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On plurigenera of hypersurface purely elliptic singularities

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Abstract

In this paper, we describe some properties of plurigenera defined for hypersurface isolated singularities, which have relations with more fundamental, well-known plurigenera $\{\gamma_m\}$, especially in case singularities are so-called purely elliptic singularities.

1. Introduction

In this paper, we will discuss the plurigenera $\{\gamma_m(X, x)\}_{m\in\mathbf{N}}$ and related subjects of hypersurface isolated singularities of dimension greater than or equal to two.

Knöller [11] showed that for a normal Gorenstein isolated singularity (X, x) and a resolution of the singularity $\pi:\tilde{X}\to X$, $(\omega_X^{\otimes m})_x/(\pi_*\omega_{\tilde{X}}^{\otimes m})_x$ is a finite dimensional vector space over the complex number field $\mathbf{C}$ and that the dimension $\gamma_m(X, x)$ of it is independent of the choice of a resolution π for any positive integer m, where ω_X (resp $\omega_{\tilde{X}}$) is the canonical sheaf of X (resp. $\tilde{X}$). Hence $\{\gamma_m(X, x)\}_{m\in\mathbf{N}}$ are regarded as invariants of the singularity (X, x). Indeed, $\{\gamma_m(X, x)\}_{m\in\mathbf{N}}$ characterize a singularity. For example, for a two-dimensional normal Gorenstein singularity (X, x), the condition that $\gamma_m(X, x)=0$ for every m is equivalent to that (X, x) is a famous *rational double point*, that is, (X, x) is analytically equivalent to the hypersurface singularity at the origin O of $\mathbf{C}^3$ defined by one of the following equations:

$$A_n(n\geq 1): x^2+y^2+z^{n+1}=0;\quad D_n(n\geq 4): x^2+y^2z+z^{n-1}=0;$$
$$E_6: x^2+y^3+z^4=0;\quad E_7: x^2+y^3+yz^3=0;\quad E_8: x^2+y^3+z^5=0.$$

On the other hand, we have another set of plurigenera $\{\delta_m(X, x)\}_{m\in\mathbf{N}}$ for a normal Gorenstein isolated singularity (X, x) — see Definition 4.1 in this paper. These give different characterization of singularities. For example, for a normal two-dimensional singularity (X, x), the condition that $\delta_m(X, x)=0$ for every m is equivalent to that (X, x) is a quotient singularity, that is, (X, x) is analytically equivalent to a quotient $(\mathbf{C}^2/G, O)$ of $\mathbf{C}^2$ by a finite subgroup G of $GL(2, \mathbf{C})$.

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$\{\gamma_m(X, x)\}_{m\in N}$ and $\{\delta_m(X, x)\}_{m\in N}$ are fundamental invariants of normal isolated singularities, which contain the most fundamental invariant of normal isolated singularities: the geometric genus $p_g=\gamma_1=\delta_1$. Including higher dimensional cases, many facts are known about these two plurigenera. We refer the reader to Ishii [7] on this subject.

As in the study of algebraic varieties, in the study of singularities, "hypersurface" singularities are investigated more deeply than the general cases. But now we cannot calculate γ_m explicitly, while, for example, we can calculate the Hodge numbers of toric hypersurfaces [1]. Our aim in this paper is to make some progress in this subject.

Let us give an outline of this paper:

In § 3, we will define one more set of plurigenera $\{\tilde{\gamma}_m(X, x)\}_{m\in N}$ for a hypersurface isolated singularity (X, x) and show that $\gamma_m(X, x)\leq\tilde{\gamma}_m(X, x)$ for every m. In [15], a formula to calculate $\delta_m(X, x)$ for a hypersurface isolated singularity (X, x) is already given. The definitions of $\{\tilde{\gamma}_m(X, x)\}_{m\in N}$ and the proof of the inequalities above are inspired by this formula. And we have a conjecture that under a suitable condition, $\gamma_m(X, x)=\tilde{\gamma}_m(X, x)$ holds for every m although we cannot identify such a condition now — see Remark 3.4.3.

In § 4, we will give several facts about $\{\tilde{\gamma}_m(X, x)\}_{m\in N}$ in case (X, x) is a so-called *purely elliptic singularity*, which is a singularity such that $\delta_m(X, x)=1$ for every m, and will give some example to $\{\gamma_m\}_{m\in N}$ and $\{\tilde{\gamma}_m\}_{m\in N}$, where $\gamma_m=\tilde{\gamma}_m$ holds for every m. Finally, we will give a note on $\{\tilde{\gamma}_m\}_{m\in N}$ for purely elliptic singularities of a special type.

2. Preliminaries

2.1. General settings. Let $(X, x)=(\mathrm{V}(f), \mathbf{0})$ be a hypersurface isolated singularity of dimension r defined by a polynomial $f=\sum_{\lambda\in(\mathbf{Z}_{\geq 0})^{r+1}}a_\lambda\cdot\mathbf{z}^\lambda\in\mathbf{C}[z_0, z_1, \ldots, z_r]$, where $x=\mathbf{0}$ is the origin of the $(r+1)$-dimensional affine space $\mathbf{C}^{r+1}$ and $\mathbf{z}^\lambda=\prod_{i=0}^{r}z_i^{\lambda_i}$ for each nonnegative integral vector $\lambda=(\lambda_0, \lambda_1, \ldots, \lambda_r)$. We alway assume that r is greater than or equals to two.

In this paper, we also always assume that the polynomial f is nondegenerate: The convex hull of the set $\bigcup_{a_\lambda\neq 0}(\lambda+(\mathbf{R}_{\geq 0})^{r+1})$ is called the *Newton diagram* of f and denoted by $\Gamma_+(f)$. The union of the compact faces of $\Gamma_+(f)$ is called the *Newton boundary* of $\Gamma_+(f)$ and denoted by $\Gamma(f)$. We associate a polynomial $f_\Delta:=\sum_{\lambda\in\Delta\cap(\mathbf{Z}_{\geq 0})^{r+1}}a_\lambda\cdot\mathbf{z}^\lambda$ with each face Δ of $\Gamma(f)$. A polynomial f is said to be *nondegenerate* if f_Δ has no solutions in $(\mathbf{C}^*)^{r+1}$ for any face Δ of $\Gamma(f)$.

We regard $\mathbf{C}^{r+1}$ as an affine toric variety V_Σ associated with the fan Σ in $N_\mathbf{R}\cong N\otimes_\mathbf{Z}\mathbf{R}$ for $N\cong\mathbf{Z}^{r+1}$ consisting of the faces of the cone $(\mathbf{R}_{\geq 0})^{r+1}$. Then the set of the exponents of monomials in $\mathbf{C}[z_0, z_1, \ldots, z_r]$ is naturally identified with the set of vectors with nonnegative entries in the dual space $M:=\mathrm{Hom}_\mathbf{Z}(N, \mathbf{Z})$. We refer the reader to Fulton [2] or Oda [12] on the theory of toric varieties.

For each vector $\mathbf{n}$ in $(\mathbf{R}_{\geq 0})^{r+1}$, define $l_f(\mathbf{n}):=\min\{\langle\lambda, \mathbf{n}\rangle \mid \lambda\in\Gamma_+(f)\}$ and

$$\Delta(\mathbf{n}):=\Gamma_+(f)\cap\{\lambda\in N_{\mathbf{R}} \mid \langle\lambda, \mathbf{n}\rangle=l(\mathbf{n})\}.$$

After these definitions, we define the dual diagram of each face Δ of $\Gamma_+(f)$:

$$\Delta^*:=\{\mathbf{n}\in N_{\mathbf{R}} \mid \Delta(\mathbf{n})=\Delta\}.$$

Then the closure of Δ^* in $N_{\mathbf{R}}$ is a cone and the set of the closures of the dual diagrams of all the faces of $\Gamma_+(f)$ forms a fan $\Sigma(f)$. This is a subdivision of the fan Σ and called the *dual fan* of the Newton diagram $\Gamma_+(f)$ of f.

Take a nonsingular subdivision $\hat{\Sigma}(f)$ of $\Sigma(f)$ and denote by $V_{\hat{\Sigma}(f)}$ the toric variety associated with $\hat{\Sigma}(f)$. We can associate a map of fans $(N, \hat{\Sigma}(f)) \to (N, \Sigma)$ with the subdivision $\hat{\Sigma}(f)$ of Σ. This map of fans induces a proper, birational morphism of toric varieties:

$$\Pi: V_{\hat{\Sigma}(f)} \to V_{\Sigma}\cong\mathbf{C}^{r+1}.$$

In the following, we assume that the subdivision $\hat{\Sigma}(f)$ is the one satisfying that the morphism Π induces an isomorphism

$$V_{\hat{\Sigma}(f)}\backslash\Pi^{-1}(x) \cong V_{\Sigma}\backslash\{x\}.$$

Then it is well-known that the restriction $\pi:\tilde{X}\to X$ of Π on the proper transform $\tilde{X}$ of X with respect to Π is a good resolution of the singularity (X, x) — see Kempf et al. [10].

3. The plurigenera $\{\gamma_m(X, x)\}_{m\in\mathbf{N}}$ and $\{\hat{\gamma}_m(X, x)\}_{m\in\mathbf{N}}$ for a hypersurface isolated singularity (X, x)

Let $(X, x)=(\mathbf{V}(f), 0)$, $\hat{\Sigma}(f)$ and $\Pi: V_{\hat{\Sigma}(f)} \to \mathbf{C}^{r+1}$ as in § 2.

3.1. Differential forms around a singularity

3.1.1. Let $\Omega^{r+1}_{\mathbf{C}^{r+1}}(X)$ be the sheaf of meromorphic differential forms on $\mathbf{C}^{r+1}$ with single pole along X. For any open subset U, we have

$$\Omega^{r+1}_{\mathbf{C}^{r+1}}(X)(U)=\mathcal{O}_{\mathbf{C}^{r+1}}(U)\cdot\eta,$$

where $\eta=dz_0\wedge dz_1\wedge\cdots\wedge dz_r/f$.

Let $(\Omega^{r+1}_{\mathbf{C}^{r+1}}(X))^{\otimes m}$ represent the m-times tensor product of $\Omega^{r+1}_{\mathbf{C}^{r+1}}(X)$ for each positive integer m.

3.1.2. Let U be an open neighborhood of the origin $x=0$ of $\mathbf{C}^{r+1}$ and let Ψ be an element of $(\Omega^{r+1}_{\mathbf{C}^{r+1}}(X))^{\otimes m}(U)$. Then Ψ is written as $\Psi=\psi\cdot\eta^{\otimes m}$, where $\psi\in\mathcal{O}_{\mathbf{C}^{r+1}}(U)$, that is a holomorpic function on U.

Now expand ψ in a power series at the origin x, say, $\psi=\sum_{\lambda\in(\mathbf{Z}_{\geq 0})^{r+1}}c_\lambda\cdot\mathbf{z}^\lambda$, where $\mathbf{z}^\lambda:=\prod_{i=0}^{r} z_i^{\lambda_i}$. This gives rise to an isomorphism

$$(\Omega^{r+1}_{\mathbf{C}^{r+1}}(X))^{\otimes m}_x \to \mathbf{C}\{z_0, z_1, \dots, z_r\}\cdot\eta_x^{\otimes m},$$

where $\mathbf{C}\{z_0, z_1, \dots, z_r\}$ is the ring of convergent power series and η_x is the germ of η at x. Denote by g_Ψ the image of each element $\Psi_x\in(\Omega^{r+1}_{\mathbf{C}^{r+1}}(X))^{\otimes m}_x$ by this isomorphism.

We define a diagram $\Gamma_+(g)$ for each power series g in $\mathbf{C}\{z_0, z_1, \dots, z_r\}$ to be the convex hull of the set $\bigcup_{c_\lambda\neq 0}(\lambda+(\mathbf{R}_{\geq 0})^{r+1})$ in $M_{\mathbf{R}}\cong\mathbf{R}^{r+1}$. For g and for each vector $\mathbf{n}$ of N, define

$$l_g(\mathbf{n}):=\min\{\langle\lambda, \mathbf{n}\rangle\,|\,\lambda\in\Gamma_+(g)\}.$$

3.1.3. The morphism $\Pi: V_{\hat{\Sigma}(f)}\to\mathbf{C}^{r+1}$ induces a homomorphism of sheaves:

$$\Pi^*: (\Omega^{r+1}_{\mathbf{C}^{r+1}}(X))^{\otimes m}\to\Pi_*\tilde{I}_*(\Omega^{r+1}_{V_{\hat{\Sigma}(f)}\setminus\Pi^{-1}(x)}(\tilde{X}\setminus\Pi^{-1}(x))^{\otimes m}),$$

which is defined by the pull-backs of differential forms, where $\tilde{I}: V_{\hat{\Sigma}(f)}\setminus\Pi^{-1}(x)\to V_{\hat{\Sigma}(f)}$ is the inclusion. Then the image of an element of $(\Omega^{r+1}_{\mathbf{C}^{r+1}}(X))^{\otimes m}(U)$ by Π^*, for an open neighborhood U of x, can be regarded as a meromorphic differential form with single pole along $\tilde{X}$ and with poles along irreducible components of $\Pi^{-1}(x)$. We note that $\Pi^{-1}(x)=\bigcup_{\hat{\rho}\in\hat{\Sigma}(f)(1)\setminus\Sigma(1)}D_{\hat{\rho}}$, where $\Sigma(1)$ (resp. $\hat{\Sigma}(f)(1)$) denotes the set of one-dimensional cones of Σ (resp. $\hat{\Sigma}(f)$) and $D_{\hat{\rho}}$ denotes the invariant divisor of the toric variety $V_{\hat{\Sigma}(f)}$ corresponding to one-dimensional cone $\hat{\rho}$.

Proposition 3.1. *Let Ψ_x be an element of $(\Omega^{r+1}_{\mathbf{C}^{r+1}}(X))^{\otimes m}_x$. Then $\Pi^*(\Psi_x)$ has zeros of order $l_{g_{\Psi_x}}(\mathbf{n}(\hat{\rho}))+m(\langle\mathbf{1}, \mathbf{n}(\hat{\rho})\rangle-1-l_f(\mathbf{n}(\hat{\rho})))$ along the invariant divisor $D_{\hat{\rho}}$ corresponding to $\hat{\rho}\in\hat{\Sigma}(f)(1)\setminus\Sigma(1)$, where $\mathbf{1}:=(1, 1, \dots, 1)\in M$.*

Proof. This is an easy modification of [15], Lemma 2.1. □

3.1.4. The homomorphism $(\Omega^{r+1}_{V_{\hat{\Sigma}(f)}}(\tilde{X}))^{\otimes m}\to\tilde{I}_*(\Omega^{r+1}_{V_{\hat{\Sigma}(f)}\setminus\Pi^{-1}(x)}(\tilde{X}\setminus\Pi^{-1}(x))^{\otimes m})$ induced by the inclusion $\tilde{I}: V_{\hat{\Sigma}(f)}\setminus\Pi^{-1}(x)\to V_{\hat{\Sigma}(f)}$ is injective, so that we may regard the former as a subsheaf of the latter.

By Proposition 3.1, for an element $\Psi_x\in(\Omega^{r+1}_{\mathbf{C}^{r+1}}(X))^{\otimes m}_x$, $\Pi^*(\Psi_x)\in\Pi_*(\Omega^{r+1}_{V_{\hat{\Sigma}(f)}}(\tilde{X}))^{\otimes m}_x$ if and only if $l_{g_{\Psi_x}}(\mathbf{n}(\hat{\rho}))+m(\langle\mathbf{1}, \mathbf{n}(\hat{\rho})\rangle-1-l_f(\mathbf{n}(\hat{\rho})))\geq 0$ for any $\hat{\rho}\in\hat{\Sigma}(f)(1)\setminus\Sigma(1)$. Here we define a subset $m\Delta_+(\hat{\Sigma}(f))$ of $M_{\mathbf{R}}$ to be the set:

$$\{\lambda\in(\mathbb{R}_{\geq 0})^{r+1}\,|\,\langle\frac{1}{m}\lambda, \mathbf{n}(\hat{\rho})\rangle\geq l_f(\mathbf{n}(\hat{\rho}))-\langle\mathbf{1}, \mathbf{n}(\hat{\rho})\rangle+1 \textit{ for any } \hat{\rho}\in\hat{\Sigma}(f)(1)\setminus\Sigma(1)\}.$$

Then we have the following:

Lemma 3.2. $\Pi^*(\Psi_x)\in\Pi_*(\Omega^{r+1}_{V_{\hat{\Sigma}(f)}}(\tilde{X}))^{\otimes m}_x$ *if and only if* $\Gamma_+(g_{\Psi_x})\subset m\Delta_+(\hat{\Sigma}(f))$.

Proof. As we mentioned above, $\Pi^*(\Psi_x)\in\Pi_*(\Omega^{r+1}_{V_{\hat{\Sigma}(f)}}(\tilde{X}))^{\otimes m}_x$ if and only if $l_{g_{\Psi_x}}(\mathbf{n}(\hat{\rho}))+m(\langle\mathbf{1}, \mathbf{n}(\hat{\rho})\rangle-1-l_f(\mathbf{n}(\hat{\rho})))\geq 0$ for any $\hat{\rho}\in\hat{\Sigma}(f)\setminus\Sigma(1)$.

Now assume that $\Pi^*(\Psi_x)\in\Pi_*(\Omega^{r+1}_{V_{\hat{\Sigma}(f)}}(\tilde{X}))^{\otimes m}_x$. Since $l_{g_{\Psi_x}}(\mathbf{n}(\hat{\rho}))=\min\{\langle\lambda, \mathbf{n}(\hat{\rho})\rangle\,|\,\lambda\in\Gamma_+(g_{\Psi_x})\}$, we have $\langle\lambda, \mathbf{n}(\hat{\rho})\rangle+m(\langle\mathbf{1}, \mathbf{n}\rangle-1-l_f(\mathbf{n}(\hat{\rho})))\geq 0$, so that

$$\langle\frac{1}{m}\lambda, \mathbf{n}(\hat{\rho})\rangle\geq l_f(\mathbf{n}(\hat{\rho}))-\langle\mathbf{1}, \mathbf{n}(\hat{\rho})\rangle+1$$

holds for any $\lambda\in\Gamma_+(g_{\Psi_x})$. Therefore, we obtain $\Gamma_+(g_{\Psi_x})\subset m\Delta_+(\hat{\Sigma}(f))$.

The converse is obvious. □

3.1.5. Define the $\mathbb{C}$-submodule $A^{(m)}_+$ of $(\Omega^{r+1}_{\mathbb{C}^{r+1}}(X))^{\otimes m}_x=\mathbb{C}\{z_0, z_1, \ldots, z_r\}\cdot\eta^{\otimes m}_x$ to be

$$\{g\cdot\eta^{\otimes m}_x\,|\,g\in m\Delta_+(\hat{\Sigma}(f))\}.$$

Lemma 3.3. *The homomorphism*

$$\Pi^*: A^{(m)}_+\to\Pi_*(\Omega^{r+1}_{V_{\hat{\Sigma}(f)}}(\tilde{X}))^{\otimes m}_x$$

is an isomorphism.

Proof. Recall that we have an inclusion $\Pi_*(\Omega^{r+1}_{V_{\hat{\Sigma}(f)}}(\tilde{X}))^{\otimes m}_x\hookrightarrow\Pi_*\tilde{I}_*(\Omega^{r+1}_{V_{\hat{\Sigma}(f)}\setminus\Pi^{-1}(x)}(\tilde{X}\setminus\Pi^{-1}(x)))^{\otimes m}_x$ given by the restriction map. Also note that we have an isomorphism $\Pi^*:(\Omega^{r+1}_{\mathbb{C}^{r+1}}(X))^{\otimes m}_x\cong(\Omega^{r+1}_{\mathbb{C}^{r+1}\setminus\{x\}}(X\setminus\{x\}))^{\otimes m}_x\xrightarrow{\sim}\Pi_*(\Omega^{r+1}_{V_{\hat{\Sigma}(f)}}(\tilde{X}\setminus\Pi^{-1}(x)))^{\otimes m}_x$ since $r+1\geq 3$ and $\Pi|_{V_{\hat{\Sigma}(f)}}: V_{\hat{\Sigma}(f)}\setminus\Pi^{-1}(x)\to\mathbb{C}^{r+1}\setminus\{x\}$ is an isomorphism.

Then it follows that for any element of $\Pi_*(\Omega^{r+1}_{V_{\hat{\Sigma}(f)}}(X))^{\otimes m}_x$, there exists an element $\Psi_x=\psi_x\cdot\eta^{\otimes m}_x$ in $(\Omega^{r+1}_{\mathbb{C}^{r+1}}(X))^{\otimes m}_x$. By Proposition 3.2, $\Gamma_+(\psi_x)\subset m\Delta_+(\hat{\Sigma}(f))$, so that $\Psi_x\in A^{(m)}_+$. Hence the map in the lemma is surjective.

The injectivity of the map is obvious. □

3.2. Poincaré residue maps

3.2.1. Define the sheaf Ω^r_X to be the direct image $i_*\Omega^r_{X\setminus\{x\}}$ of $\Omega^r_{X\setminus\{x\}}$ with respect to the inclusion

$i : X \backslash \{x\} \to X$.

Let U be an open subset of $\mathbf{C}^{r+1}$. We have a natural map, called the *Poincaré residue map*,

$$\mathrm{Res}_X : \Omega^{r+1}_{\mathbf{C}^{r+1}}(X)(U) \to \Omega^r_X(X \cap U)$$

by sending $\Psi = \psi \cdot \eta$ to $\mathrm{Res}_X(\Psi) := \psi|_{X\backslash\{x\}} \cdot \mathrm{Res}_X(\eta)$, where $\eta = dz_0 \wedge dz_1 \wedge \cdots \wedge dz_r / f$ and $\mathrm{Res}_X(\eta)$ $:= (-1)^{j-1}(dz_0 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_r)/(\partial f/\partial z_j)$ (if $\partial f/\partial z_j \neq 0$).

This induces the maps between the tensor products:

$$\mathrm{Res}^{\otimes m}_X : (\Omega^{r+1}_{\mathbf{C}^{r+1}}(X))^{\otimes m}(U) \to (\Omega^r_X)^{\otimes m}(X \cap U).$$

Therefore, we have a homomorphism of sheaves for any positive integer m:

$$\mathrm{Res}^{\otimes m}_X : (\Omega^{r+1}_{\mathbf{C}^{r+1}}(X))^{\otimes m} \to (\Omega^r_X)^{\otimes m}.$$

Lemma 3.4. *The homomorphism of sheaves* $\mathrm{Res}^{\otimes m}_X$ *is surjective for every* m.

Proof. This lemma follows the adjunction formula for a nonsingular variety and a Cartier divisor on it.

3.2.2. On the other hand, since $\tilde{X}$ is a nonsingular divisor of a nonsingular variety $V_{\Sigma(f)}$, we have another Poincaré residue map

$$\mathrm{Res}^{\otimes m}_{\tilde{X}} : (\Omega^{r+1}_{V_{\Sigma(f)}}(\tilde{X}))^{\otimes m} \to (\Omega^r_{\tilde{X}})^{\otimes m}$$

for each positive integer m.

3.3. The definition of $\{\gamma_m(X, x)\}_{m \in \mathbf{N}}$

3.3.1. Let (X, x) be a normal isolated Gorenstein singularity of dimension r and let $\pi : \tilde{X} \to X$ be a good resolution of the singularity (X, x). Then, the sequence

$$0 \to (\Omega^r_{\tilde{X}})^{\otimes m} \to \tilde{i}_*(\Omega^r_{\tilde{X}\backslash\pi^{-1}(x)})^{\otimes m}$$

is exact, where $\tilde{i} : \tilde{X}\backslash\pi^{-1}(x) \to \tilde{X}$ is the inclusion, so that

$$0 \to \pi_*(\Omega^r_{\tilde{X}})^{\otimes m} \to (\Omega^r_X)^{\otimes m} \cong \pi_* \tilde{i}_*(\Omega^r_{\tilde{X}\backslash\pi^{-1}(x)})^{\otimes m}$$

is also exact. Therefore, we can regard $\pi_*(\Omega^r_{\tilde{X}})^{\otimes m}$ as a subsheaf of $(\Omega^r_X)^{\otimes m}$.

Here we give the definition of the plurigenera $\{\gamma_m\}_{m \in \mathbf{N}}$ introduced by Knöller [11]:

Proposition and Definition 3.5 (Knöller). *For a normal isolated Gorenstein singularity (X, x) and a positive integer m, $(\omega_X^{\otimes m})_x/(\pi_*\omega_{\tilde{X}}^{\otimes m})_x$ is a finite dimensional vector space over $\mathbf{C}$, where $\pi: \tilde{X} \to X$ is a good resolution of the singularity (X, x).*

The dimension of this vector space is independent of the choice of a resolution and is denoted by $\gamma_m(X, x)$, that is,

$$\gamma_m(X, x) := \dim_{\mathbf{C}}(\omega_X^{\otimes m})_x/(\pi_*\omega_{\tilde{X}}^{\otimes m})_x = \dim_{\mathbf{C}}(\Omega_X^r)_x^{\otimes m}/\pi_*(\Omega_{\tilde{X}}^r)_x^{\otimes m},$$

where ω_X (resp. $\omega_{\tilde{X}}$) is the canonical sheaf of X (resp. $\tilde{X}$).

We sometimes use the symbol γ_m for $\gamma_m(X, x)$ for simplicity.

3.4. The definition of $\{\tilde{\gamma}_m(X, x)\}_{m\in\mathbf{N}}$ of a hypersurface isolated singularity (X, x) and relations to $\{\gamma_m(X, x)\}_{m\in\mathbf{N}}$

3.4.1. Next we will define the plurigenera $\{\tilde{\gamma}_m(X, x)\}_{m\in\mathbf{N}}$ for a hypersurface isolated singularity (X, x). The definition of $\{\tilde{\gamma}_m\}_{m\in\mathbf{N}}$ is inspired by the formula for another set of the plurigenera $\{\delta_m(X, x)\}_{m\in\mathbf{N}}$ in [15], Thorem 2.2. See [14] and [15] for details on $\{\delta_m\}_{m\in\mathbf{N}}$. So we assume that $(X, x)=(\mathrm{V}(f), \mathbf{0})$, where $f\in\mathbf{C}[z_0, z_1, ..., z_r]$ is a nondegenerate polynomial and $\mathbf{0}$ is the origin of $\mathbf{C}^{r+1}$ and we use the notation in § 2.

Recall that the definition of the diagram $m\Delta_+(\hat{\Sigma}(f))$ for each $m\in\mathbf{N}$. Define $m\Delta_-(\hat{\Sigma}(f)) := (\mathbf{R}_{\geq 0})^{r+1}\setminus m\Delta_+(\hat{\Sigma}(f))$.

Now we define $\{\tilde{\gamma}_m(X, x)\}_{m\in\mathbf{N}}$ as follows:

Definition 3.6. Let $(X, x)=(\mathrm{V}(f), \mathbf{0})$ be an isolated singularity at the origin of $\mathbf{C}^{r+1}$ defined by a nondegenerate polynomial $f\in\mathbf{C}[z_0, z_1, ..., z_r]$. Then we define $\tilde{\gamma}_m(X, x)$ to be

$$\#(m\Delta_-(\hat{\Sigma}(f))\cap M) - \#\{\lambda\in(\mathbf{Z}_{\geq 0})^{r+1}\subset M \mid (\lambda+\Gamma_+(f))\cap m\Delta_-(\hat{\Sigma}(f))\neq\emptyset\}$$

for each $m\in\mathbf{N}$, where $\#A$ denotes the cardinality of a set A.

3.4.2.

Theorem 3.7. *We have the following inequality between $\gamma_m(X, x)$ and $\tilde{\gamma}_m(X, x)$ for every $m\in\mathbf{N}$:*

$$\gamma_m(X, x) \leq \tilde{\gamma}_m(X, x).$$

The following proof is a modification of [15], Thorem 2.2 for the formula for δ_m.

Proof. Let θ be an element of $(\Omega_X^r)_x^{\otimes m}$. By Lemma 3.4, the homomorphism $\mathrm{Res}_X^{\otimes m}:(\Omega_{\mathbf{C}^{r+1}}^{r+1}(X))_x^{\otimes m}\to(\Omega_X^r)_x^{\otimes m}$ between the stalks at x is surjective. Hence there exists an element $\Psi_x=\psi_x\cdot\eta_x^{\otimes m}$ such that $\mathrm{Res}_X^{\otimes m}(\Psi_x)=\theta$, where $\psi_x\in\mathcal{O}_{X,x}$. We may regard ψ_x as a convergent power series: $\sum_{\lambda\in(\mathbf{Z}_{\geq 0})^{r+1}}c_\lambda\cdot\mathbf{z}^\lambda$.

We write ψ_x as a sum of two series: $\psi_{x,-}+\psi_{x,+}$, where $\psi_{x,+}:=\sum_{\lambda\in m\Delta_+(\hat{\Sigma}(f))\cap(\mathbf{Z}_{\geq 0})^{r+1}}c_\lambda\cdot\mathbf{z}^\lambda$ and $\psi_{x,-}:=\psi_x-\psi_{x,+}$. Define $\Psi_{x,+}:=\psi_{x,+}\cdot\eta_x^{\otimes m}$ and $\Psi_{x,-}:=\psi_{x,-}\cdot\eta_x^{\otimes m}$. Then $\Psi_x=\Psi_{x,-}+\Psi_{x,+}$. Since $\psi_{x,+}\in A_+^{(m)}$, we have $\Pi^*(\Psi_{x,+})\in\Pi_*(\Omega_{V\hat{\Sigma}(f)}^{r+1}(\tilde{X}))_x^{\otimes m}$ by Lemma 3.2.

Since the diagram:

$$\begin{array}{ccc}
A_+^{(m)} & \xrightarrow{\mathrm{Res}_X^{\otimes m}} & (\Omega_X^r)_x^{\otimes m}=i_*(\Omega_{X\setminus\{x\}}^r)_x^{\otimes m} \\
\Pi^*\downarrow & & \downarrow\pi^* \\
\Pi_*(\Omega_{\tilde{X}}^{r+1}(\tilde{X}))_x^{\otimes m} & \xrightarrow[\Pi_*(\mathrm{Res}_{\tilde{X}}^{\otimes m})]{} & \pi_*\tilde{i}_*(\Omega_{\tilde{X}\setminus\pi^{-1}(x)}^r)_x^{\otimes m}
\end{array}$$

commutes, $\pi^*(\mathrm{Res}_X^{\otimes m}(\Psi_{x,+}))\in\pi_*(\Omega_{\tilde{X}}^r)_x^{\otimes m}$. Therefore, $(\Omega_X^r)_x^{\otimes m}/\pi_*(\Omega_{\tilde{X}}^r)_x^{\otimes m}$ is generated by the elements $\{\mathrm{Res}_X^{\otimes m}(\mathbf{z}^\lambda\cdot\omega_X^{\otimes m})\mid\lambda\in m\Delta_-(\hat{\Sigma}(f))\cap(\mathbf{Z}_{\geq 0})^{r+1}\}$.

Next assume that $\pi^*(\mathrm{Res}_X^{\otimes m}(\Psi_{x,-}))$ is contained by the image of the homomorphism:

(1) $$\Pi_*(\mathrm{Res}_{\tilde{X}}^{\otimes m}):\Pi_*(\Omega_{V\hat{\Sigma}(f)}^{r+1}(\tilde{X}))^{\otimes m}\to\Pi_*(\Omega_{\tilde{X}}^r)^{\otimes m}.$$

Then, by Lemma 3.3, there exists an element $\Phi_x=\phi_x\cdot\eta_x^{\otimes m}\in A_+^{(m)}$ such that the image of the composite of Π^* and $\Pi_*(\mathrm{Res}_{\tilde{X}}^{\otimes m})$ is just $\pi^*(\mathrm{Res}_X^{\otimes m}(\Psi_{x,-}))$. Then we have $\pi^*(\psi_{x,-}|_X)=\pi^*(\phi_x|_X)$ in $\pi_*(\mathcal{O}_{\tilde{X}})_x$. Since $\pi^*:\mathcal{O}_{X,x}\to\pi_*(\mathcal{O}_{\tilde{X}})_x$ is injective, $\psi_{x,-}|_X=\phi_x|_X$ holds, so that there exists a power series $\sum_{\lambda\in(\mathbf{Z}_{\geq 0})^{r+1}}d_\lambda\cdot\mathbf{z}^\lambda$ such that $\psi_{x,-}-\phi_x=(\sum_{\lambda\in(\mathbf{Z}_{\geq 0})^{r+1}}d_\lambda\cdot\mathbf{z}^\lambda)\cdot f_x$. It follows that $\psi_{x,-}=\sum_{\lambda\in(\mathbf{Z}_{\geq 0})^{r+1}}d_\lambda\cdot(\mathbf{z}^\lambda\cdot f_x)_-$. This completes the proof. □

3.4.3. *Remark.* In the proof of the theorem just above, if the homomorphism (1) is surjective, we obtain an equation:

$$\gamma_m(X,x)=\tilde{\gamma}_m(X,x).$$

Until now, we cannot state the condition under which the homomorphism (1) is surjective.

3.5. The diagram $\Delta_+(\hat{\Sigma}(f))$

3.5.1. We can rewrite $m\Delta_+(\hat{\Sigma}(f))$ as

$$\{\lambda\in(\mathbf{Z}_{\geq 0})^{r+1}\subset M\mid\frac{1}{m}\lambda\in\Delta_+(\hat{\Sigma}(f))\}$$

and we have

$$m\Delta_-(\hat{\Sigma}(f))=(\mathbb{R}_{\geq 0})^{r+1}\backslash m\Delta_+(\hat{\Sigma}(f)).$$

Hence we obtain the following:

Corollary 3.8. *$\tilde{\gamma}_m(X, x)$ is determined by the diagram $\Delta_+(\hat{\Sigma}(f))$.*

3.5.2. *Remark.* Here, we note that the diagram $\Delta_+(\hat{\Sigma}(f))$ is the same $\Gamma_{\hat{\Sigma}(f)}$ introduced by Ishii [8] in order to construct the canonical model of the hypersurface isolated singularity $(X, x)=(\mathrm{V}(f), \mathbf{0})$. Indeed, Ishii obtained the following result:

Theorem 3.9 (Ishii). *Let Σ_0 be the dual fan associated with the polytope $\Delta_+(\hat{\Sigma}(f))$ and let $\varphi: X_0 \to X$ be the birational morphism induced by the subdivision Σ_0 of Σ. Denote by $\eta: X_0' \to X_0$ be the normalization.*

Then the composite $\varphi \circ \eta: X_0' \to X$ is the canonical modification.

4. Application to purely elliptic singularities

4.1. Preliminaries

4.1.1. In this section, we will apply the result obtained in § 3 to a particular class of singularities, called purely elliptic singularities. First of all, we recall the definition of the plurigenera $\{\delta_m(X, x)\}_{m\in\mathrm{N}}$ to introduce the notion of purely elliptic singularities.

Proposition and Definition 4.1 ([14]). *For a normal isolated Gorenstein singularity (X, x) and a positive integer m, $(\omega_X^{\otimes m})_x/\pi_*(\omega_{\tilde{X}}^{\otimes m}((m-1)E))_x$ is a finite dimensional vector space over $\mathbf{C}$, where $\pi: \tilde{X} \to X$ is a good resolution of the singularity (X, x), ω_X (resp. $\omega_{\tilde{X}}$) is the canonical sheaf of X (resp. $\tilde{X}$), and E is the exceptional divisor.*

The dimension of the vector space is independent of the choice of a good resolution π and denoted by $\delta_m(X, x)$, that is,

$$\delta_m(X, x):=\dim_{\mathbf{C}}(\omega_X^{\otimes m})_x/(\pi_*\omega_{\tilde{X}}^{\otimes m}((m-1)E))_x.$$

Definition 4.2 ([14]). A normal isolated Gorenstein singularity (X, x) is said to be a *purely elliptic singularity* if $\delta_m=1$ for all $m\in\mathbf{N}$.

4.1.2. There is a criterion whether an isolated singularity defined by a polynomial is a purely elliptic singularity:

Proposition 4.3 ([15]). *Let $(X, x)=(\mathrm{V}(f), 0)$ be an r-dimensional isolated singularity defined by a nondegenerate polynomial $f\in\mathbf{C}[z_0, z_1, \dots, z_r]$.*

Then (X, x) is a purely elliptic singularity if and only if the Newton boundary $\Gamma(f)$ contains the vector $\mathbf{1}=(1, 1, \dots, 1)\in M$.

4.2. The essential cone and the diagram $m\Delta_+(\hat{\Sigma}(f))$

4.2.1. Let $(X, x)=(\mathrm{V}(f), 0)$ be an r-dimensional purely elliptic singularity defined by a nondegenerate polynomial $f\in\mathbf{C}[z_0, z_1, \dots, z_r]$.

Let $\hat{\Sigma}(f)$ be a nonsingular subdivision of the dual fan $\Sigma(f)$ of f such that the primitive integral generators of one-dimensional cones are all positive vectors.

Recall that $m\Delta_+(\hat{\Sigma}(f))$ is the set:

$$\{\lambda\in(\mathbf{Z}_{\geq 0})^{r+1}\subset M \mid \langle\frac{1}{m}\lambda, \mathbf{n}(\hat{\rho})\rangle\geq l_f(\mathbf{n}(\hat{\rho}))-\langle\mathbf{1}, \mathbf{n}(\hat{\rho})\rangle+1 \text{ for any } \hat{\rho}\in\hat{\Sigma}(f)(1)\backslash\Sigma(1)\}.$$

We will divide the set $\hat{\Sigma}(f)(1)\backslash\Sigma(1)$ into two classes. To do this, we introduce the notion of the essential cone due to Ishii [6]:

Definition 4.4 (Ishii). The set

$$C_1(f):=\{\mathbf{n}\in N_{\mathbf{R}} \mid l_f(\mathbf{n})\geq\langle\mathbf{1}, \mathbf{n}\rangle\}$$

is a cone in $N_{\mathbf{R}}$ and called the *essential cone.*

4.2.2. In case $(X, x)=(\mathrm{V}, 0)$ is a purely elliptic singularity, $l_f(\mathbf{n})\leq\langle\mathbf{1}, \mathbf{n}\rangle$ holds for any $\mathbf{n}\in N_{\mathbf{R}}$ since the Newton boundary $\Gamma(f)$ contains $\mathbf{1}\in M$. Hence in this case, we have

$$C_1(f)=\{\mathbf{n}\in N_{\mathbf{R}} \mid l_f(\mathbf{n})=\langle\mathbf{1}, \mathbf{n}\rangle\}.$$

If $\mathbf{n}\in(\mathbb{R}_{\geq 0})^{r+1}$ but $\mathbf{n}\notin C_1(f)$, then $l_f(\mathbf{n})<\langle\mathbf{1}, \mathbf{n}\rangle$, so that $l_f(\mathbf{n})-\langle\mathbf{1}, \mathbf{n}\rangle+1\leq 0$. Hence

$$\langle\frac{1}{m}\lambda, \mathbf{n}\rangle\geq l_f(\mathbf{n})-\langle\mathbf{1}, \mathbf{n}\rangle+1$$

holds for any $\lambda\in(\mathbf{Z}_{\geq 0})^{r+1}$. Thus, we can write $m\Delta_+(\hat{\Sigma}(f))$ as follows:

Proposition 4.5. *If $(X, x)=(\mathrm{V}(f), 0)$ is a purely elliptic singularity, then $m\Delta_+(\hat{\Sigma}(f))$ equals to the set:*

$$\{\lambda\in(\mathbf{Z}_{\geq 0})^{r+1}\subset M\,|\,\langle\frac{1}{m}\lambda,\mathbf{n}(\hat{\rho})\rangle\geq 1\ \textit{for any}\ \hat{\rho}\in(\hat{\Sigma}(f)(1)\backslash\Sigma(1))\cap C_1(f)(1)\}$$

or

$$\{\lambda\in(\mathbf{Z}_{\geq 0})^{r+1}\subset M\,|\,\langle\lambda,\mathbf{n}(\hat{\rho})\rangle\geq m\ \textit{for any}\ \hat{\rho}\in(\hat{\Sigma}(f)(1)\backslash\Sigma(1))\cap C_1(f)(1)\}.$$

When $(X, x)=(\mathbf{V}(f), \mathbf{0})$ is a purely elliptic singularity, by this proposition, we have only to take a nonsingular subdivision of the essential cone $C_1(f)$ in order to obtain $m\Delta_+(f)$. An algorithm to get a nonsingular subdivision is given by Oka [13].

4.3. $\hat{\gamma}_m$ and special vectors: systems of weights

4.3.1. Here we investigate the most simple case, that is the diagram $\Delta_+(\hat{\Sigma}(f))$ has only one compact face. In this case, there exists a special integral vector in the essential cone $C_1(f)$, which Ishii [6] calls the absolutely minimal vector in $C_1(f)$: Denote by $C_1(f)[1]$ the set of primitive integral vectors in $C_1(f)$. Then a primitive integral vector $\mathbf{p}=(p_0, p_1, \dots, p_r)$ is said to be *absolutely minimal* if for any element $\mathbf{q}=(q_0, q_1, \dots, q_r)\in C_1(f)[1]$, $p_i\leq q_i$ holds for $i=0, 1, \dots, r$.

The following is easy to check.

Proposition 4.6. *If there exists an absolutely minimal vector* $\mathbf{p}$ *in* $C_1(f)[1]$, *then the diagram* $m\Delta_+(\hat{\Sigma}(f))$ *has only one compact face, more precisely,*

$$\begin{aligned} m\Delta_+(\hat{\Sigma}(f)) &= m\Delta_+(\mathbf{p}) \\ &:=\{\lambda\in(\mathbf{R}_{\geq 0})^{r+1}\,|\,\langle\lambda,\mathbf{p}\rangle\geq m\} \end{aligned}$$

and its unique compact face is

$$\{\lambda\in(\mathbf{R}_{\geq 0})^{r+1}\,|\,\langle\lambda,\mathbf{p}\rangle=m\}.$$

Conversely, if $\Delta_+(\hat{\Sigma}(f))$ *has only one compact face* δ_0, *then the primitive integral generator* $\mathbf{p}=\mathbf{n}(\delta_0^*)$ *of the dual cone* δ_0^* *is the absolutely minimal vector in* $C_1(f)[1]$.

4.3.2. *Remark.* Due to Ishii [6], if $\Delta_+(\hat{\Sigma}(f))$ has a unique compact face δ_0, then the canonical model of the purely elliptic singularity $(X, x)=(\mathbf{V}(f), \mathbf{0})$ is obtained by the weighted blow-up with respect to the integral vector $\mathbf{p}=\mathbf{n}(\delta_0^*)$.

The statement that the essential cone of a hypersurface purely elliptic singularity contains the absolutely minimal vector is false in general. Counter examples are given by Ishii [6] § 4.1. Nevertheless, many examples of hypersurface purely elliptic singularities whose essential cones contains the absolutely minimal vectors are known.

4.3.3. For a positive vector $\mathbf{p}=(p_0, p_1, \ldots, p_r)\in M$, denote $p_0+p_1+\cdots+p_r$ by deg $\mathbf{p}$. Then we can write down the formula for $\bar{\gamma}_m(X, x)$ as the following simple form:

Theorem 4.7. *Let $(X, x)=(\mathbf{V}(f), \mathbf{0})$ be a purely elliptic singularity defined by a nondegenerate polynomial $f\in\mathbf{C}[z_0, z_1, \ldots, z_r]$.*

Assume that $\Delta_+(\hat{\Sigma}(f))$ has a unique compact face δ_0 and let $\mathbf{p}:=\mathbf{n}(\delta_0^*)$ be the primitive integral generator of the dual cone δ_0^*. Then we have

$$\bar{\gamma}_m(X, x)=\#(m\Delta_-(\mathbf{p})\cap M)-\#\{\lambda\in(\mathbf{Z}_{\geq 0})^{r+1}\subset M\,|\,(\lambda+\Gamma_+(\mathbf{p}))\cap m\Delta_-(\mathbf{p})\neq\emptyset\} \tag{2}$$

$$=\#(m\Delta_-(\mathbf{p})\cap M)-\#((m-\deg\mathbf{p})\Delta_-(\mathbf{p})\cap M), \tag{3}$$

where $m\Delta_-(\mathbf{p}):=(\mathbb{R}_{\geq 0})^{r+1}\backslash m\Delta_+(\mathbf{p})$ and $\Gamma_+(\mathbf{p}):=(\langle\mathbf{1}, \mathbf{p}\rangle)\Delta_+(\mathbf{p})$.

Proof. We have only to show that the set

$$\{\lambda\in(\mathbf{Z}_{\geq 0})^{r+1}\,|\,(\lambda+\Gamma_+(f))\cap m\Delta_-(\mathbf{p})\neq\emptyset\}$$

equals to the sets

$$\{\lambda\in(\mathbf{Z}_{\geq 0})^{r+1}\,|\,(\lambda+\Gamma_+(\mathbf{p}))\cap m\Delta_-(\mathbf{p})\neq\emptyset\}$$

and

$$(m-\deg\mathbf{p})\Delta_-(\mathbf{p})\cap M.$$

For $\lambda\in(\mathbf{Z}_{\geq 0})^{r+1}$, if $(\lambda+\Gamma_+(f))\cap m_-(\mathbf{p})\neq\emptyset$, then $(\lambda+\Gamma_+(\mathbf{p}))\cap m\Delta_-(\mathbf{p})\neq\emptyset$ since $\Gamma_+(f)\subset\Gamma_+(\mathbf{p})$ holds. Conversely, assume that for $\lambda\in(\mathbf{Z}_{\geq 0})^{r+1}$, $(\lambda+\Gamma_+(\mathbf{p}))\cap m\Delta_-(\mathbf{p})\neq\emptyset$. Then since the compact face of $\Gamma_+(\mathbf{p})$ is parallel to the boundary of $m\Delta_+(\mathbf{p})$ and contains $\mathbf{1}\in M$, $\lambda+\mathbf{1}\in(\lambda+\Gamma_+(\mathbf{p}))\cap m\Delta_-(\mathbf{p})$. Note that the Newton diagram $\Gamma_+(f)$ of a purely elliptic singularity contains $\mathbf{1}\in M$, so that $\lambda+\mathbf{1}\in(\lambda+\Gamma_+(f))\cap m\Delta_-(\mathbf{p})\neq\emptyset$. Thus the first set equals to the second one.

Next note that $(\lambda+\Gamma_+(\mathbf{p}))\cap m\Delta_-(\mathbf{p})\neq\emptyset$ if and only if $\lambda+\mathbf{1}\in m\Delta_-(\mathbf{p})$ for $\lambda\in(\mathbf{Z}_{\geq 0})^{r+1}$, which is equivalent to $\langle\lambda+\mathbf{1}, \mathbf{p}\rangle<m$, that is, $\langle\lambda, \mathbf{p}\rangle<m-\deg\mathbf{p}$. Thus the second set equals to the third one. □

A set of positive integers $(p_0, p_1, \ldots, p_r)$ is sometimes called a *system of weights* or *weight system*. We may assume that the absolutely minimal vector $\mathbf{p}=(p_0, p_1, \ldots, p_r)$ has positive integers as its entries, so that we can regard it as a system of weights.

By the previous theorem, we have the following:

Corollary 4.8. *Let $(X, x)=(\mathrm{V}(f), 0)$ be a purely elliptic singularity defined by a nondegenerate polynomial $f\in\mathbf{C}[z_0, z_1, \dots, z_r]$.*

Assume that $\Delta_+(\hat{\Sigma}(f))$ has a unique compact face δ_0 and let $\mathbf{p}:=\mathbf{n}(\delta_0^*)$ be the primitive integral generator of the dual cone δ_0^*. Then $\tilde{\gamma}_m(X, x)$ is determined completely by the weight system $\mathbf{p}=(p_0, p_1, \dots, p_r)$.

4.3.4. *Examples.* It is well-known that a two-dimensional hypersurface purely elliptic singularity is analytically equivalent to an isolated singularity $(X, x)=(\mathrm{V}(f), 0)$ defined by the polynomial either case (1):

$$z_0^{3+p}+z_1^{3+q}+z_2^{3+r}+\lambda z_0z_1z_2, \quad (p, q, r\geq 0, \lambda\neq 0),$$

the case (2):

$$z_0^2+z_1^{4+q}+z_2^{4+r}+\lambda z_0z_1z_2, \quad (p, r\geq 0, \lambda\neq 0),$$

or the case (3):

$$z_2^2+z_1^3+z_2^{6+r}+\lambda z_0z_1z_2, \quad (r\geq 0, \lambda\neq 0).$$

In the case (1), $\mathbf{p}:=(1, 1, 1)\in M$ is the absolutely minimal vector in the essential cone $C_1(f)$ and we have

$$\#(m\Delta_-(\mathbf{p})\cap M)=\frac{1}{6}m(m+1)(m+2).$$

Now that $\deg\mathbf{p}=3$, we have

$$\tilde{\gamma}_m(X, x)=\#(m\Delta_-(\mathbf{p})\cap M)-\#((m-\deg\mathbf{p})\Delta_-(\mathbf{p})\cap M)=\frac{3}{2}(m^2-m)+1.$$

In the case (2), $\mathbf{p}:=(2, 1, 1)\in M$ is the absolutely minimal vector in $C_1(f)$ and we have

$$\#(m\Delta_-(\mathbf{p})\cap M)=\begin{cases}\frac{1}{24}m(m+2)(2m+5), & m \text{ even},\\ \frac{1}{24}(m+1)(m+3)(2m+1), & m \text{ odd}.\end{cases}$$

Then since $\deg\mathbf{p}=4$, when m is even or or odd, we have

$$\tilde{\gamma}_m(X, x)=\#(m\Delta_-(\mathbf{p})\cap M)-\#((m-\deg\mathbf{p})\Delta_-(\mathbf{p})\cap M)=\frac{2}{2}(m^2-m)+1.$$

Finally, in the case (3), $\mathbf{p}=(3, 2, 1)$ is the absolutely minimal vector in $C_1(f)$ and $\#(m\Delta_-(\mathbf{p})$

$\cap M)$ equals to

$$\begin{cases} \frac{1}{72}m(m+6)(2m+3)+\frac{1}{6}m, & m\equiv 0 \mod 6, \\ \frac{1}{72}(m-1)(m+5)(2m+1)+\frac{1}{12}(m-1)(m+5)+\frac{1}{6}(m-1)+1, & m\equiv 1 \mod 6, \\ \frac{1}{72}(m-2)(m+4)(2m-1)+\frac{1}{6}(m-1)(m+4)+\frac{1}{6}(m-2)+1, & m\equiv 2 \mod 6, \\ \frac{1}{72}(m-3)(m+3)(2m-3)+\frac{1}{4}(m-1)(m+3)+\frac{1}{6}(m-3)+1, & m\equiv 3 \mod 6, \\ \frac{1}{72}(m-4)(m+2)(2m-5)+\frac{1}{3}(m-1)(m+2)+\frac{1}{6}(m-4)+1, & m\equiv 4 \mod 6, \\ \frac{1}{72}(m-5)(m+1)(2m-7)+\frac{5}{12}(m-1)(m+1)+\frac{1}{6}(m-5)+1, & m\equiv 5 \mod 6. \end{cases}$$

Although $\#(m\Delta_-(\mathbf{p})\cap M)$ are given by the six polynomials in m as above, $\#(m\Delta_-(\mathbf{p})\cap M)-\#((m-\deg\mathbf{p})\Delta_-(\mathbf{p})\cap M)$ are represented by a polynomial

$$\tilde{\gamma}_m(X, x)=\#(m\Delta_-(\mathbf{p})\cap M)-\#((m-\deg\mathbf{p})\Delta_-(\mathbf{p})\cap M)=\frac{1}{2}(m^2-m)+1.$$

On the other hand, we have the following formula which give γ_m for a Gorenstein surface singularity:

Proposition 4.9 (Kato [9]). *Let (X, x) be a Gorenstein singularity of dimension two and π: $(\tilde{X}, E) \to (X, x)$ be the minimal resolution.*

Then

$$\gamma_m(X, x)=-K^2(m^2-m)+p_g(X, x),$$

where K is a canonical divisor on $\tilde{X}$ and $p_g(X, x)$ is the geometric genus of the singularity (X, x),

Now $p_g(X, x)=\delta_1(X, x)=1$ holds for the singularity (X, x) defined by a polynomial as in (1), (2) of (3) since (X, x) is a purely elliptic singularity. Moreover it is well-known that $K^2=-3$ for the singularity defined by a polynomial as in (1), that $K^2=-2$ for the singularity defined by a polynomial as in (2), and that $K^2=-1$ for the singularity defined by a polynomial as in (3).

These are examples where $\gamma_m(X, x)=\tilde{\gamma}_m(X, x)$ for every m.

4.4. Types of purely elliptic singularities

4.4.1. Let (X, x) be an r-dimentional purely elliptic singularity and $\pi: \tilde{X} \to X$ be a good resolution of (X, x). Then it is known that $K_{\tilde{X}} = \pi^* K_X + E_I - E_J$ holds, where K_X (resp. $K_{\tilde{X}}$) is a canonical divisor of X (resp. $\tilde{X}$), E_I is a positive divisor and E_J is a reduced divisor.

Ishii [5] classified the purely elliptic singularities into r classes by means of the mixed Hodge structures of the cohomologies of the support of E_J:

Proposition 4.10 (Ishii).

$$\mathbf{C} \cong H^{r-1}(E_J, \mathcal{O}_J) \cong \mathrm{Gr}_F^0 H^{r-1}(E_J) = \bigoplus_{i=0}^{r-1} H_{r-1}^{0,i}(E_J),$$

where $H_m^{i,j}(*)$ *is the* (i, j)*-component of* $\mathrm{Gr}_{i+j}^W H^m(*)$.

Hence, in particular, for a unique i $(0 \le i \le r-1)$,

$$H^{r-1}(E_J, \mathcal{O}_J) = H_{r-1}^{0,i}(E_J) \cong \mathbf{C}.$$

Definition 4.11 (Ishii). A purely elliptic singularity (X, x) is said to be *of type* $(0, i)$ if $H^{r-1}(E_J, \mathcal{O}_{E_J})$ consists of the $(0, i)$-Hodge component.

In case $(X, x) = (\mathrm{V}(f), \mathbf{0})$ is a purely elliptic singularity defined by a nondegenerate polynomial $f \in \mathbf{C}[z_0, z_1, \ldots, z_r]$, we also have a criterion of what type the singularity (X, x) is:

Proposition 4.12 ([15]). *Let* $(X, x) = (\mathrm{V}(f), \mathbf{0})$ *be an r-dimensional purely elliptic singularity defined by a nondegenerate polynomial f. Then* (X, x) *is of type* $(0, \dim \Delta_1(f) - 1)$ *if* $\dim \Delta_1(f) \ge 2$ *and of type* $(0, 0)$ *if* $\dim \Delta_1(f) = 1$ *or* 0, *where* $\Delta_1(f)$ *is the face of the Newton boundary* $\Gamma(f)$ *containing* $\mathbf{1} \in M$ *in its relative interior.*

We call the face $\Delta_1(f)$ in the proposition just above the *initial face* of $\Gamma(f)$.

4.4.2. Let $(X, x) = (\mathrm{V}(f), \mathbf{0})$ be a purely elliptic singularity of type $(0, r-1)$. Then by the criterion of Watanabe's 4.12, the initial face $\Delta_1(f)$ is of r-dimension.

It follows that the essential cone $C_1(f)$ is a one-dimensional cone and its primitive integral generator $\mathbf{p}$ is the absolutely minimal vector in $C_1(f)$. Thus, we can attach a system of weights $\mathbf{p} = (p_0, p_1, \ldots, p_r)$ to each hypersurface purely elliptic singularity of type $(0, r-1)$.

Therefore, we can apply Theorem 4.7 to every r-dimensional hypersurface purely elliptic singularity of type $(0, r-1)$ and hence $\{\bar{\gamma}_m(X, x)\}_{m \in \mathbf{N}}$ are determined completely by the system of weights $\mathbf{p}$.

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