_{2A}+d_{5B}+d_{7B}=42<48,$$
$$d_{2A}+d_{5C}+d_{7B}=46<48,$$
$$d_{2B}+d_{5B}+d_{7B}=46<48,$$
$$d_{2C}+d_{5B}+d_{7B}=46<48.$$

and hence, by Scott's theorem, these are non-generating triples of Co_1. Now, our main proof will consider a number of cases.

Case $(2C, 5A, 7B)$. In this case, $\Delta(Co_1)=1274$ and the maximal subgroups of Co_1 with non-empty intersection with all the conjugacy classes in this triple are, up to isomorphisms, $2_+^{1+8}.O_8^+(2)$, $2^{2+12}:(A_8\times S_3)$, $A_9\times S_3$ and $(A_7\times L_2(7)):2$. We calculate that $\Sigma(2_+^{1+8}.O_8^+(2))=210$, $\Sigma(2^{2+12}:(A_8\times S_3))=224$, $\Sigma(A_9\times S_3)=14$ and $\Sigma((A_7\times L_2(7)):2)=0$. Our calculations give, $\Delta^*(Co_1)\leq 1274-224=1050<|C_{Co_1}(7B)|=1176$, and therefore Co_1 is not $(2C, 5A, 7B)$-generated.

Case $(2C, 5B, 7A)$. The maximal subgroups of Co_1 that may contain $(2C, 5B, 7A)$-generated proper subgroups are isomorphic to $3.Suz.2$, $(A_4\times G_2(4)):2$, $3^2.U_4(3).D_8$ and $(A_5\times J_2):2$. We can see that $\Sigma(3.Suz.2)=\Sigma((A_4\times G_2(4)):2)=\Sigma(3^2.U_4(3).D_8)=\Sigma((A_5\times J_2):2)=0$. Therefore, $\Delta^*(Co_1)=\Delta(Co_1)=17640$, and so Co_1 is $(2C, 5B, 7A)$-generated.

Case $(2B, 5C, 7A)$. In this case $\Delta(Co_1)=17640$ and $(A_5\times J_2):2$ is the unique maximal subgroups of Co_1, up to isomorphisms, with non-empty intersection with each of the classes $2B$, $5C$ and $7A$. However, $\Sigma((A_5\times J_2):2)=0$, proving the $(2B, 5C, 7A)$-generation of Co_1.

Case $(2B, 5C, 7B)$. The only maximal subgroups of Co_1 that may contain $(2B, 5C, 7B)$-generated proper subgroups are isomorphic to $2_+^{1+8}.O_8^+(2)$. Moreover, $\Delta(Co_1)=189336$ and $\Sigma(2_+^{1+8}.O_8^+(2))=336$. On the other hand, $\Delta^*(Co_1)\geq\Delta(Co_1)-21.336>0$ and so Co_1 is $(2B, 5C, 7B)$-generated.

Case $(2C, 5C, 7A)$. In this case $\Delta(Co_1)=423360$ and $(A_5\times J_2):2$ is the unique maximal subgroups of Co_1, up to isomorphisms, with non-empty intersection with each of the classes $2C$, $5C$ and $7A$. However, $\Sigma((A_5\times J_2):2)=0$, proving the $(2C, 5C, 7A)$-generation of Co_1.

Case $(2C, 5C, 7B)$. The maximal subgroups of Co_1 that may contain $(2C, 5C, 7B)$-generated proper subgroups are isomorphic to Co_2, Co_3, and $2_+^{1+8}.O_8^+(2)$. We calculate $\Delta(G)=522732$, $\Sigma(Co_2)=9576$, $\Sigma(Co_3)=1512$ and $\Sigma(2_+^{1+8}.O_8^+(2))=3556$. Our calculations give, $\Delta^*(Co_1)\geq\Delta(Co_1)-21.9576-28.1512-21.3556>0$, and so Co_1 is $(2C, 5C, 7B)$-generated. This completes the proof. □

3.2. (2, 5, 11)-generation of Co_1. In this section we obtain all the $(2X, 5Y, 11Z)$-generation of Co_1. In fact, we prove that,

Lemma 3.2. *The group Co_1 is $(2X, 5Y, 11A)$-generated if and only if $(X, Y) \in \{(A, C), (B, B), (B, C), (C, A), (C, B), (C, C)\}$.*

Proof. Set $R=\{(A, C), (B, B), (B, C), (C, A), (C, B), (C, C)\}$ and $S=\{(B, A), (A, B)\}$. If $(X, Y) \not\in R \cup S$, then we can see that $\Delta(Co_1) < |C_{Co_1}(11A)|$ and so Co_1 is not $(2X, 5Y, 11A)$-generated. We prove that Co_1 is $(2X, 5Y, 11A)$-generated if and only if $(X, Y) \in R$. Our main proof will consider a number of cases.

Case $(2B, 5A, 11A)$. In this case $\Delta(Co_1)=220$ and $3.Suz.2$ is the unique maximal subgroups of Co_1, up to isomorphisms, with non-empty intersection with each of the classes $2B$, $5A$ and $11A$. We calculate, $\Delta(Co_1)=\Sigma(3.Suz.2)=220$. This shows that $\Delta^*(Co_1)=0$, and so Co_1 is not $(2B, 5A, 11A)$-generated.

Case $(2A, 5B, 11A)$. Using Table III, we have $d_{2A}+d_{5B}+d_{11A}=44<48$ and hence by Scott's theorem Co_1 is not $(2A, 5B, 11A)$-generated.

Case $(2A, 5C, 11A)$. The maximal subgroups of Co_1 that may contain $(2A, 5C, 11A)$-generated proper subgroups are isomorphic to Co_2 and Co_3. We calculate $\Delta(G)=2178$, $\Sigma(Co_2)=165$ and $\Sigma(Co_3)=88$. Our calculations give, $\Delta^*(Co_1) \geq \Delta(Co_1)-6.165-6.88>0$, and so Co_1 is $(2A, 5C, 11A)$-generated.

Case $(2B, 5B, 11A)$. From the list of maximal subgroups of Co_1 we observe that, up to isomorphisms, $3.Suz.2$ and $2^{11}:M_{24}$ are the only maximal subgroups of Co_1 that admit $(2B, 5B, 11A)$-generated subgroups. From the structure constant we calculate $\Delta(Co_1)=36300$, $\Sigma(3.Suz.2)=2277$ and $\Sigma(2^{11}:M_{24})=1628$. Thus, $\Delta^*(Co_1) \geq \Delta(Co_1)-2277-3.1628>0$. This shows that the Conway's group Co_1 is $(2B, 5B, 11A)$-generated.

Case $(2B, 5C, 11A)$. In this case $\Delta(Co_1)=157938$ and there is no maximal subgroups of Co_1 that contains $(2B, 5C, 11A)$-generated proper subgroups. Therefore, $\Delta^*(Co_1)=\Delta(Co_1)>0$, and so Co_1 is $(2B, 5C, 11A)$-generated.

Case $(2C, 5A, 11A)$. In this case $\Delta(Co_1)=1848$ and $3.Suz.2$ is the unique maximal subgroups of Co_1, up to isomorphisms, with non-empty intersection with each of the classes $2C$, $5A$ and $11A$. Now since, $\Sigma(3.Suz.2)=0$, hence $\Delta^*(Co_1)=\Delta(Co_1)$ and Co_1 is $(2C, 5A, 11A)$-generated.

Case $(2C, 5B, 11A)$. From the list of maximal subgroups of Co_1 we observe that, up to isomorphisms, Co_2, $3.Suz.3$, $2^{11}:M_{24}$, Co_3 and $U_6(2).S_3$ are the only maximal subgroups of Co_1 with order divisible by 11. We calculate $\Delta(Co_1)=467214$, $\Sigma(3.Suz.2)=0$, $\Sigma(Co_2)=41712$, $\Sigma(2^{11}:$

M_{24})=6050, $\Sigma(Co_3)=15026$, and $\Sigma(U_6(2).S_3)=17524$. Also, a fixed element of order 7 is contained in 6 conjugates of Co_2, 3 conjugates of $2^{11}:M_{24}$, 12 conjugates of Co_3 and 2 conjugates of $U_6(2).S_3$. Therefore, $\Delta^*(Co_1)\geq\Delta(Co_1)-6.41712-3.6050-12.15026-2.7524>0$, which proves the group Co_1 is $(2C, 5B, 11A)$-generated.

Case $(2C, 5C, 11A)$. Amongst the maximal subgroups of Co_1 with order divisible by $2\times 5\times 11$, the only subgroups with non-empty intersection with any conjugacy class in this triple are isomorphic to Co_2 and Co_3. We can see that $\Delta(Co_1)=634392$, $\Sigma(Co_2)=10428$ and $\Sigma(Co_3)=0$. Our calculations give, $\Delta^*(Co_1)\geq\Delta(Co_1)-6\cdot 10428>0$, proving the result. □

3.3. (2, 5, 13)-generations of Co_1. In this section we obtain all of the $(2X, 5Y, 13A)$-generations of Co_1. In fact, we prove that,

Lemma 3.3. *The group Co_1 is $(2X, 5Y, 13A)$-generated if and only if $(X, Y)\in\{(A, B), (A, C), (B, A), (B, B), (B, C), (C, B), (C, C)\}$.*

Proof. Set $T=\{(A, B), (A, C), (B, A), (B, B), (B, C), (C, B), (C, C)\}$. If $(X, Y)\notin T$. then $\Delta(Co_1)<|C_{Co_1}(13A)|$ and by Lemma 1.2, Co_1 is not $(2X, 3Y, 13A)$-generated. Moreover, if $(X, Y)=(A, C)$, (B, C) or (C, C), then $\Delta(Co_1)\neq 0$ and Co_1 does not have any maximal subgroup which contains these triples. Therefore, for these triples Co_1 is $(2X, 3Y, 13A)$-generated. Now we continue the proof case by case.

Case $(2A, 5B, 13A)$. Amongst the maximal subgroups of Co_1 with order divisible by $2\times 5\times 13$, the only subgroups with non-empty intersection with any conjugacy class in this triple are isomorphic to $3.Suz.2$ and $(A_4\times G_2(4)):2$. We can see that $\Delta(Co_1)=754$, $\Sigma(3.Suz.2)=130$ and $\Sigma((A_4\times G_2(4)):2)=26$. Our calculations give, $\Delta^*(Co_1)\geq\Delta(Co_1)-4.130-26>0$, proving the generation of Co_1 by this triple.

Case $(2B, 5A, 13A)$. In this case, $\Delta(Co_1)=6084$ and the only maximal subgroups with non-empty intersection with any conjugacy class in this triple are isomorphic to, $3.Suz.2$ and $(A_4\times G_2(4)):2$. We calculate, $\Sigma(3.Suz.2)=1053$ and $\Sigma((A_4\times G_2(4)):2)=416$. Our calculations give, $\Delta^*(G)\geq\Delta(G)-4.1053-416>0$. Therefore, Co_1 is $(2B, 5A, 13A)$-generated.

Case $(2B, 5B, 13A)$. In this case, $\Delta(Co_1)=109252$ and the only maximal subgroups with non-empty intersection with any conjugacy class in this triple are isomorphic to, $3.Suz.2$ and $(A_4\times G_2(4)):2$. We calculate, $\Sigma(3.Suz.2)=3484$ and $\Sigma((A_4\times G_2(4)):2)=416$. Our calculations give, $\Delta^*(G)\geq\Delta(G)-4.3484-416>0$. Therefore, Co_1 is $(2B, 5B, 13A)$-generated.

Case $(2C, 5B, 13A)$. From the list of maximal subgroups of Co_1 we observe that, up to isomorphisms, $3.Suz.2$ and $(A_4\times G_2(4)):2$ are the only maximal subgroups of Co_1 that admit $(2C, 5B, 13A)$-generated subgroups. From the structure constants we calculate $\Delta(Co_1)=93288$, $\Sigma(3.Suz.2)=0$ and $\Sigma((A_4\times G_2(4)):2)=416$. Thus, $\Delta^*(Co_1)\geq\Delta(Co_1)-416>0$. This shows that the Conway's group Co_1 is $(2C, 5B, 13A)$-generated and the lemma is proved. □

3.4. (2, 5, 23)-generation of Co_1. In this section we obtain all of the $(2X, 5Y, 23Z)$-generation of the Conway's group Co_1. Since $23A^{-1}=23B$, hence, the group Co_1 is $(2X, 5Y, 23A)$-generated if and only if it is $(2X, 5Y, 23B)$-generated. Therefore, it is enough to investigate the $(2X, 5Y, 23A)$-generation of Co_1.

Lemma 3.4. *The group Co_1 is $(2X, 5Y, 23Z)$-generated if and only if $(X, Y)\in\{(A, C), (B, A), (B, B), (B, C), (C, A), (C, B), (C, C)\}$.*

Proof. Set $T=\{(A, C), (B, A), (B, B), (B, C), (C, A), (C, B), (C, C)\}$. If $(X, Y)\notin T\cup\{(A, B)\}$, then $\Delta(Co_1)<|C_{Co_1}(23A)|$, and so Co_1 is not $(2X, 5Y, 23A)$-generated. Also, if $(X, Y)=(A, B)$, then $d_{2A}+d_{5B}+d_{23A}=46$ and by Scott's theorem $(2A, 5B, 23A)$ is not a generating triple for Co_1. We now assume that $(X, Y)\in T$ and show that Co_1 is $(2X, 5Y, 23A)$-generated. First of all, if $(X, Y)=(B, A)$, (B, C) or (C, A), then there is no maximal subgroup of Co_1 that admit $(2X, 5Y, 23A)$-generated subgroups. Now for such a triple $(2X, 5Y, 23A)$, $\Delta(Co_1)\neq 0$ and so Co_1 is $(2X, 5Y, 23A)$-generated. We investigate another triples case by case.

Case $(2A, 5C, 23A)$. From the list of maximal subgroups of Co_1 we observe that, up to isomorphisms, Co_2 and Co_3 are the only maximal subgroups of Co_1 that admit $(2A, 5C, 23A)$-generated subgroups. From the structure constants we calculate $\Delta(Co_1)=3082$, $\Sigma(Co_2)=345$ and $\Sigma(Co_3)=115$. Thus, $\Delta^*(Co_1)\geq\Delta(Co_1)-345-115>0$. This shows that the Conway's group Co_1 is $(2A, 5C, 23A)$-generated.

Case $(2B, 5B, 23A)$. Amongst the maximal subgroups of Co_1 with order divisible by $2\times 5\times 23$, the only maximal subgroups with non-empty intersection with any conjugacy class in this triple are isomorphic to $2^{11}:M_{24}$. We can see that $\Delta(Co_1)=54119$, $\Sigma(2^{11}:M_{24})=1748$. Our calculations give, $\Delta^*(Co_1)\geq\Delta(Co_1)-1748>0$, proving the generation of Co_1 by this triple.

Case $(2C, 5B, 23A)$. In this case, $\Delta(Co_1)=282210$ and the only maximal subgroups with non-empty intersection with any conjugacy class in this triple are isomorphic to Co_2, Co_3 and $2^{11}:M_{24}$. We calculate, $\Sigma(Co_2)=37720$, $\Sigma(Co_3)=6716$ and $\Sigma((2^{11}:M_{24})=4002$. Our calculations give, $\Delta^*(G)\geq\Delta(G)-37720-4002-6716>0$. Therefore, Co_1 is $(2C, 5B, 23A)$-generated.

Case $(2C, 5C, 23A)$. From the list of maximal subgroups of Co_1 we observe that, up to isomorphisms Co_2 and Co_3 are the only maximal subgroups of Co_1 that admit $(2C, 5C, 23A)$-generated subgroups. From the structure constants we calculate $\Delta(Co_1)=711804$, $\Sigma(Co_2)=9200$ and $\Sigma(Co_3)=1955$. Thus, $\Delta^*(Co_1)\geq\Delta(Co_1)-9200-1955>0$. This shows that the Conway's group Co_1 is $(2C, 5C, 23A)$-generated, which completes the proof. □

We now summarize the above results in the following theorem:

Theorem 3.5. *The Conway's group Co_1 is $(2, 5, p)$-generated for all $p \in \{7, 11, 13, 23\}$.*

Proof. The proof follows from the lemmas proved in sections 3.1-3.4. □

4. $(2, 7, p)$-Generations for Co_1

We will use the maximal subgroups of Co_1 listed in the ATLAS extensively, especially those with order divisible by 11, 13 and 23. We listed in Table IV fusion maps of these maximal subgroups into Co_1 (obtained from GAP) that will enable us to calculate $\Delta^*(2X, 7Y, pZ)$, for $p=11$, 13 and 23.

4.1. $(2, 7, 11)$-generation of Co_1. The group Co_1 has exactly five maximal subgroups with order divisible by 11 which are isomorphic to Co_2, Co_3, $3.Suz.2$, $2^{11}:M_{24}$ or $U_6(2).S_3$. In Table IV we calculate the partial fusion maps of these subgroups into Co_1, obtained from GAP. Using this calculation, we have,

Lemma 4.1. *The group Co_1 is $(2X, 7Y, 11A)$-generated if and only if $(X, Y) \in \{(A, A), (B, A), (B, B), (C, A), (C, B)\}$.*

Proof. Set $T=\{(A, A), (B, A), (B, B), (C, A), (C, B)\}$. If $(X, Y) \notin T \cup \{(A, B)\}$, then $\Delta(Co_1) < |C_{Co_1}(11A)|$, and so Co_1 is not $(2X, 7Y, 11A)$-generated. Also, if $(X, Y)=(A, B)$, then $d_{2A}+d_{7B}+d_{11A}=46<48$ and by Scott's theorem $(2A, 7B, 11A)$ is not a generating triple. Our proof will need to consider a number of cases.

Case $(2A, 7A, 11A)$. The only maximal subgroups of Co_1 with non-empty intersection with the conjugacy classes in this triple is, up to isomorphisms, $3.Suz.2$. We calculate $\Delta(Co_1)=6204$ and $\Sigma(3.Suz.2)=726$. Now since $\Delta^*(Co_1)=6204-726=5478>0$, Co_1 is $(2A, 7A, 11A)$-generated.

Case $(2B, 7A, 11A)$. In this case, $\Delta(Co_1)=100342$ and $3.Suz.2$ is the only maximal subgroups of Co_1, up to isomorphisms, with non-empty intersection with this triple. We can see that $\Sigma(3.Suz.2)=8008$, and so $\Delta^*(Co_1) \geq 100342-8008>0$. This shows the generation of Co_1 by this triple.

Case $(2B, 7B, 11A)$. From the list of maximal subgroups of Co_1, we observe that, up to isomorphisms, $2^{11}:M_{24}$ is the only maximal subgroups of Co_1 that admit $(2B, 7B, 11A)$-generated subgroups. From the character table of Co_1 and Table IV, we can see that $\Delta(Co_1)=1357092$ and $\Sigma(2^{11}:M_{24})=9504$. Therefore, $\Delta^*(Co_1) \geq \Delta(Co_1)-3.9504>0$, which proves the generation of Co_1 by this triple.

Case $(2C, 7A, 11A)$. Amongst the maximal subgroups of Co_1 with order divisible by $2\times$

7×11, up to isomorphisms, $3.Suz.2$ is the only maximal subgroup with non-empty intersection with the conjugacy classes of this triple. Now, from the character table of Co_1 and Table III, we have, $\Delta(Co_1)=490512$ and $\Sigma(3.Suz.2)=0$. This shows the generation of Co_1 by this triple.

Case $(2C, 7B, 11A)$. The only maximal subgroups of Co_1 that may contain $(2C, 7B, 11A)$-generated subgroups, are isomorphic to Co_2, Co_3, $2^{11}:M_{24}$ and $U_6(2).S_3$. We easily calculate the structure constant $\Delta(Co_1)=12004608$, $\Sigma(Co_2)=472384$, $\Sigma(Co_3)=114532$, $\Sigma(2^{11}:M_{24})=33968$ and $\Sigma(U_6(2).S_3)=35640$. Thus, $\Delta^*(Co_1)\geq\Delta(Co_1)-6\cdot 472384-12\cdot 114532-3\cdot 33968-2\cdot 35640>0$, proving the result. □

4.2. (2, 7, 13)-generation of Co_1. The group Co_1 has exactly two maximal subgroup with order divisible by 13, which are isomorphic to $3.Suz.2$ and $(A_4\times G_2(4)):2$. In Table IV, we calculate the partial fusion map of these subgroups, obtained from GAP. Using this calculation, we have,

Lemma 4.2. *The group Co_1 is $(2X, 7Y, 13A)$-generated if and only if $X\in\{A, B, C\}$ and $Y\in\{A, B\}$.*

Proof. We first assume that $(X, Y)\notin\{A, B, C\}\times\{A, B\}$. Then, $\Delta(Co_1)<|C_{Co_1}(13A)|$ and by Lemma 1.2, Co_1 is not $(2X, 7Y, 13A)$-generated. We prove the generation of Co_1 by other triples. If $(X, Y)=(A, B)$, (B, B) or (C, B), then $\Delta(Co_1)\neq 0$ and there is no maximal subgroups which contains $(2X, 7Y, 13A)$-generated subgroups. Therefore, Co_1 is $(2X, 7Y, 13A)$-generated. We continue our proof case by case.

Case $(2A, 7A, 13A)$. The only maximal subgroups of Co_1 that may contain $(2A, 7A, 13A)$-generated subgroups, are isomorphic to $3.Suz.2$. and $(A_4\times G_2(4)):2$. We easily calculate the structure constant $\Delta(Co_1)=5837$, $\Sigma(3.Suz.2)=689$ and $\Sigma((A_4\times G_2(4)):2)=221$. Hence, $\Delta^*(Co_1)\geq\Delta(Co_1)-4.689-221>0$, and the $(2A, 7A, 13A)$-generation of Co_1 follows.

Case $(2B, 7A, 13A)$. Amongst the maximal subgroups of Co_1 with order divisible by $2\times 7\times 13$, up to isomorphisms, $3.Suz.2$ and $(A_4\times G_2(4)):2$ are the only maximal subgroups with non-empty intersection with the conjugacy classes of this triple. Now, from the character table of Co_1 and Table IV, we have, $\Delta(Co_1)=437372$, $\Sigma(3.Suz.2)=17303$ and $\Sigma((A_4\times G_2(4)):2)=3120$. Hence, $\Delta^*(Co_1)\geq\Delta(Co_1)-4.17303-3120>0$, and the $(2A, 7A, 13A)$-generation of Co_1 follows.

Case $(2C, 7A, 13A)$. From the list of maximal subgroups of Co_1, Table II, we observe that, up to isomorphisms, $3.Suz.2$ and $(A_4\times G_2(4)):2$ are the only maximal subgroups of Co_1 that admit $(2C, 7A, 13A)$-generated subgroups. But, $\Delta(Co_1)=118872$ and $\Sigma(3.Suz.2)=\Sigma((A_4\times G_2(4)):2)=0$. Therefore, Co_1 is $(2C, 7A, 13A)$-generated. This completes the proof. □

4.3. (2, 7, 23)-generation of Co_1. From the list of maximal subgroups of Co_1, Table II, we can see that Co_2, Co_3 and $2^{11}:M_{24}$, are the only maximal subgroups with order divisible by 23. We use the fusion maps of these subgroups into Co_1, to investigate that $(2X, 7Y, 23Z)$ is a generation triple for Co_1 or not. Also, since $23A^{-1}=23B$, hence it is enough to consider the case $23A$. Using this information, we have,

Lemma 4.3. *The group Co_1 is $(2X, 7Y, 23A)$-generated if and only if $X \in \{A, B, C\}$ and $Y \in \{A, B\}$.*

Proof. We first assume that $(X, Y) \not\in \{A, B, C\} \times \{A, B\}$. Then, $\Delta(Co_1) < |C_{Co_1}(23A)|$ and by Lemma 1.2, Co_1 is not $(2X, 7Y, 23A)$-generated. We prove the generation of Co_1 by other triples. If $(X, Y)=(A, A)$, (B, A) or (C, A), then $\Delta(Co_1) \neq 0$ and there is no maximal subgroups which contains $(2X, 7Y, 23A)$-generated subgroups. Therefore, Co_1 is $(2X, 7Y, 23A)$-generated. We continue our proof case by case.

Case $(2A, 7B, 23A)$. The only maximal subgroups of Co_1 that may contain $(2A, 7B, 23A)$-generated subgroups, are isomorphic to Co_2, Co_3 and $2^{11}:M_{24}$. We easily calculate the structure constant $\Delta(Co_1)=39399$, $\Sigma(Co_2)=14168$, $\Sigma(Co_3)=3197$ and $\Sigma(2^{11}:M_{24})=2024$. Hence, $\Delta^*(Co_1) \geq \Delta(Co_1)-3197-2024-14168>0$, and the $(2A, 7B, 23A)$-generation of Co_1 follows.

Case $(2B, 7B, 23A)$. Amongst the maximal subgroups of Co_1 with order divisible by $2\times 7\times 23$, up to isomorphisms, $2^{11}:M_{24}$ is the only maximal subgroup with non-empty intersection with the conjugacy classes of this triple. Now, from the character table of Co_1 and Table IV, we have, $\Delta(Co_1)=1757591$, $\Sigma(2^{11}:M_{24})=13616$. Hence, $\Delta^*(Co_1) \geq \Delta(Co_1)-13616>0$, and the $(2B, 7B, 23A)$-generation of Co_1 follows.

Case $(2C, 7B, 23A)$. From the list of maximal subgroups of Co_1, Table II, we observe that, up to isomorphisms, Co_2, Co_3 and $2^{11}:M_{24}$ are the only maximal subgroups of Co_1 that admit $(2C, 7B, 23A)$-generated subgroups. But, $\Delta(Co_1)=9083160$, $\Sigma(Co_2)=471960$, $\Sigma(Co_3)=56971$ and $\Sigma(2^{11}:M_{24})=27784$. Hence, $\Delta^*(Co_1) \geq \Delta(Co_1)-56971-27784-471960>0$, and so Co_1 is $(2C, 7B, 23A)$-generated. This completes the proof. □

We now summarize the above results in the following theorem:

Theorem 4.4. *The Conway's group Co_1 is $(2, 7, p)$-generated for all $p \in \{11, 13, 23\}$.*

Proof. The proof follows from the lemmas proved in sections 4.1-4.3. □

5. (2, 11, 13)-, (2, 11, 23)- and (2, 13, 23)-Generations for Co_1

In this section we consider the $(2, p, q)$-generation of Co_1, for $(p, q)=(11, 13)$, $(11, 23)$ and $(13, 23)$.

Lemma 5.1. *The group Co_1 is $(2X, pA, qY)$-generated, for $(p, q)=(11, 13)$, $(11, 23)$ or $(13, 23)$, if and only if $X\in\{A, B, C\}$ and $Y\in\{A, B\}$.*

Proof. We first show that Co_1 is $(2X, 13A, 23Y)$-generated. To do this, it is easy to see that $\Delta(Co_1)\neq0$ and there is no maximal subgroups of Co_1 with non-empty intersection with this triple. This shows the $(2X, 13A, 23Y)$-generation of Co_1. Next we show the $(2X, 11A, 13A)$ and $(2X, 11A, 23A)$-generation of Co_1. We now consider the number of cases.

Case $(2X, 11A, 13A)$. From the list of maximal subgroups of Co_1, Table II, we can see that $3.Suz.2$ is the only maximal subgroups of Co_1 with order divisible by $2\times11\times13$. Now, it is easy to see that if $X=A$, B or C then $\Delta^*(Co_1)\geq\Delta(Co_1)-4\Sigma(3.Suz.2)>0$, which proves the generation of Co_1 by this triple.

Case $(2A, 11A, 23A)$. Amongst the maximal subgroups of Co_1 with order divisible by $2\times11\times23$, up to isomorphisms, Co_2, Co_3 and $2^{11}:M_{24}$ are the only maximal subgroup of Co_1 with non-empty intersection with the conjugacy classes of this triple. We calculate $\Delta(Co_1)=728295$, $\Sigma(Co_2)=98923$, $\Sigma(Co_3)=15456$ and $\Sigma(2^{11}:M_{24})=10304$. Hence, $\Delta^*(Co_1)\geq\Delta(Co_1)-98923-15456-10304>0$. Thus, Co_1, is $(2A, 11A, 23A)$-generated.

Case $(2B, 11A, 23A)$. The only maximal subgroup of Co_1 with non-empty intersection with the conjugacy classes of this triple is isomorphic to $2^{11}:M_{24}$. We easily calculate $\Delta(Co_1)=31654003$ and $\Sigma(2^{11}:M_{24})=38272$. Hence, $\Delta^*(Co_1)\geq\Delta(Co_1)-38272>0$ and generation of Co_1 by this triple follows.

Case $(2C, 11A, 23A)$. The maximal subgroups of Co_1 with non-empty intersection with all the classes in this triple are, up to isomorphisms, Co_2, Co_3 and $2^{11}:M_{24}$. We calculate $\Delta(Co_1)=163389240$, $\Sigma(Co_2)=2605624$, $\Sigma(Co_3)=241592$ and $\Sigma(2^{11}:M_{24})=110400$. Hence, $\Delta^*(Co_1)\geq\Delta(Co_1)-2605624-110400-241592>0$, and the result follows. □

We are now ready to state the main results of this paper:

Theorem. *The Conway's group Co_1 is $(2, p, q)$-generated for all $p, q\in\{3, 5, 7, 11, 13, 23\}$ with $p<q$, except when $(p, q)=(3, 5)$ or $(3, 7)$.*

Proof. The proof follows from the Theorems 2.5, 3.5, 4.4 and Lemma 5.1. □

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