On plurigenera of hypersurface purely elliptic singularities

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Abstract

In this paper, we describe some properties of plurigenera defined for hypersurface isolated singularities, which have relations with more fundamental, well-known plurigenera \( \{ \gamma_n \} \), especially in case singularities are so-called purely elliptic singularities.

1. Introduction

In this paper, we will discuss the plurigenera \( \{ \gamma_n(X, x) \}_{n \in \mathbb{N}} \) and related subjects of hypersurface isolated singularities of dimension greater than or equal to two.

Knöller [11] showed that for a normal Gorenstein isolated singularity \( (X, x) \) and a resolution of the singularity \( \pi : \tilde{X} \to X \), \( (\omega_{\tilde{X}}^n)_{\pi} / (\pi_* \omega_X^m)_{\pi} \) is a finite dimensional vector space over the complex number field \( \mathbb{C} \) and that the dimension \( \gamma_n(X, x) \) of it is independent of the choice of a resolution \( \pi \) for any positive integer \( m \), where \( \omega_X \) (resp. \( \omega_{\tilde{X}} \)) is the canonical sheaf of \( X \) (resp. \( \tilde{X} \)). Hence \( \{ \gamma_n(X, x) \}_{n \in \mathbb{N}} \) are regarded as invariants of the singularity \( (X, x) \).

Indeed, \( \{ \gamma_n(X, x) \}_{n \in \mathbb{N}} \) characterize a singularity. For example, for a two-dimensional normal Gorenstein singularity \( (X, x) \), the condition that \( \gamma_m(X, x) = 0 \) for every \( m \) is equivalent to that \( (X, x) \) is a famous rational double point, that is, \( (X, x) \) is analytically equivalent to the hypersurface singularity at the origin \( O \) of \( \mathbb{C}^2 \) defined by one of the following equations:

\[
A_n(n \geq 1) : x^n + y^n + z^{n+1} = 0; \quad D_n(n \geq 4) : x^2 + y^2 + z^{n-1} = 0; \\
E_6 : x^2 + y^3 + z^4 = 0; \quad E_7 : x^2 + y^3 + yz^2 = 0; \quad E_8 : x^2 + y^3 + z^2 = 0.
\]

On the other hand, we have another set of plurigenera \( \{ \delta_m(X, x) \}_{m \in \mathbb{N}} \) for a normal Gorenstein isolated singularity \( (X, x) \) — see Definition 4.1 in this paper. These give different characterization of singularities. For example, for a normal two-dimensional singularity \( (X, x) \), the condition that \( \delta_m(X, x) = 0 \) for every \( m \) is equivalent to that \( (X, x) \) is a quotient singularity, that is, \( (X, x) \) is analytically equivalent to a quotient \( (\mathbb{C}^2/G, O) \) of \( \mathbb{C}^2 \) by a finite subgroup \( G \) of \( GL(2, \mathbb{C}) \).

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\{\gamma_m(X, x)\}_{m \in \mathbb{N}} \text{ and } \{\delta_m(X, x)\}_{m \in \mathbb{N}} \text{ are fundamental invariants of normal isolated singularities, which contain the most fundamental invariant of normal isolated singularities: the geometric genus } p_g = \gamma_1 = \delta_1. \text{ Including higher dimensional cases, many facts are known about these two plurigenera. We refer the reader to Ishii [7] on this subject.}

As in the study of algebraic varieties, in the study of singularities, "hypersurface" singularities are investigated more deeply than the general cases. But now we cannot calculate \(\gamma_m\) explicitly, while, for example, we can calculate the Hodge numbers of toric hypersurfaces [1]. Our aim in this paper is to make some progress in this subject.

Let us give an outline of this paper:

In § 3, we will define one more set of plurigenera \(\{\tilde{\gamma}_m(X, x)\}_{m \in \mathbb{N}}\) for a hypersurface isolated singularity \((X, x)\) and show that \(\gamma_m(X, x) \leq \tilde{\gamma}_m(X, x)\) for every \(m\). In [15], a formula to calculate \(\delta_m(X, x)\) for a hypersurface isolated singularity \((X, x)\) is already given. The definitions of \(\{\tilde{\gamma}_m(X, x)\}_{m \in \mathbb{N}}\) and the proof of the inequalities above are inspired by this formula. And we have a conjecture that under a suitable condition, \(\gamma_m(X, x) = \tilde{\gamma}_m(X, x)\) holds for every \(m\) although we cannot identify such a condition now — see Remark 3.4.3.

In § 4, we will give several facts about \(\{\tilde{\gamma}_m(X, x)\}_{m \in \mathbb{N}}\) in case \((X, x)\) is a so-called purely elliptic singularity, which is a singularity such that \(\delta_m(X, x) = 1\) for every \(m\), and will give some example to \(\{\gamma_m\}_{m \in \mathbb{N}}\) and \(\{\tilde{\gamma}_m\}_{m \in \mathbb{N}}\), where \(\gamma_m = \tilde{\gamma}_m\) holds for every \(m\). Finally, we will give a note on \(\{\tilde{\gamma}_m\}_{m \in \mathbb{N}}\) for purely elliptic singularities of a special type.

2. Preliminaries

2.1. General settings. Let \((X, x) = (V(f), 0)\) be a hypersurface isolated singularity of dimension \(r\) defined by a polynomial \(f = \sum_{a \in \mathbb{Z}^{r+1}} a \cdot z^a \in \mathbb{C}[z_0, z_1, \ldots, z_r]\), where \(x = 0\) is the origin of the \((r+1)\)-dimensional affine space \(\mathbb{C}^{r+1}\) and \(z^a = \prod_{i \in \mathbb{Z}} z_i^{a_i}\) for each nonnegative integral vector \(a = (a_0, a_1, \ldots, a_r)\). We always assume that \(r\) is greater than or equals to two.

In this paper, we also always assume that the polynomial \(f\) is nondegenerate: The convex hull of the set \(\cup_{a \in \mathbb{N}} (\lambda + (\mathbb{R}_{>0})^{r+1})\) is called the Newton diagram of \(f\) and denoted by \(\Gamma(f)\). The union of the compact faces of \(\Gamma_+(f)\) is called the Newton boundary of \(\Gamma_+(f)\) and denoted by \(\Gamma(f)\). We associate a polynomial \(f_\Delta := \sum_{a \in \mathbb{N}} a \cdot z^a\) with each face \(\Delta\) of \(\Gamma(f)\). A polynomial \(f\) is said to be nondegenerate if \(f_\Delta\) has no solutions in \((\mathbb{C}^*)^{r+1}\) for any face \(\Delta\) of \(\Gamma(f)\).

We regard \(\mathbb{C}^{r+1}\) as an affine toric variety \(V_\Sigma\) associated with the fan \(\Sigma\) in \(\mathbb{N} \cong \mathbb{N} \otimes_{\mathbb{Z}} \mathbb{R}\) for \(N \cong \mathbb{Z}^{r+1}\) consisting of the faces of the cone \((\mathbb{R}_{>0})^{r+1}\). Then the set of the exponents of monomials in \(\mathbb{C}[z_0, z_1, \ldots, z_r]\) is naturally identified with the set of vectors with nonnegative entries in the dual space \(M := \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})\). We refer the reader to Fulton [2] or Oda [12] on the theory of toric varieties.
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For each vector $n$ in $(\mathbb{R}_{\geq 0})^{r+1}$, define $l(n) := \min\{\langle \lambda, n \rangle | \lambda \in \Gamma_+(f) \}$ and

$$\Delta(n) := \Gamma_+(f) \cap \{\lambda \in \mathbb{N}_\mathbb{R} | \langle \lambda, n \rangle = l(n) \}.$$ 

After these definitions, we define the dual diagram of each face $\Delta$ of $\Gamma_+(f)$:

$$\Delta^* := \{n \in \mathbb{N}_\mathbb{R} | \Delta(n) = 0 \}.$$ 

Then the closure of $\Delta^*$ in $\mathbb{N}_\mathbb{R}$ is a cone and the set of the closures of the dual diagrams of all the faces of $\Gamma_+(f)$ forms a fan $\Sigma(f)$. This is a subdivision of the fan $\Sigma$ and called the dual fan of the Newton diagram $\Gamma_+(f)$ of $f$.

Take a nonsingular subdivision $\hat{\Sigma}(f)$ of $\Sigma(f)$ and denote by $V_{\hat{\Sigma}(f)}$ the toric variety associated with $\hat{\Sigma}(f)$. We can associate a map of fans $(N, \hat{\Sigma}(f)) \to (N, \Sigma)$ with the subdivision $\hat{\Sigma}(f)$ of $\Sigma$. This map of fans induces a proper, birational morphism of toric varieties:

$$\Pi : V_{\hat{\Sigma}(f)} \to V_{\Sigma} \cong \mathbb{C}^{r+1}.$$ 

In the following, we assume that the subdivision $\hat{\Sigma}(f)$ is the one satisfying that the morphism $\Pi$ induces an isomorphism

$$V_{\hat{\Sigma}(f)} / \Pi^{-1}(x) \cong V_{\Sigma} / \{x\}.$$ 

Then it is well-known that the restriction $\pi : \tilde{X} \to X$ of $\Pi$ on the proper transform $\tilde{X}$ of $X$ with respect to $\Pi$ is a good resolution of the singularity $(X, x)$ — see Kempf et al. [10].

3. The plurigenera $\{\gamma_n(X, x)\}_{n \in \mathbb{N}}$ and $\{\tilde{\gamma}_n(X, x)\}_{n \in \mathbb{N}}$ for a hypersurface isolated singularity $(X, x)$

Let $(X, x) = (V(f), 0)$, $\hat{\Sigma}(f)$ and $\Pi : V_{\hat{\Sigma}(f)} \to \mathbb{C}^{r+1}$ as in § 2.

3.1. Differential forms around a singularity

3.1.1. Let $\Omega^{r+1}_{\mathbb{C}}(X)$ be the sheaf of meromorphic differential forms on $\mathbb{C}^{r+1}$ with single pole along $X$. For any open subset $U$, we have

$$\Omega^{r+1}_{\mathbb{C}}(X)(U) = \mathcal{O}_{\mathbb{C}^{r+1}}(U) \cdot \eta,$$

where $\eta = dz_0 \wedge dz_1 \wedge \cdots \wedge dz_r/f$.

Let $(\Omega^{r+1}_{\mathbb{C}}(X))^{*m}$ represent the $m$-times tensor product of $\Omega^{r+1}_{\mathbb{C}}(X)$ for each positive integer $m$. 
3.1.2. Let $U$ be an open neighborhood of the origin $x = 0$ of $\mathbb{C}^{r+1}$ and let $\psi$ be an element of $(\Omega^{r+1}_{\mathbb{C}}(X))^{*m}(U)$. Then $\Psi$ is written as $\Psi = \psi \cdot \eta^{*m}$, where $\psi \in \Omega^{r+1}_{\mathbb{C}}(U)$, that is a holomorphic function on $U$.

Now expand $\psi$ in a power series at the origin $x$, say, $\psi = \sum_{k \in \mathbb{N}_0} c_k \cdot x^k$, where $x^k = \prod_{i=0}^k z_i^k$. This gives rise to an isomorphism

$$(\Omega^{r+1}_{\mathbb{C}}(X))^{*m}_z \cong \mathbb{C}(z_0, z_1, ..., z_r) \cdot \eta^{*m}_z,$$

where $\mathbb{C}(z_0, z_1, ..., z_r)$ is the ring of convergent power series and $\eta_z$ is the germ of $\eta$ at $x$. Denote by $g_x$ the image of each element $\Psi_x \in (\Omega^{r+1}_{\mathbb{C}}(X))^{*m}$ by this isomorphism.

We define a diagram $\Gamma_\ast(g)$ for each power series $g$ in $\mathbb{C}(z_0, z_1, ..., z_r)$ to be the convex hull of the set $\bigcup_{\lambda \in \mathbb{R}_+^r}(\lambda + (\mathbb{R}_0)^{r+1})$ in $M = \mathbb{R}^{r+1}$. For $g$ and for each vector $n$ of $N$, define

$$I_\lambda(n) := \min \langle \lambda, n \rangle | \lambda \in \Gamma_\ast(g) \rangle.$$

3.1.3. The morphism $\Pi : V_{\mathbb{R}^r} \to \mathbb{C}^{r+1}$ induces a homomorphism of sheaves:

$$\Pi^* : (\Omega^{r+1}_{\mathbb{C}}(X))^{*m} \to \Pi^\ast I_x(\Omega^{r+1}_{\mathbb{R}^r}|_{\mathbb{R}^r}(X \setminus \Pi^{-1}(x))^{*m}),$$

which is defined by the pull-backs of differential forms, where $I : V_{\mathbb{R}^r} \setminus \Pi^{-1}(x) \to V_{\mathbb{R}^r}$ is the inclusion. Then the image of an element of $(\Omega^{r+1}_{\mathbb{C}}(X))^{*m}(U)$ by $\Pi^*$, for an open neighborhood $U$ of $x$, can be regarded as a meromorphic differential form with single pole along $\bar{X}$ and with poles along irreducible components of $\Pi^{-1}(x)$. We note that $\Pi^{-1}(x) = \bigcup_{\lambda \in \mathbb{R}_+^r}(\lambda + (\mathbb{R}_0)^{r+1})D_{\lambda}$, where $\Sigma(1)$ (resp. $\tilde{\Sigma}(1)$) denotes the set of one-dimensional cones of $\Sigma$ (resp. $\tilde{\Sigma}(f)$) and $D_{\lambda}$ denotes the invariant divisor of the toric variety $V_{\mathbb{R}^r}$ corresponding to one-dimensional cone $\lambda$.

**Proposition 3.1.** Let $\Psi_x$ be an element of $(\Omega^{r+1}_{\mathbb{C}}(X))^{*m}_x$. Then $\Pi^*(\Psi_x)$ has zeros of order $lg_{\Psi_x}(n(\bar{\rho})) + m(1, n(\bar{\rho})) - 1 - I_\lambda(n(\bar{\rho}))$ along the invariant divisor $D_{\lambda}$ corresponding to $\bar{\rho} \in \tilde{\Sigma}(f)(1) \setminus \Sigma(1)$, where $1 := (1, 1, ..., 1) \in M$.

**Proof.** This is an easy modification of [15], Lemma 2.1. \hfill $\square$

3.1.4. The homomorphism $(\Omega^{r+1}_{\mathbb{R}^r}(\bar{X}))^{*m} \to \tilde{I}_x(\Omega^{r+1}_{\mathbb{R}^r}|_{\mathbb{R}^r}(\bar{X} \setminus \Pi^{-1}(x))^{*m})$ induced by the inclusion $I : V_{\mathbb{R}^r} \setminus \Pi^{-1}(x) \to V_{\mathbb{R}^r}$ is injective, so that we may regard the former as a subsheaf of the latter.

By Proposition 3.1, for an element $\Psi_x \in (\Omega^{r+1}_{\mathbb{C}}(X))^{*m}_x$, $\Pi^*(\Psi_x) \in \Pi^\ast(\Omega^{r+1}_{\mathbb{R}^r}(\bar{X}))^{*m}$ if and only if $lg_{\Psi_x}(n(\bar{\rho})) + m(1, n(\bar{\rho})) - 1 - I_\lambda(n(\bar{\rho})) \geq 0$ for any $\bar{\rho} \in \tilde{\Sigma}(f)(1) \setminus \Sigma(1)$. Here we define a subset $m\Delta_\ast(\tilde{\Sigma}(f))$ of $M$ to be the set:
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\[ \lambda \in (\mathbb{R}_{>0})^{r+1} \mid \langle \frac{1}{m} \lambda, n(\rho) \rangle \geq l(\rho, n(\rho)) - \langle 1, n(\rho) \rangle + 1 \text{ for any } \rho \in \hat{\Sigma}(f)(1) \backslash \Sigma(1). \]

Then we have the following:

**Lemma 3.2.** \( \Pi^{*}(\Psi_{x}) \subset \Pi_{*}(\Omega_{\mathbb{V}_{\mathbb{E}_{0}}(\hat{X})})^{\eta_{m}} \) if and only if \( \Gamma_{*}(g_{\Psi}) \subset m\Delta_{\cdot}(\hat{S}(f)). \)

**Proof.** As we mentioned above, \( \Pi^{*}(\Psi_{x}) \subset \Pi_{*}(\Omega_{\mathbb{V}_{\mathbb{E}_{0}}(\hat{X})})^{\eta_{m}} \) if and only if \( l_{\Psi_{x}}(n(\rho)) + m(\langle 1, n(\rho) \rangle - 1 - l(\rho, n(\rho))) \geq 0 \) for any \( \rho \in \hat{S}(f) \backslash \Sigma(1). \)

Now assume that \( \Pi^{*}(\Psi_{x}) \subset \Pi_{*}(\Omega_{\mathbb{V}_{\mathbb{E}_{0}}(\hat{X})})^{\eta_{m}}. \) Since \( l_{\Psi_{x}}(n(\rho)) = \min \{ \langle \lambda, n(\rho) \rangle \mid \lambda \in \Gamma_{*}(g_{\Psi}) \}, \) we have \( \langle \lambda, n(\rho) \rangle + m(\langle 1, n \rangle - 1 - l(\rho, n(\rho))) \geq 0, \) so that

\[ \langle \frac{1}{m} \lambda, n(\rho) \rangle \geq l(\rho, n(\rho)) - \langle 1, n(\rho) \rangle + 1 \]

holds for any \( \lambda \in \Gamma_{*}(g_{\Psi}). \) Therefore, we obtain \( \Gamma_{*}(g_{\Psi}) \subset m\Delta_{\cdot}(\hat{S}(f)). \)

The converse is obvious. \( \square \)

3.1.5. Define the \( \mathbb{C} \)-submodule \( A^{(m)} \) of \( (\Omega^{r+1}_{\mathbb{C}^{r+1}}(X))^{\eta_{m}} = \mathbb{C} \{ z_{0}, z_{1}, ..., z_{r} \} \cdot \eta_{m} \) to be

\[ \{ g \cdot \eta_{m} \mid g \in m\Delta_{\cdot}(\hat{S}(f)) \}. \]

**Lemma 3.3.** The homomorphism

\[ \Pi^{*} : A^{(m)} \rightarrow \Pi_{*}(\Omega_{\mathbb{V}_{\mathbb{E}_{0}}(\hat{X})})^{\eta_{m}} \]

is an isomorphism.

**Proof.** Recall that we have an inclusion \( \Pi_{*}(\Omega_{\mathbb{V}_{\mathbb{E}_{0}}(\hat{X})})^{\eta_{m}} \rightarrow \Pi_{*} \hat{I}_{*}(\Omega_{\mathbb{V}_{\mathbb{E}_{0}}(\hat{X})})^{\eta_{m}} \) given by the restriction map. Also note that we have an isomorphism \( \Pi_{*} : (\Omega^{r+1}_{\mathbb{C}^{r+1}}(X))^{\eta_{m}} \cong (\Omega_{\mathbb{C}^{r+1}_{\mathbb{V}_{\mathbb{E}_{0}}}}(X \backslash \{ x \}))^{\eta_{m}} \cong \Pi_{*}(\Omega_{\mathbb{V}_{\mathbb{E}_{0}}(\hat{X})})^{\eta_{m}} \) since \( r + 1 \geq 3 \) and \( \Pi_{|\mathbb{V}_{\mathbb{E}_{0}}}(X \backslash \{ x \}) \rightarrow \mathbb{C}^{r+1} \backslash \{ x \} \) is an isomorphism.

Then it follows that for any element of \( \Pi_{*}(\Omega^{r+1}_{\mathbb{V}_{\mathbb{E}_{0}}}(X))^{\eta_{m}}, \) there exists an element \( \Psi_{x} = \psi_{x} \cdot \eta_{m} \) in \( (\Omega^{r+1}_{\mathbb{C}^{r+1}}(X))^{\eta_{m}}. \) By Proposition 3.2, \( \Gamma_{*}(\psi_{x}) \subset m\Delta_{\cdot}(\hat{S}(f)). \) Hence the map in the lemma is surjective.

The injectivity of the map is obvious. \( \square \)

3.2. **Poincaré residue maps**

3.2.1. Define the sheaf \( \Omega_{X}^{\cdot} \) to be the direct image \( i_{*} \Omega_{X}^{\cdot}(\mathcal{U}) \) of \( \Omega_{X}^{\cdot}(\mathcal{U}) \) with respect to the inclusion
\(i: X \setminus \{x\} \to X\).

Let \(U\) be an open subset of \(\mathbb{C}^{r+1}\). We have a natural map, called the Poincaré residue map,

\[
\text{Res}_x : \Omega^{r+1}_x(X)(U) \to \Omega^r_x(X \cap U)
\]

by sending \(\Psi = \psi \cdot \eta\) to \(\text{Res}_x(\Psi) := \psi \mid_{X \setminus \{x\}} \cdot \text{Res}_x(\eta)\), where \(\eta = dz_0 \wedge dz_2 \wedge \cdots \wedge dz_r / f\) and \(\text{Res}_x(\eta) := (-1)^r \cdot (dz_0 \wedge \cdots \wedge dz_j \wedge \cdots \wedge dz_r) / (\partial f / \partial z_j)\) (if \(\partial f / \partial z_j \neq 0\)).

This induces the maps between the tensor products:

\[
\text{Res}_x^m : (\Omega^{r+1}_x(X))^* \to (\Omega^r_x)^*.
\]

Therefore, we have a homomorphism of sheaves for any positive integer \(m\):

\[
\text{Res}_x^m : (\Omega^{r+1}_x(X))^* \to (\Omega^r_x)^*.
\]

**Lemma 3.4.** The homomorphism of sheaves \(\text{Res}_x^m\) is surjective for every \(m\).

**Proof.** This lemma follows the adjunction formula for a nonsingular variety and a Cartier divisor on it.

3.2.2. On the other hand, since \(\tilde{X}\) is a nonsingular divisor of a nonsingular variety \(V_{3/1}\), we have another Poincaré residue map

\[
\text{Res}_{\tilde{x}}^m : (\Omega^{r+1}_{3/1}(\tilde{X}))^* \to (\Omega^r_{\tilde{x}})^*
\]

for each positive integer \(m\).

3.3. **The definition of \((\gamma_n(X, x))_{n \in \mathbb{N}}\)**

3.3.1. Let \((X, x)\) be a normal isolated Gorenstein singularity of dimension \(r\) and let \(\pi : \tilde{X} \to X\) be a good resolution of the singularity \((X, x)\). Then, the sequence

\[
0 \to (\Omega^r_{\tilde{x}})^* \to \tilde{i}_*(\Omega_{\tilde{X} \setminus \{\pi(x)\}})^*
\]

is exact, where \(\tilde{i} : \tilde{X} \setminus \pi^{-1}(x) \to \tilde{X}\) is the inclusion, so that

\[
0 \to \pi_* (\Omega^r_{\tilde{x}})^* \to (\Omega^r_{x})^* \cong \tilde{i}_* (\Omega_{\tilde{X} \setminus \{\pi(x)\}})^*
\]

is also exact. Therefore, we can regard \(\pi_* (\Omega^r_{\tilde{x}})^*\) as a subsheaf of \((\Omega^r_x)^*\).

Here we give the definition of the plurigenera \((\gamma_n)_{n \in \mathbb{N}}\) introduced by Knöller [11]:
Proposition and Definition 3.5 (Knöller). For a normal isolated Gorenstein singularity \((X, x)\) and a positive integer \(m\), \((\omega_X^m)_x/((\pi_*\omega_X^m)_x)\) is a finite dimensional vector space over \(\mathbb{C}\), where \(\pi: \tilde{X} \to X\) is a good resolution of the singularity \((X, x)\).

The dimension of this vector space is independent of the choice of a resolution and is denoted by \(\gamma_m(X, x)\), that is,
\[
\gamma_m(X, x) := \dim_{\mathbb{C}} ((\omega_X^m)_x/((\pi_*\omega_X^m)_x) = \dim_{\mathbb{C}} (\Omega_x^m)_x/((\pi_*\Omega_x^m)_x)_{x}^m,
\]
where \(\omega_X\) (resp. \(\omega_{\tilde{X}}\)) is the canonical sheaf of \(X\) (resp. \(\tilde{X}\)).

We sometimes use the symbol \(\gamma_m\) for \(\gamma_m(X, x)\) for simplicity.

3.4. The definition of \(\{\tilde{\gamma}_m(X, x)\}_{m \in \mathbb{N}}\) of a hypersurface isolated singularity \((X, x)\) and relations to \(\{\gamma_m(X, x)\}_{m \in \mathbb{N}}\)

3.4.1. Next we will define the plurigenera \(\{\tilde{\gamma}_m(X, x)\}_{m \in \mathbb{N}}\) for a hypersurface isolated singularity \((X, x)\). The definition of \(\{\tilde{\gamma}_m\}_{m \in \mathbb{N}}\) is inspired by the formula for another set of the plurigenera \(\{\delta_m(X, x)\}_{m \in \mathbb{N}}\) in [15], Thorem 2.2. See [14] and [15] for details on \(\{\delta_m\}_{m \in \mathbb{N}}\). So we assume that \((X, x) = (V(f), 0)\), where \(f \in \mathbb{C}[z_0, z_1, ..., z_r]\) is a nondegenerate polynomial and \(0\) is the origin of \(\mathbb{C}^{r+1}\) and we use the notation in § 2.

Recall that the definition of the diagram \(m\Delta_+ (\tilde{\mathbb{S}}(f))\) for each \(m \in \mathbb{N}\). Define \(m\Delta_- (\tilde{\mathbb{S}}(f)) := (\mathbb{R}_{\geq 0})^{r+1} \setminus m\Delta_+ (\tilde{\mathbb{S}}(f))\).

Now we define \(\{\tilde{\gamma}_m(X, x)\}_{m \in \mathbb{N}}\) as follows:

**Definition 3.6.** Let \((X, x) = (V(f), 0)\) be an isolated singularity at the origin of \(\mathbb{C}^{r+1}\) defined by a nondegenerate polynomial \(f \in \mathbb{C}[z_0, z_1, ..., z_r]\). Then we define \(\tilde{\gamma}_m(X, x)\) to be
\[
\#(m\Delta_-(\tilde{\mathbb{S}}(f)) \cap M) - \#(\lambda \in (\mathbb{Z}_{\geq 0})^{r+1} \subset M \mid (\lambda + \Gamma_+(f)) \cap m\Delta_-(\tilde{\mathbb{S}}(f)) \neq \emptyset)
\]
for each \(m \in \mathbb{N}\), where \(#A\) denotes the cardinality of a set \(A\).

3.4.2.

**Theorem 3.7.** We have the following inequality between \(\gamma_m(X, x)\) and \(\tilde{\gamma}_m(X, x)\) for every \(m \in \mathbb{N}\):
\[
\gamma_m(X, x) \leq \tilde{\gamma}_m(X, x).
\]

The following proof is a modification of [15], Thorem 2.2 for the formula for \(\delta_m\).
Proof. Let \( \theta \) be an element of \((\Omega^e_\Lambda)^{\otimes m}\). By Lemma 3.4, the homomorphism \( \text{Res}^{\otimes m}_x : (\Omega^e_{\Lambda^+1}(X))^{\otimes m}_x \to (\Omega^e_\Lambda)^{\otimes m}_x \) between the stalks at \( x \) is surjective. Hence there exists an element \( \Psi_x = \psi_x \cdot \eta^{\otimes m}_x \) such that \( \text{Res}^m_x(\Psi_x) = \theta \), where \( \psi_x \in \mathcal{O}_X \). We may regard \( \psi_x \) as a convergent power series: \( \sum_{i \in \mathbb{Z}_{\geq 0}^{r+1}} c_i \cdot x^i \).

We write \( \psi_x \) as a sum of two series: \( \psi_{x,+} + \psi_{x,-} \), where \( \psi_{x,+} = \sum_{i \in \mathbb{Z}_{\geq 0}^{r+1}} c_i \cdot x^i \) and \( \psi_{x,-} = \psi_x - \psi_{x,+} \). Define \( \Psi_{x,+} = \psi_x \cdot \eta^{\otimes m}_x \) and \( \Psi_{x,-} = \psi_x \cdot \eta^{\otimes m}_x \). Then \( \Psi_x = \Psi_{x,+} + \Psi_{x,-} \). Since \( \psi_{x,+,x} \in A^{(m)} \), we have \( \Pi^*(\Psi_{x,+}) \in \Pi^*(\Omega^e_{V_{\Lambda}^{(m)}}(\hat{X}))^{\otimes m}_x \) by Lemma 3.2.

Since the diagram:

\[
\begin{array}{ccc}
A^{(m)} & \xrightarrow{\text{Res}^m_x} & (\Omega^e_\Lambda)^{\otimes m}_x \\
\Pi^* \downarrow & & \downarrow \pi^* \\
\Pi_*((\Omega^e_{\Lambda^+1}(\hat{X}))^{\otimes m}_x) & \xrightarrow{\pi_* \text{Res}^m_x} & \pi_*\pi_*(\Omega^e_{\Lambda^+1}(\hat{X}))^{\otimes m}_x \\
\end{array}
\]

commutes, \( \pi^*(\text{Res}^m_x(\Psi_{x,+})) \in \pi_*((\Omega^e_\Lambda)^{\otimes m}_x) \). Therefore, \( (\Omega^e_\Lambda)^{\otimes m}_x / \pi_*((\Omega^e_\Lambda)^{\otimes m}_x) \) is generated by the elements \( (\text{Res}^m_x(\phi_x \cdot \eta^{\otimes m}_x)) \mid \lambda \in m \Delta(-\mathcal{F}(f)) \cap (\mathbb{Z}_{\geq 0})^{r+1} \).

Next assume that \( \pi^*(\text{Res}^m_x(\Psi_{x,-})) \) is contained by the image of the homomorphism:

\( \Pi_*((\Omega^e_\Lambda)^{\otimes m}_x) : \Pi_*((\Omega^e_{V_{\Lambda}^{(m)}}(\hat{X}))^{\otimes m}_x) \to \Pi_*((\Omega^e_\Lambda)^{\otimes m}_x) \).

Then, by Lemma 3.3, there exists an element \( \Phi_x = \phi_x \cdot \eta^{\otimes m}_x \in A^{(m)} \) such that the image of the composite of \( \Pi^* \) and \( \Pi_*((\Omega^e_\Lambda)^{\otimes m}_x) \) is just \( \pi^*(\text{Res}^m_x(\Psi_{x,-})) \). Then we have \( \pi^*(\psi_{x,-} \mid x) = \pi^*(\phi_x \mid x) \) in \( \pi_*((\mathcal{O}_x)_x) \). Since \( \pi^* : \mathcal{O}_x \to \pi_*((\mathcal{O}_x)_x) \) is injective, \( \psi_{x,-} \mid x = \phi_x \mid x \) holds, so that there exists a power series \( \sum_{i \in \mathbb{Z}_{\geq 0}^{r+1}} d_i \cdot x^i \) such that \( \psi_{x,-} - \phi_x = (\sum_{i \in \mathbb{Z}_{\geq 0}^{r+1}} d_i \cdot x^i) \cdot f_x \). It follows that \( \psi_{x,-} = (\sum_{i \in \mathbb{Z}_{\geq 0}^{r+1}} d_i \cdot (x^i \cdot f_x)) \). This completes the proof.

\[ \square \]

3.4.3. Remark. In the proof of the theorem just above, if the homomorphism (1) is surjective, we obtain an equation:

\[ \gamma^e_m(X, x) = \tilde{\gamma}^e_m(X, x). \]

Until now, we cannot state the condition under which the homomorphism (1) is surjective.

3.5. The diagram \( \Delta_*(\mathcal{F}(f)) \)

3.5.1. We can rewrite \( m \Delta_*(\mathcal{F}(f)) \) as

\[ \{ \lambda \in (\mathbb{Z}_{\geq 0})^{r+1} \subset M \mid \frac{1}{m} \lambda \in \Delta_*(\mathcal{F}(f)) \} \]
and we have

\[ m\Delta_-(\mathcal{F}(f)) = (R_{\mathcal{L}})^{-1}\mathcal{L}(\mathcal{F}(f)). \]

Hence we obtain the following:

**Corollary 3.8.** \( \tilde{\gamma}_n(X, x) \) is determined by the diagram \( \Delta_-(\mathcal{F}(f)) \).

3.5.2. **Remark.** Here, we note that the diagram \( \Delta_-(\mathcal{F}(f)) \) is the same \( \Gamma_{\mathcal{L}(f)} \) introduced by Ishii [8] in order to construct the canonical model of the hypersurface isolated singularity \((X, x) = (V(f), 0)\). Indeed, Ishii obtained the following result:

**Theorem 3.9 (Ishii).** Let \( \Sigma_0 \) be the dual fan associated with the polytope \( \Delta_-(\mathcal{F}(f)) \) and let \( \varphi : X_0 \rightarrow X \) be the birational morphism induced by the subdivision \( \Sigma_0 \) of \( \Sigma \). Denote by \( \eta : X_0 \rightarrow X_0 \) be the normalization.

Then the composite \( \varphi \circ \eta : X_0 \rightarrow X \) is the canonical modification.

4. **Application to purely elliptic singularities**

4.1. **Preliminaries**

4.1.1. In this section, we will apply the result obtained in §3 to a particular class of singularities, called purely elliptic singularities. First of all, we recall the definition of the plurigenera \( \{\delta_m(X, x)\}_{m\in \mathbb{N}} \) to introduce the notion of purely elliptic singularities.

**Proposition and Definition 4.1 ([14]).** For a normal isolated Gorenstein singularity \((X, x)\) and a positive integer \(m\), \( (\omega_X^{\otimes m})_x / \pi^* (\omega_X^{\otimes (m-1)E})_x \) is a finite dimensional vector space over \(\mathbb{C}\), where \(\pi : \bar{X} \rightarrow X\) is a good resolution of the singularity \((X, x)\), \(\omega_X\) (resp. \(\omega_{\bar{X}}\)) is the canonical sheaf of \(X\) (resp. \(\bar{X}\)), and \(E\) is the exceptional divisor.

The dimension of the vector space is independent of the choice of a good resolution \(\pi\) and denoted by \(\delta_m(X, x)\), that is,

\[ \delta_m(X, x) := \dim_{\mathbb{C}} (\omega_X^{\otimes m})_x / (\pi^* \omega_X^{\otimes (m-1)E})_x. \]

**Definition 4.2 ([14]).** A normal isolated Gorenstein singularity \((X, x)\) is said to be a purely elliptic singularity if \(\delta_m = 1\) for all \(m \in \mathbb{N}\).

4.1.2. There is a criterion whether an isolated singularity defined by a polynomial is a purely
elliptic singularity:

**Proposition 4.3 ([15]).** Let \((X, x) = (V(f), 0)\) be an \(r\)-dimensional isolated singularity defined by a nondegenerate polynomial \(f \in \mathbb{C}[z_0, z_1, \ldots, z_r]\).

Then \((X, x)\) is a purely elliptic singularity if and only if the Newton boundary \(\Gamma(f)\) contains the vector \(1 = (1, 1, \ldots, 1) \in M\).

4.2. The essential cone and the diagram \(m \Delta_\ast(\hat{\Sigma}(f))\)

4.2.1. Let \((X, x) = (V(f), 0)\) be an \(r\)-dimensional purely elliptic singularity defined by a nondegenerate polynomial \(f \in \mathbb{C}[z_0, z_1, \ldots, z_r]\).

Let \(\hat{\Sigma}(f)\) be a nonsingular subdivision of the dual fan \(\Sigma(f)\) of \(f\) such that the primitive integral generators of one-dimensional cones are all positive vectors.

Recall that \(m \Delta_\ast(\hat{\Sigma}(f))\) is the set:

\[
\{ \lambda \in (\mathbb{Z}_{\geq 0})^{r+1} \subset M \mid \frac{1}{m} \lambda, n(\hat{\rho}) - n(\rho) \geq \langle 1, n(\hat{\rho}) \rangle + 1 \text{ for any } \hat{\rho} \in \hat{\Sigma}(f)(1) \setminus \Sigma(1) \}.
\]

We will divide the set \(\hat{\Sigma}(f)(1) \setminus \Sigma(1)\) into two classes. To do this, we introduce the notion of the essential cone due to Ishii [6]:

**Definition 4.4 (Ishii).** The set

\[
C_\ast(f) := \{ n \in N_\mathbb{R} \mid \lambda(n) \geq \langle 1, n \rangle \}
\]

is a cone in \(N_\mathbb{R}\) and called the essential cone.

4.2.2. In case \((X, x) = (V, 0)\) is a purely elliptic singularity, \(\lambda(n) \leq \langle 1, n \rangle\) holds for any \(n \in N_\mathbb{R}\) since the Newton boundary \(\Gamma(f)\) contains \(1 \in M\). Hence in this case, we have

\[
C_\ast(f) = \{ n \in N_\mathbb{R} \mid \lambda(n) = \langle 1, n \rangle \}.
\]

If \(n \in (\mathbb{R}_{>0})^{r+1}\) but \(n \not\in C_\ast(f)\), then \(\lambda(n) < \langle 1, n \rangle\), so that \(\lambda(n) - \langle 1, n \rangle + 1 \leq 0\). Hence

\[
\frac{1}{m} \lambda, n \geq \lambda(n) - \langle 1, n \rangle + 1
\]

holds for any \(\lambda \in (\mathbb{Z}_{>0})^{r+1}\). Thus, we can write \(m \Delta_\ast(\hat{\Sigma}(f))\) as follows:

**Proposition 4.5.** If \((X, x) = (V(f), 0)\) is a purely elliptic singularity, then \(m \Delta_\ast(\hat{\Sigma}(f))\) equals to the set:

...
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\{λ ∈ (Z_{≥0})^{r+1} ⊂ M | \langle \frac{1}{m}λ, n(\bar{ρ}) \rangle ≥ 1 \text{ for any } \bar{ρ} ∈ (\hat{S}(f)(1) \setminus Σ(1)) \cap C_1(f)(1) \}

or

\{λ ∈ (Z_{≥0})^{r+1} ⊂ M | \langle λ, n(\bar{ρ}) \rangle ≥ m \text{ for any } \bar{ρ} ∈ (\hat{S}(f)(1) \setminus Σ(1)) \cap C_1(f)(1) \}.

When \((X, x) = (V(f), 0)\) is a purely elliptic singularity, by this proposition, we have only to take a nonsingular subdivision of the essential cone \(C_1(f)\) in order to obtain \(mΔ_*(f)\). An algorithm to get a nonsingular subdivision is given by Oka [13].

### 4.3. \(\tilde{γ}_m\) and special vectors: systems of weights

#### 4.3.1. Here we investigate the most simple case, that is the diagram \(Δ_*(\hat{S}(f))\) has only one compact face. In this case, there exists a special integral vector in the essential cone \(C_1(f)\), which Ishii [6] calls the absolutely minimal vector in \(C_1(f)\): Denote by \(C_1(f)[1]\) the set of primitive integral vectors in \(C_1(f)\). Then a primitive integral vector \(p = (p_0, p_1, ..., p_r)\) is said to be absolutely minimal if for any element \(q = (q_0, q_1, ..., q_r) ∈ C_1(f)[1]\), \(p_i ≤ q_i\) holds for \(i = 0, 1, ..., r\).

The following is easy to check.

**Proposition 4.6.** If there exists an absolutely minimal vector \(p\) in \(C_1(f)[1]\), then the diagram \(mΔ_*(\hat{S}(f))\) has only one compact face, more precisely,

\[mΔ_*(\hat{S}(f)) = mΔ_*(p) = \{λ ∈ (R_{≥0})^{r+1} | \langle λ, p \rangle ≥ m \}

and its unique compact face is

\[\{λ ∈ (R_{≥0})^{r+1} | \langle λ, p \rangle = m \}.

Conversely, if \(Δ_*(\hat{S}(f))\) has only one compact face \(δ_0\), then the primitive integral generator \(p = n(δ^*_0)\) of the dual cone \(δ^*_0\) is the absolutely minimal vector in \(C_1(f)[1]\).

#### 4.3.2. Remark. Due to Ishii [6], if \(Δ_*(\hat{S}(f))\) has a unique compact face \(δ_0\), then the canonical model of the purely elliptic singularity \((X, x) = (V(f), 0)\) is obtained by the weighted blow-up with respect to the integral vector \(p = n(δ^*_0)\).

The statement that the essential cone of a hypersurface purely elliptic singularity contains the absolutely minimal vector is false in general. Counter examples are given by Ishii [6] § 4.1. Nevertheless, many examples of hypersurface purely elliptic singularities whose essential cones
contains the absolutely minimal vectors are known.

4.3.3. For a positive vector $p = (p_0, p_1, ..., p_r) \in M$, denote $p_0 + p_1 + \cdots + p_r$ by $\deg p$. Then we can write down the formula for $\tilde{\gamma}_n(X, x)$ as the following simple form:

Theorem 4.7. Let $(X, x) = (V(f), 0)$ be a purely elliptic singularity defined by a nondegenerate polynomial $f \in \mathbb{C}[z_0, z_1, ..., z_r]$. Assume that $\Delta_*(\tilde{\Sigma}(f))$ has a unique compact face $\delta_0$ and let $p := n(\delta_0^\circ)$ be the primitive integral generator of the dual cone $\delta_0^\circ$. Then we have

\begin{align*}
\tilde{\gamma}_n(X, x) &= \#(m \Delta_-(p) \cap M) - \#(\lambda \in (\mathbb{Z}_{\geq 0})^{r+1} \cap M \setminus (\lambda + \Gamma_+(p)) \cap m \Delta_-(p) \neq \emptyset) \\
&= \#(m \Delta_-(p) \cap M) - \#((m - \deg p) \Delta_-(p) \cap M),
\end{align*}

where $m \Delta_-(p) := (\mathbb{Z}_{\geq 0})^{r+1} \setminus m \Delta_-(p)$ and $\Gamma_+(p) := \langle 1, p \rangle \Delta_+(p)$.

Proof. We have only to show that the set

$$\{\lambda \in (\mathbb{Z}_{\geq 0})^{r+1} | (\lambda + \Gamma_+(p)) \cap m \Delta_-(p) \neq \emptyset\}$$

equals the sets

$$\{\lambda \in (\mathbb{Z}_{\geq 0})^{r+1} | (\lambda + \Gamma_+(p)) \cap m \Delta_-(p) \neq \emptyset\}$$

and

$$m - \deg p \Delta_-(p) \cap M.$$

For $\lambda \in (\mathbb{Z}_{\geq 0})^{r+1}$, if $(\lambda + \Gamma_+(p)) \cap m \Delta_-(p) \neq \emptyset$, then $(\lambda + \Gamma_+(p)) \cap m \Delta_-(p) \neq \emptyset$ since $\Gamma_+(f) \subset \Gamma_+(v)$ holds. Conversely, assume that for $\lambda \in (\mathbb{Z}_{\geq 0})^{r+1}$, $(\lambda + \Gamma_+(p)) \cap m \Delta_-(p) \neq \emptyset$. Then since the compact face of $\Gamma_+(p)$ is parallel to the boundary of $m \Delta_+(p)$ and contains $1 \in M$, $\lambda + 1 \in (\lambda + \Gamma_+(p)) \cap m \Delta_-(p)$. Note that the Newton diagram $\Gamma_+(f)$ of a purely elliptic singularity contains $1 \in M$, so that $\lambda + 1 \in (\lambda + \Gamma_+(f)) \cap m \Delta_-(p) \neq \emptyset$. Thus the first set equals to the second one.

Next note that $(\lambda + \Gamma_+(p)) \cap m \Delta_-(p) \neq \emptyset$ if and only if $\lambda + 1 \in m \Delta_-(p)$ for $\lambda \in (\mathbb{Z}_{\geq 0})^{r+1}$, which is equivalent to $\langle \lambda + 1, p \rangle < m$, that is, $\langle \lambda, p \rangle < m - \deg p$. Thus the second set equals to the third one.

A set of positive integers $(p_0, p_1, ..., p_r)$ is sometimes called a system of weights or weight system. We may assume that the absolutely minimal vector $p = (p_0, p_1, ..., p_r)$ has positive integers as its entries, so that we can regard it as a system of weights.

By the previous theorem, we have the following:
Corollary 4.8. Let \((X, x) = (V(f), 0)\) be a purely elliptic singularity defined by a nondegenerate polynomial \(f \in \mathbb{C}[z_0, z_1, \ldots, z_r]\).

Assume that \(\Delta_{*}(\mathfrak{S}(f))\) has a unique compact face \(\delta_0\) and let \(p := n(\delta_0)\) be the primitive integral generator of the dual cone \(\delta_0^*\). Then \(\hat{\gamma}_n(X, x)\) is determined completely by the weight system \(p = (p_0, p_1, \ldots, p_r)\).

4.3.4. Examples. It is well-known that a two-dimensional hypersurface purely elliptic singularity is analytically equivalent to an isolated singularity \((X, x) = (V(f), 0)\) defined by the polynomial either case (1):

\[z_0^{p+q} + z_1^{r+q} + z_2^{r+q} + \lambda z_0 z_1 z_2, \quad (p, q, r \geq 0, \lambda \neq 0),\]

the case (2):

\[z_0^{p+q} + z_1^{r+q} + z_2^{r+q} + \lambda z_0 z_1 z_2, \quad (p, q, r \geq 0, \lambda \neq 0),\]

or the case (3):

\[z_0^{p+q} + z_1^{r+q} + \lambda z_0 z_1 z_2, \quad (r \geq 0, \lambda \neq 0).\]

In the case (1), \(p := (1, 1, 1) \in M\) is the absolutely minimal vector in the essential cone \(C_1(f)\) and we have

\[\#(m\Delta_-(p) \cap M) = \frac{1}{6} m(m+1)(m+2).\]

Now that \(\text{deg}p = 3\), we have

\[\hat{\gamma}_n(X, x) = \#(m\Delta_-(p) \cap M) - \#((m-\text{deg}p)\Delta_-(p) \cap M) = \frac{3}{2}(m^2 - m) + 1.\]

In the case (2), \(p := (2, 1, 1) \in M\) is the absolutely minimal vector in \(C_1(f)\) and we have

\[\#(m\Delta_-(p) \cap M) = \begin{cases} \frac{1}{24} m(m+2)(2m+5), & m \text{ even}, \\ \frac{1}{24} (m+1)(m+3)(2m+1), & m \text{ odd}. \end{cases}\]

Then since \(\text{deg}p = 4\), when \(m\) is even or or odd, we have

\[\hat{\gamma}_n(X, x) = \#(m\Delta_-(p) \cap M) - \#((m-\text{deg}p)\Delta_-(p) \cap M) = \frac{2}{2}(m^2 - m) + 1.\]

Finally, in the case (3), \(p = (3, 2, 1)\) is the absolutely minimal vector in \(C_1(f)\) and \(\#(m\Delta_-(p)}\)
\[ \cap M \) equals to

\[
\begin{align*}
\frac{1}{72} m(m+6)(2m+3) + \frac{1}{6} m, & \quad m \equiv 0 \mod 6, \\
\frac{1}{72} (m-1)(m+5)(2m+1) + \frac{1}{12} (m-1)(m+5) + \frac{1}{6} (m-1) + 1, & \quad m \equiv 1 \mod 6, \\
\frac{1}{72} (m-2)(m+4)(2m-1) + \frac{1}{6} (m-1)(m+4) + \frac{1}{6} (m-2) + 1, & \quad m \equiv 2 \mod 6, \\
\frac{1}{72} (m-3)(m+3)(2m-3) + \frac{1}{4} (m-1)(m+3) + \frac{1}{6} (m-3) + 1, & \quad m \equiv 3 \mod 6, \\
\frac{1}{72} (m-4)(m+2)(2m-5) + \frac{1}{3} (m-1)(m+2) + \frac{1}{6} (m-4) + 1, & \quad m \equiv 4 \mod 6, \\
\frac{1}{72} (m-5)(m+1)(2m-7) + \frac{5}{12} (m-1)(m+1) + \frac{1}{6} (m-5) + 1, & \quad m \equiv 5 \mod 6.
\end{align*}
\]

Although \(#(m\Delta_- (p) \cap M)\) are given by the six polynomials in \(m\) as above, \(#(m\Delta_- (p) \cap M) - #(m - \deg p)\Delta_- (p) \cap M)\) are represented by a polynomial

\[
\hat{\gamma}_m(X, x) = #(m\Delta_- (p) \cap M) - #(m - \deg p)\Delta_- (p) \cap M) = \frac{1}{2} (m^2 - m) + 1.
\]

On the other hand, we have the following formula which give \(\gamma_n\) for a Gorenstein surface singularity:

**Proposition 4.9 (Kato [9]).** Let \((X, x)\) be a Gorenstein singularity of dimension two and \(\pi: (\tilde{X}, E) \to (X, x)\) be the minimal resolution.

Then

\[
\gamma_n(X, x) = -K^2(m^2 - m) + p_g(X, x),
\]

where \(K\) is a canonical divisor on \(\tilde{X}\) and \(p_g(X, x)\) is the geometric genus of the singularity \((X, x)\).

Now \(p_g(X, x) = \delta_1(X, x) = 1\) holds for the singularity \((X, x)\) defined by a polynomial as in (1), (2) of (3) since \((X, x)\) is a purely elliptic singularity. Moreover it is well-known that \(K^2 = -3\) for the singularity defined by a polynomial as in (1), that \(K^2 = -2\) for the singularity defined by a polynomial as in (2), and that \(K^2 = -1\) for the singularity defined by a polynomial as in (3).

These are examples where \(\gamma_n(X, x) = \hat{\gamma}_n(X, x)\) for every \(m\).
4.4. Types of purely elliptic singularities

4.4.1. Let \((X, x)\) be an \(r\)-dimensional purely elliptic singularity and \(\pi: \tilde{X} \rightarrow X\) be a good resolution of \((X, x)\). Then it is known that \(K_{\tilde{X}} = \pi^*K_X + E_I - E_f\) holds, where \(K_X\) (resp. \(K_{\tilde{X}}\)) is a canonical divisor of \(X\) (resp. \(\tilde{X}\)), \(E_I\) is a positive divisor and \(E_f\) is a reduced divisor.

Ishii [5] classified the purely elliptic singularities into \(r\) classes by means of the mixed Hodge structures of the cohomologies of the support of \(E_I\):

**Proposition 4.10** (Ishii).

\[
C \cong H^{r-1}(E_I, \mathcal{O}_I) \cong \text{Gr}^W_{r-1}H^r(E_I) = \bigoplus_{i=0}^{r-1} H^{r-1}_i(E_I),
\]

where \(H^k_i(\ast)\) is the \((i, j)\)-component of \(\text{Gr}^W_{r-1}H^r(\ast)\).

Hence, in particular, for a unique \(i\) \((0 \leq i \leq r-1)\),

\[
H^{r-1}(E_I, \mathcal{O}_I) = H^{r-1}_i(E_I) \cong C.
\]

**Definition 4.11** (Ishii). A purely elliptic singularity \((X, x)\) is said to be of type \((0, i)\) if \(H^{r-1}(E_I, \mathcal{O}_I)\) consists of the \((0, i)\)-Hodge component.

In case \((X, x) = (V(f), 0)\) is a purely elliptic singularity defined by a nondegenerate polynomial \(f \in \mathbb{C}[z_0, z_1, \ldots, z_r]\), we also have a criterion of what type the singularity \((X, x)\) is:

**Proposition 4.12** ([15]). Let \((X, x) = (V(f), 0)\) be an \(r\)-dimensional purely elliptic singularity defined by a nondegenerate polynomial \(f\). Then \((X, x)\) is of type \((0, \text{dim}\Delta_1(f) - 1)\) if \(\text{dim}\Delta_1(f) \geq 2\) and of type \((0, 0)\) if \(\text{dim}\Delta_1(f) = 1\) or \(0\), where \(\Delta_1(f)\) is the face of the Newton boundary \(\Gamma(f)\) containing \(1 \in M\) in its relative interior.

We call the face \(\Delta_1(f)\) in the proposition just above the initial face of \(\Gamma(f)\).

4.4.2. Let \((X, x) = (V(f), 0)\) be a purely elliptic singularity of type \((0, r-1)\). Then by the criterion of Watanabe's 4.12, the initial face \(\Delta_1(f)\) is of \(r\)-dimension.

It follows that the essential cone \(C_1(f)\) is a one-dimensional cone and its primitive integral generator \(p\) is the absolutely minimal vector in \(C_1(f)\). Thus, we can attach a system of weights \(p = (p_0, p_1, \ldots, p_r)\) to each hypersurface purely elliptic singularity of type \((0, r-1)\).

Therefore, we can apply Theorem 4.7 to every \(r\)-dimensional hypersurface purely elliptic singularity of type \((0, r-1)\) and hence \(\tilde{g}_a(X, x)\) are determined completely by the system of weights \(p\).
References


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