Kumamoto J. Math.
Vol. 13, 49-57 March (2000)

On affine translation surfaces of constant mean curvature

Huafei Sun

(Received December 21, 1999
Revised January 28, 2000)

1. Introduction

Let E^3 be a 3-dimensional Euclidean space. A surface in E^3 is called a translation surface if it is obtained as a graph of a function $F(x, y)=p(x)+q(y)$, where $p(x)$ and $q(y)$ are differentiable functions. And a surface is said to be minimal if its mean curvature is zero identically. As well known, for the minimal translation surfaces in the 3-dimensional Euclidean space E^3, F. Scherk proved the following classical theorem: Let M be a minimal translation surface in E^3, then it must be a plane or a surface which is a graph of a function is as $az=\ln\cos(ax)-\ln\cos(ay)$, where a is a non-zero constant.

It is natural to consider the similar problem that affine minimal translation surfaces in the affine space R^3. F. Manhart [1] proved:

Theorem A. *Let M be an nondegenerate affine minimal translation surface in R^3. Then M is one of the graph of the following functions under affine transformations:*

(i)
$$z=x^2\pm y^2$$
$$z=x^{\frac{2}{3}}\pm y^{\frac{2}{3}}$$
$$z=x^2\pm y^{\frac{2}{3}},$$

(ii)
$$z=\ln x-\ln y$$
$$z=\pm\ln x\pm(1+\cos\mathrm{h}t),\quad t+\sin\mathrm{h}t=y$$
$$z=\pm\ln x\pm(1-\cos t),\quad t-\sin t=y$$
$$z=\pm(1+\cos\mathrm{h}t)\pm(1+\cos\mathrm{h}s),\quad t+\sin\mathrm{h}t=x,\quad s+\sin\mathrm{h}s=y$$
$$z=\pm(1+\cos\mathrm{h}t)\pm(1-\cos s),\quad t+\sin\mathrm{h}t=x,\quad s-\sin\mathrm{h}s=y$$
$$z=\pm(1-\cos t)\pm(1-\cos s),\quad t-\sin t=x,\quad s-\sin s=y.$$

In [4], M. Magid studied nondegenerate affine surfaces which are both affine minimal and

The subject is supported by JSPS

timelike minimal, and obtained a complete classification. In the present paper, we study affine translation surfaces of constant affine mean curvature and obtain:

Theorem. *Let M be a nondegenerate affine translation surface of constant affine mean curvature in R^3. Then besides (i) and (ii) in Theorem A, M is the graph of the following function up to equiaffine transformations:*

(iii) $$z=\frac{1}{2}Ax^2+3^{-\frac{4}{3}}A^{\frac{1}{3}}\int_{y_0}^{y}\left\{\int_{v_0}^{v}(Hs^2+C)^{-\frac{3}{4}}ds\right\}dv,$$

where A and C are constant and $H(\neq 0)$ is the affine mean curvature of M.

Acknowledgement. I would like to thank Professor K. Yamada for his advice and encouragement and Professor T. Kurose for his valuable suggestions. I also would like to express my thanks to Professor T. Sasaki for useful help.

2. Preliminaries

Let $f: M \to R^3$ be an immersion of a connected differentiable 2-manifold into the affine space R^3 equipped with usual flat connection D and the canonical volume element ω, and ξ an arbitrary vector field along f transversal to $f(M)$. For any vector fields X and Y on M, we write

$$D_X f_*(Y)=f_*(\nabla_X Y)+h(X, Y)\xi, \tag{2.1}$$

$$D_X\xi=-f_*(SX)+\tau(X)\xi, \tag{2.2}$$

thus definiting an affine connection ∇, a symmetric tensor h of type (0, 2) and a tensor S of type (1, 1) which is called the affine shape operator, and a 1-form τ, called the transversal connection form. We call h the affine fundamental form. We define by $H=\frac{1}{2}$ trace S the affine mean curvature of M. We call f affine minimal if H is zero identically. We define a volume elenment θ on M by

$$\theta(X_1, X_2)=\omega(f_*(X_1), f_*(X_2), \xi)=\det(f_*(X_1), f_*(X_2), \xi), \tag{2.3}$$

for any tangent vectors X_1, X_2 of M.

We say that f is nondegenerate if h is nondegenerate. This condition does not depand on choise of ξ. In this case, it is known that there is a unique choice of ξ such that the corresponding induced connection ∇, the nondegenerate metric h, and the induced volume element θ satisfy

(i) $\nabla\theta=0$, thus (∇, θ) is an equiaffine structure on M;

(ii) $\theta=\omega_h$, where $\omega_h(X_1, X_2)=|\det(h(X_i, X_j))|^{\frac{1}{2}}$ (the volume element given by h). We call such a pair (f, ξ) a Blaschke immersion. We call ∇ the induced connection and h the affine metric. Condition (i) implies that $\tau=0$ so that $D_X\xi=-f_*(SX)$.

Let $x^3=F(x^1, x^2)$ be a differentiable function on a domain $D\subset R^2$. We consider an immersion

$$f: D\ni(x^1, x^2)\in M \mapsto (x^1, x^2, F(x^1, x^2))\in R^3.$$

We start with a tentative choice of transversal field $\xi_0=(0, 0, 1)$. Since $D_{\partial_i}\xi_0=0$, we have $\tau=0$. Denote by ∂_j the coordinate vector field $\partial/\partial x^j$, then we have

$$f_*(\partial_1)=(1, 0, F_1), \quad f_*(\partial_2)=(0, 1, F_2),$$

where $F_j=\partial F/\partial x^j$. Thus we get

$$D_{\partial_i}(f_*(\partial_j))=(0, 0, F_{ij})=F_{ij}\xi, \quad F_{ij}=\frac{\partial^2 F}{\partial x^i\partial x^j},$$

and then the induced connection ∇^0 and the affine fundamental form h^0 with respect to ξ_0 are given as

$$\nabla^0_{\partial_i}(\partial_j)=0, \quad h^0(\partial_i, \partial_j)=F_{ij}.$$

Thus the immersion is nondegenerate if and only if $\det(F_{ij})\neq 0$. Taking $\phi=|\det(F_{ij})|^{\frac{1}{4}}$, we set

$$\xi=-\sum_{k,j}(F^{kj}\phi_j)f_*(\partial_k)+\phi\xi_0,$$

where $\phi_j=\partial\phi/\partial x^j$ and (F^{ij}) is the inverse matrix of the matrix (F_{ij}). From which we have

$$D_{\partial_i}\xi=-\sum_{j,k}\partial_i(F^{kj}\phi_j)f_*(\partial_k).$$

Then (f, ξ) is a Blaschke immersion, and the affine shape operator is

$$S(\partial_i)=\sum_{j,k}\partial_i(F^{kj}\phi_j)\partial_k.$$

Hence we see that the affine mean curvature of M satisfies

$$H=\frac{1}{2}\sum_{i,j}\partial_i(F^{ij}\phi_j). \tag{2.4}$$

3. Proof of the main Theorem

From now on, we assume that M is a translation surface i.e. it is obtained as the graph of function $F(x, y)=p(x)+q(y)$, where $p(x)$ and $q(y)$ are differentiable functions. Hence, we have

$$(F_{ij})=(h_{ij})=\begin{pmatrix} p''(x) & 0 \\ 0 & q''(y) \end{pmatrix}, \quad (F^{ij})=(F_{ij})^{-1}=\begin{pmatrix} p''(x)^{-1} & 0 \\ 0 & q''(y)^{-1} \end{pmatrix},$$

and

$$\phi=|\det(F_{ij})|^{\frac{1}{4}}=|p''(x)q''(y)|^{\frac{1}{4}}\neq 0.$$

In the first, we assume that $p''(x)>0$ and $q''(y)>0$. By a direct calculation we have

$$\begin{aligned} 2H=&\frac{-p'''^2q''}{4p''^2}(p''q'')^{-\frac{3}{4}}+\frac{1}{4p''}\left[-\frac{3}{4}(q''p'')^{-\frac{7}{4}}p'''^2q''^2+(p''q'')^{-\frac{3}{4}}p^{(4)}q''\right] \\ &+\frac{-q'''^2p''}{4q''^2}(p''q'')^{-\frac{3}{4}}+\frac{1}{4q''}\left[-\frac{3}{4}(q''p'')^{-\frac{7}{4}}q'''^2p''^2+(p''q'')^{-\frac{3}{4}}q^{(4)}p''\right], \end{aligned}$$

i.e.,

$$8H=\left(-\frac{7}{4}q'''^2+q''q^{(4)}\right)q''^{-\frac{11}{4}}p''^{\frac{1}{4}}+\left(-\frac{7}{4}p'''^2+p''p^{(4)}\right)p''^{-\frac{11}{4}}q''^{\frac{1}{4}}.$$

Let $p''(x)=f(x)$, $q''(y)=g(y)$. Then

$$\begin{aligned} 8H&=\left(-\frac{7}{4}f'^2+ff''\right)f^{-\frac{11}{4}}g^{\frac{1}{4}}+\left(-\frac{7}{4}g'^2+gg''\right)g^{-\frac{11}{4}}f^{\frac{1}{4}} \\ &=Q(x)g^{\frac{1}{4}}+R(y)f^{\frac{1}{4}}, \end{aligned} \tag{3.1}$$

where

$$Q(x)=\left(-\frac{7}{4}f'^2+ff''\right)f^{-\frac{11}{4}}, \quad R(y)=\left(-\frac{7}{4}g'^2+gg''\right)g^{-\frac{11}{4}}.$$

In order to prove our Theorem we need the following Lemma:

Lemma. *If $H=$constant and $f'g'\neq 0$, then $H\equiv 0$.*

proof. Since $H=$constant differentiating (3.1) with respect to x and y we get

$$0=Q'g^{\frac{1}{4}}+\frac{1}{4}Rf^{-\frac{3}{4}}f', \tag{3.2}$$

and

$$0=\frac{1}{4}Qg^{-\frac{3}{4}}g'+R'f^{\frac{1}{4}}. \tag{3.3}$$

From (3.2) and (3.3) we get

$$\frac{Q'}{f^{-\frac{3}{4}}f'}=-\frac{R}{4g^{\frac{1}{4}}}=\lambda=constant, \tag{3.4}$$

and

$$\frac{R'}{g^{-\frac{3}{4}}g'}=-\frac{Q}{4f^{\frac{1}{4}}}=\mu=constant. \tag{3.5}$$

From (3.4) and (3.5) we get

$$\begin{aligned} &Q'=\lambda f^{-\frac{3}{4}}f' \\ &R=-4\lambda g^{\frac{1}{4}} \\ &R'=-\lambda g^{-\frac{3}{4}}g' \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &R'=\mu g^{-\frac{3}{4}}g' \\ &Q=-4\mu f^{\frac{1}{4}} \\ &Q'=-\mu f^{-\frac{3}{4}}f'. \end{aligned} \tag{3.7}$$

From (3.6) and (3.7) we get

$$\lambda=-\mu$$

since $f\neq 0$ and $f'\neq 0$. Then we have

$$\begin{aligned} &R=-4\lambda g^{\frac{1}{4}} \\ &Q=4\lambda f^{\frac{1}{4}}. \end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.1) we get

$$8H=4\lambda f^{\frac{1}{4}}g^{\frac{1}{4}}-4\lambda f^{\frac{1}{4}}g^{\frac{1}{4}}=0.$$

This completes the proof of the Lemma. □

When $f'(x)=0$ and $g'(y)=0$, we can easily get that

$$p(x)=ax^2+bx+c$$
$$q(y)=dy^2+ey+f, \tag{3.9}$$

where a, b, c, d, e, f and are constnt. The solution (3.9) is included in the results of [1].

The remainder is to treat the cases that $f'(x)=0$, $g'(y)\neq 0$. We see that $f(x)=q''(x)=C_1$ $=$constant and $Q(x)=0$. Thus from (3.1) we have

$$C=8HC_1^{-\frac{1}{4}}=8Hf^{-\frac{1}{4}}=\left(-\frac{7}{4}g'^2+gg''\right)g^{-\frac{11}{4}},$$

namely

$$-\frac{7}{4}g'^2+gg''=Cg^{\frac{11}{4}}. \tag{3.10}$$

Setting

$$g'=\frac{dg}{dy}=s,$$

then we have

$$g''=\frac{ds}{dg}s=\frac{1}{2}\frac{ds^2}{dg}.$$

Thus from (3.10) we have

$$\frac{ds^2}{dg}-\frac{7}{2}\frac{1}{g}s^2=2Cg^{\frac{7}{4}}. \tag{3.11}$$

The equation (3.11) has the solution

$$s^2=g^{\frac{7}{2}}\left(-\frac{8}{3}Cg^{-\frac{3}{4}}+C_2\right),$$

i.e.,

$$\frac{dg}{dy}=\pm g^{\frac{7}{4}}(ag^{-\frac{3}{4}}+b)^{\frac{1}{2}}, \tag{3.12}$$

where $a=-\frac{8}{3}C$, $b=C_2$ are constant.

Let $g^{-\frac{3}{4}}=m$. We get from (3.12) that $dm=\pm\frac{3}{4}(am+b)^{\frac{1}{2}}dy$, and

$$\int\frac{dm}{(am+b)^{\frac{1}{2}}}=\pm\frac{3}{4}y+C_3, \tag{3.13}$$

C_3 being a constant.

From which we get

$$g^{-\frac{3}{4}}=m=\frac{9a}{64}\left(y\pm\frac{4}{3}C_3\right)^2-\frac{b}{a},$$

i.e.,

$$q''(y)=g(y)=m^{-\frac{3}{4}}=\left[-3C_1^{-\frac{1}{4}}H\left(y\pm\frac{4}{3}C_3\right)^2+\frac{3C_1^{\frac{1}{4}}C_2}{64H}\right]^{-\frac{4}{3}}.$$

Setting $A_1=-3C_1^{-\frac{1}{4}}$, $A_2=\frac{4}{3}C_3$, $A_3=\frac{3C_1^{\frac{1}{4}}C_2}{64H}$, from which we get

$$\begin{aligned} q''(y)&=[A_1H(y\pm A_2)^2+A_3]^{-\frac{4}{3}} \\ &=A_1^{-\frac{4}{3}}\left[H(y\pm A_2)^2+\frac{A_3}{A_1}\right]^{-\frac{4}{3}}. \end{aligned} \tag{3.14}$$

From (3.14) we get

$$\begin{aligned} q(y)&=\int_{y_0}^{y}\left\{\int_{t_0}^{t}A_1^{-\frac{4}{3}}\left[H(s\pm A_2)^2+\frac{A_3}{A_1}\right]^{-\frac{4}{3}}ds\right\}dt \\ &=A_1^{-\frac{4}{3}}\int_{y_0}^{y}\left\{\int_{v_0}^{v}\left(Hs^2+\frac{A_3}{A_1}\right)^{-\frac{4}{3}}ds\right\}dv. \end{aligned}$$

Therefore when $f'(x)=0$, $g'(y)\neq0$, we obtain the solution of equation (3.1):

$$\begin{aligned} p(x)&=\frac{1}{2}C_1x^2+bx+c, \\ q(y)&=(-3)^{-\frac{4}{3}}C_1^{\frac{1}{3}}\int_{y_0}^{y}\left\{\int_{v_0}^{v}\left(Hs^2+\frac{A_3}{A_1}\right)^{-\frac{4}{3}}ds\right\}dv, \end{aligned} \tag{3.15}$$

and so

$$z=\frac{1}{2}C_1x^2+ax+b+3^{-\frac{4}{3}}C_1^{\frac{1}{3}}\int_{y_0}^{y}\left\{\int_{v_0}^{v}(Hs^2+c)^{-\frac{4}{3}}ds\right\}dv, \tag{3.16}$$

where a, b, $c=\frac{A_3}{A_1}$ are constant.

Therefore under equiaffine transformation

$$\bar{x}=x,\quad \bar{y}=y,\quad \bar{z}=z-ax-b,$$

from (3.16) we get

$$z=\frac{1}{2}Ax^2+3^{-\frac{4}{3}}A^{\frac{1}{3}}\int_{y_0}^{y}\left\{\int_{v_0}^{v}(Hs^2+c)^{-\frac{4}{3}}ds\right\}dv, \tag{3.17}$$

where $A=C_1$ and C is a constant.

For the case of $p''(x)q''(y)<0$, without loss of the generality, we may assume that $p''(x)<0$ and $q''(y)>0$. Thus by setting $p_1(x)=-p(x)>0$ and $q_1=q(y)$ we see that $p_1''(x)q_1''(y)>0$. Using the same method as above we can get

$$z=-\frac{1}{2}Ax^2+3^{-\frac{4}{3}}A^{\frac{1}{3}}\int_{y_0}^{y}\left\{\int_{v_0}^{v}(Hs^2+c)^{-\frac{4}{3}}ds\right\}dv.$$

Thus we completes the proof of Theorem. □

Now we consider a special case. Taking $C=0$ from (3.17) we can get

$$z=\frac{1}{2}Ax^2+\frac{9}{10}A^{\frac{1}{3}}(3H)^{-\frac{4}{3}}y^{-\frac{2}{3}}+P_1y+P_2, \tag{3.18}$$

where P_1 and P_2 are constant. By an equiaffine transformation

$$\overline{x}=x, \quad \overline{y}=y, \quad \overline{z}=z-P_1y-P_2$$

we get from (3.18) that

$$z=\frac{1}{2}Ax^2+\frac{9}{10}A^{\frac{1}{3}}(3H)^{-\frac{4}{3}}y^{-\frac{2}{3}}$$

and a surface

$$\left(x,\, y,\, \frac{1}{2}Ax^2+\frac{9}{10}A^{\frac{1}{3}}(3H)^{-\frac{4}{3}}y^{-\frac{2}{3}}\right).$$

References

[1] F. Manhart, *Die affinminimalrückungflächen*, Arch. Math. **44** (1985), 547-556.

[2] W. Blaschke, *Vorlesungen über Differentialgeometrie* II, Berlin (1923).

[3] W. Yang and J. Nie, *On affine minimal rotation surfaces in* A^3, J. of Math. (PCR) **7** (1987), 205-210.

[4] M. A. Magid, *Timelike Thomsen surfaces*, Results in Mathematics **20** (1991), 691-697.

[5] K. Nomizu and T.Sasaki, *Affine differential geometry*, Cambridge University Press, Cambridge, New York (1994).

[6] A. -M. Li, U. Simon and G. Zhao. *Global affine differential geometry of hypersurfaces*, W. de Gruyer, Berlin and New York (1993).

Huafei Sun
Department of Mathematics
Kumamoto University
Kumamoto 860-8555
Japan