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On the existence of nonnegative radial solutions for elliptic system

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Abstract

The existence of nonnegative radial solutions for some systems of $m(m\geq 1)$ semilinear elliptic equations is proved by a simple application of a fixed point theorem in cones.

1. Introduction and Main Result

This paper can be regarded as a continuation of both [2] and [1].

In this paper, we study the existence of nontrivial nonnegative radial solutions to the system of m quasilinear elliptic equations

$$\Delta u+p(r)f(u)=0,\quad 0<A<r<B \tag{1.1}$$

with one of the following three sets of boundary conditions

$$u=0 \text{ on } r=A \text{ and } r=B \qquad (1.2)_a$$

$$u=0 \text{ on } r=A \text{ and } \frac{\partial u}{\partial r}=0 \text{ on } r=B \qquad (1.2)_b$$

$$\frac{\partial u}{\partial r}=0 \text{ on } r=A \text{ and } u=0 \text{ on } r=B \qquad (1.2)_c$$

where $\{x\in\mathbf{R}^n;\ A<r<B\}$ is an annulus, $r:=\sqrt{x_1^2+\ldots+x_n^2}$, $n\geq 1$, and $p(r)f(u):=(p_1(r)f_1(u),\ldots,p_m(r)f_m(u))$, $m\geq 1$.

Throughout this paper, we make the following hypotheses

(H_1) $q(t)=(q_1(t),\ldots,q_m(t))$, $q_j(t):=h^2[w(t)]^{2n-2}p_j(w(t))$, $j=1,2,\ldots,m$, is a nonnegative measurable m-dimensional vector function defined on $[0,1]$ and satisfies either

$$0<\int_0^1 s(1-s)q(s)ds<+\infty, \qquad (1.3)_a$$

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or

$$0<\int_0^1 sq(s)ds<+\infty \tag{1.3$_b$}$$

or

$$0<\int_0^1 (1-s)q(s)ds<+\infty \tag{1.3$_c$}$$

where the function $r=w(t)$ is the inverse function to

$$t=v(r):=\frac{1}{h}\int_A^r \frac{ds}{s^{n-1}}, \quad h:=\int_A^B \frac{ds}{s^{n-1}}. \tag{1.4}$$

$(H_2) f(u)=(f_1(u), \ldots, f_m(u))$ is a nonnegative continuous m-dimensional vector function defined on $\mathbf{R}_+^m$, $\mathbf{R}_+=[0, +\infty)$, and satisfies either

(i) $f_0=0$ and $f_\infty=+\infty$ (superlinear), or

(ii) $f_0=+\infty$ and $f_\infty=0$ (sublinear),

where

$$f_0:=\lim_{|u|\downarrow 0}\frac{f(u)}{|u|}, \quad f_\infty:=\lim_{|u|\uparrow\infty}\frac{f(u)}{|u|}.$$

Here and henceforth, we denote the norm of $u\in\mathbf{R}^m$ by

$$|u|:=\max\{|u_j|; j=1, \ldots, m\}$$

and write $z>y(z\geq y)$ if $z-y\in\mathring{\mathbf{R}}_+^m$, $\mathring{\mathbf{R}}_+=(0, +\infty)$, $(z-y\in\mathbf{R}_+^m)$. Further, we say that a vector y is positive (nonnegative) if $y>0(y\geq 0)$.

The hypothesis (H_1) allows $q(t)\equiv 0$ on some subintervals of $[0, 1]$ and to have singularity at $t=0$ and $t=1$. For example,

$$q_j(t)=t^{-\alpha_j}(1-t)^{-\beta_j}(|\cos 2\pi t|+\cos 2\pi t), \quad j=1, 2, \ldots, m$$

satisfies $(1.3)_a$ provided α_j, $\beta_j\in(0, 2)$.

If $u=u(r)$ is a nonnegative radial solution to the problem (1.1)-(1.2), then it satisfies the problem

$$[r^{n-1}u'(r)]'+r^{n-1}p(r)f(u(r))=0, \quad A<r<B, \tag{1.5}$$

$$u(A)=u(B)=0, \quad \text{or} \tag{1.6$_a$}$$

$$u(A)=u'(B)=0, \quad \text{or} \tag{1.6$_b$}$$

$$u'(A)=u(B)=0, \tag{1.6$_c$}$$

Let us introduce the change of variables

$$r=w(t) \text{ and } y(t)=u(w(t)),$$

where $w(t)$ is determined by (H_1). Then the problem (1.5)-(1.6) can be written as

$$y''(t)+q(t)f(y(t))=0, \quad 0<t<1 \tag{1.7}$$

$$y(0)=y(1)=0, \quad \text{or} \tag*{(1.8)$_a$}$$

$$y(0)=y'(1)=0, \quad \text{or} \tag*{(1.8)$_b$}$$

$$y'(0)=y(1)=0, \tag*{(1.8)$_c$}$$

Conversely, if $y(t)$ is a nonnegative solution to the problem (1.7)-(1.8), then $u(r):=y(v(r))$ is a nonnegative radial solution to the problem (1.1)-(1.2), where the function $v(r)$ is defined by (1.4). Therefore, we concentrate on the problem (1.7)-(1.8) in the sequel.

Here we say that a function $y(t)=(y_1(t), ..., y_m(t))$ is a nonnegative solution to the problem (1.7)-(1.8) if it satisfies the following conditions:

(i) $y(t)\in C([0, 1]; \mathbf{R}^m_+)\cap C^1((0, 1); \mathbf{R}^m)$,

(ii) $y'(t)$ is locally absolutely continuous in $(0, 1)$, or in $(0, 1]$, or in $[0, 1)$,

(iii) $y''(t)=-q(t)f(y(t))$ for a.e. $t\in[0, 1]$, $y(0)=y(1)=0$, or $y(0)=y'(1)=0$, or $y'(0)=y(1)=0$.

It is clear that $y(t)\equiv 0$ is a trivial solution to the problem (1.7)-(1.8) when $f(0)=0$. Because of the physical background of the problem above, we are mainly interested in nontrivial nonnegative solutions.

The main purpose this paper is to extend and improve the existence results in both [1] and [2]. The paper [1] deals with only the case of $m=1$ and the paper [2] only the case when $m=2$ and $f(u)$ is sublinear (i.e. $f_0=+\infty$, $f_{+\infty}=0$). In [2], conditions on $f(u)$ appear to prolix. Other papers related to this note can be found in [3-8] and the references therein.

The main result of this paper is stated as follows

Theorem 1. *Let* (H_1) *and* (H_2) *hold. Then the problem (1.7)-(1.8) has a nontrivial nonnegative solution* $y(t)$, *i.e., the problem (1.1)-(1.2) has a nontrivial nonnegative radial solution on the annulus domain* $\{x\in\mathbf{R}^n; A<r<B\}$.

The proof of Theorem 1 will be based on an application of the following fixed point theorem due to Krasnoselskii (which is quoted from [1])

Theorem 2 (Krasnoselskii). *Let* E *be a Banach space, and let* $K\subset E$ *be a cone in* E. *Assume* Ω_1, Ω_2 *are open subset of* E *with* $0\in\Omega_1$, $\overline{\Omega}_1\subset\Omega_2$, *and let*

$$\Phi : K\cap(\overline{\Omega}_2\backslash\Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|\Phi_y\|\le\|y\|\forall y\in K\cap\partial\Omega_1$ *and* $\|\Phi_y\|\ge\|y\|\forall y\in K\cap\partial\Omega_2$; *or*

(ii) $\|\Phi_y\|\ge\|y\|\forall y\in K\cap\partial\Omega_1$ *and* $\|\Phi_y\|\le\|y\|\forall y\in K\cap\partial\Omega_2$.

Then Φ *has a fixed point in* $K\cap(\overline{\Omega}_2\backslash\Omega_1)$.

2. Proof of Theorem 1

In this section, we only consider the problem (1.7)-$(1.8)_a$, since the problems (1.7)-$(1.8)_b$ or (1.7)-$(1.8)_c$ can be studied in the similar way. We will distinguish two case to discuss the problem (1.7)-$(1.8)_a$.

Superliner case. In this case, $f_0=0$ and $f_\infty=+\infty$. We wish to prove the existence of a nontrivial nonegative solution for the problem (1.7)-$(1.8)_a$. It is obvious that the problem (1.7)-(1.8)a has a solution $y(t)$ if and only if $y(t)$ solves the integral equation

$$y(t)=\int_0^1 G_a(t,s)q(s)f(y(s))ds=:(\Phi_a y)(t), \tag{2.1}$$

where

$$G_a(t,s):=\begin{cases} s(1-t), & 0\le s\le t\le 1,\\ t(1-s), & 0\le t\le s\le 1\end{cases}$$

It is clear that $G_a(t,s)\le s(1-s)$ on $[0,1]\times[0,1]$.

Let us define an operator $\Phi_a : K_a \to K_a$ by the right hand-side of (2.1), where

$$K_a:=\{y\in E;\ y_j(t) \text{ is concave on } [0,1] \text{ with } y_j(0)=y_j(1)=0,\quad j=1,2,\dots,m\},$$

is a cone in E, while E is the Banach space of continuous m-dimensional vector functions defined on $[0,1]$ with the norm

$$\|y\|:=\max\{\|y_j\|:j=1,\dots,m\},\quad \|y_j\|:=\max\{|y_j(t)|:0\le t\le 1\}.$$

Whence it follows that, for any $y_j(t)\in K_a$, we have

$$y_j(t)\ge\|y_j\|t(1-t) \text{ on } [0,1],\quad j=1,2,\dots,m. \tag{2.2}$$

From the definition of Φ_a, we have for any $y\in K_a$

$$(\Phi_a y)_j'(t) = -\int_0^t s q_j(s) f_j(y(s))ds + \int_t^1 (1-s) q_j(s) f_j(y(s))ds,$$
$$0<t<1,$$

$(\Phi_a y)_j''(t) = -q_j(t) f_j(y(t))$, a.e. in $(0, 1)$, and

$(\Phi_a y)_j(0) = (\Phi_a y)_j(1) = 0$, $j=1, 2, \ldots, m$.

Therefore, $\Phi_a(K_a) \subset K_a$. Moreover, from the continuity of $f(y)$ on $\mathbf{R}_m^+$, we also can prove that Φ_a is completely continuous on K_a.

Now, since $f_0=0$, we can choose a $\rho_1>0$ such that

$$f_j(u) \leq \varepsilon |u| \text{ whenever } 0 \leq |u| \leq \rho_1, \quad j=1, \ldots, m \tag{2.4}$$

where the constant $\varepsilon>0$ satisfies the condition

$$0 < \varepsilon \int_0^1 s(1-s) q_j(s) ds \leq \frac{1}{2}, \quad j=1, 2, \ldots, m. \tag{2.5}$$

Thus if $y \in K_a$ with $\|y\| = \rho_1$, then from (2.4) and (2.5)

$$\begin{aligned} 0 \leq (\Phi_a y)_j(t) &\leq \int_0^1 s(1-s) q_j(s) f_j(y(s)) ds \\ &\leq \varepsilon \|y\| \int_0^1 s(1-s) q_j(s) ds \\ &\leq \|y\|, \quad j=1, 2, \ldots, m, \end{aligned}$$

i.e.

$$\|\Phi_a y\| < \|y\| \quad \forall y \in K_a \cap \partial\Omega_1,$$

where $\Omega_1 := \{y \in E; \|y\| < \rho_1\}$.

Further, since $f_\infty = +\infty$, there exists a $\rho_2 > \rho_1$ such that

$$f_j(u) \geq M|u| \text{ whenever } |u| > \delta^2 \rho_2, \quad j=1, 2, \ldots, m \tag{2.6}$$

where the constants $\delta \in \left(0, \frac{1}{4}\right)$ satisfies

$$\alpha := \min\left\{\int_\delta^{1-\delta} s(1-s) q_j(s) ds; \quad j=1, 2, \ldots, m\right\} > 0. \tag{2.7}$$

In the sequel, δ is always chosen to satisfy (2.7). Moreover, $M>0$ is chosen so that

$$\delta^2 \alpha M > 2. \tag{2.8}$$

Let $\Omega_2 := \{y \in E : \|y\| < \rho_2\}$, then $y \in K_a$ with $\|y\| = \rho_2$ implies that

$$y_j(t) \geq \delta^2 \rho_2 \text{ on } [\delta, 1-\delta] \text{ for some } j \in \{1, 2, \ldots, m\}, \tag{2.9}$$

and so

$$\begin{aligned}
(\Phi_a y)_j\left(\frac{1}{2}\right) &= \int_0^1 G_a\left(\frac{1}{2}, s\right) q_j(s) f_j(y(s)) ds \\
&\geq \frac{1}{2} \int_\delta^{1-\delta} s(1-s) q_j(s) f_j(y(s)) ds \\
&\geq \frac{1}{2} a \delta^2 \rho_2 M \\
&> \rho_2 = \| y \| \quad j = 1, \ldots, m.
\end{aligned}$$

Here we have used (2.6), (2.7), (2.8) and (2.9). This shows that

$$\| \Phi_a y \| > \| y \| \quad \forall y \in K_a \cap \partial \Omega_2.$$

Therefore, from the first part of Theorem 2, we conclude that Φ_a has a fixed point in $K_a \cap (\overline{\Omega_2} \backslash \Omega_1)$. Let $y(t)$ be the fixed point. Then it satisfies the integral equation (2.1) and $0 < \rho_1 \leq \| y \| \leq \rho_2$. This shows that the fixed point $y(t)$ is a nontrivial nonnegative solution to the problem (1.7)-$(1.8)_a$.

Sublinear case. In this case $f_0 = +\infty$ and $f_\infty = 0$.

Since $f_0 = +\infty$, we may choose $\rho_1 > 0$ such that

$$f_j(u) \geq M |u| \text{ whenever } 0 \leq |u| \leq \rho_1, \quad j = 1, 2, \ldots, m,$$

where the constant M satisfies (2.8). Thus, for $y \in K_a$ with $\| y \| = \rho_1$,

$$\begin{aligned}
(\Phi_a y)_j\left(\frac{1}{2}\right) &\geq \frac{1}{2} \int_\delta^{1-\delta} s(1-s) q_j(s) f_j(y(s)) ds \\
&\geq \frac{1}{2} a \delta^2 \rho_1 M \\
&> \rho_1 = \| y \| \quad j = 1, \ldots, m.
\end{aligned}$$

i.e.

$$\| \Phi_a y \| > \| y \| \quad \forall y \in K_a \cap \partial \Omega_1,$$

where $\Omega_1 := \{ y \in E; \| y \| < \rho_1 \}$.

Further, Since $f_\infty = 0$, there exists an $N > \rho_1$ such that

$$f_j(u) \leq \varepsilon |u| \text{ whenever } |u| \geq N, \quad j = 1, \ldots, m.$$

where the constant ε satisfies (2.5). Let

$$\rho_2 \geq 2N + \max\{f_j(u);\ 0 \leq |u| \leq N\}\left(\int_0^1 2s(1-s)q_j(s)ds\right), \quad j=1, 2, \ldots, m$$

Then for $y \in K_a$ with $\|y\| = \rho_2$, we have

$$\begin{aligned}(\Phi_a y)_j(t) &\leq \int_0^1 s(1-s)q_j(s)f_j(y(s))ds \\ &< N + \max\{f_j(u);\ 0 \leq |u| \leq N\}\left(\int_0^1 s(1-s)q_j(s)ds\right) \\ &\quad + \varepsilon\|y\|\int_0^1 s(1-s)q_j(s)ds \\ &\leq \frac{1}{2}\rho_2 + \frac{1}{2}\|y\| = \|y\|\end{aligned}$$

i.e,

$$\|\Phi_a y\| < \|y\| \quad \forall y \in K_a \cap \partial\Omega_2,$$

where $\Omega_2 := \{y \in E;\ \|y\| < \rho_2\}$.

Therefore, by the second part to Theorem 2, it follows that Φ_a has a fixed point $y(t)$ in $K_a \cap (\overline{\Omega}_2 \backslash \Omega_1)$, which is a nontrivial nonnegative solution to the problem (1.7)-(1.8)$_a$. This completes the proof of Theorem 1 for the problem (1.7)-(1.8)$_a$.

Finally, we consider the problem (1.7)-(1.8)$_b$ and (1.7)-(1.8)$_c$, It is easy to check that (1.7)-(1.8)$_b$ and (1.7)-(1.8)$_c$ are equivalent to the integral equation

$$y(t) = \int_0^t G_b(t, s)q(s)f(y(s))ds$$

and

$$y(t) = \int_t^1 G_c(t, s)q(s)f(y(s))ds$$

respectively, where

$$G_b(t, s) := \begin{cases} s, & 0 \leq s \leq t \leq 1 \\ t, & 0 \leq t \leq s \leq 1, \end{cases}$$

and

$$G_c(t, s) := \begin{cases} 1-t, & 0 \leq s \leq t \leq 1 \\ 1-s, & 0 \leq t \leq s \leq 1. \end{cases}$$

For the problem (1.7)-$(1.8)_b$, let K_b be the cone given by

$$K_b=\{y\in E: y_j(t) \text{ is concave on } [0,1],\ y_j(0)=0 \text{ is the minimum and } y_j(1) \text{ is the maximum},\quad j=1,2,\dots,m\}.$$

For the problem (1.7)-$(1.8)_c$, let K_c be the cone given by

$$K_c=\{y\in E: y_j(t) \text{ is concave on } [0,1],\ y_j(0)=0 \text{ is the maximum and } y_j(1)=0 \text{ is the minimum},\quad j=1,2,\dots,m\}$$

By the above method, Theorem 2 is employed to prove that both (1.7)-$(1.8)_b$ and (1.7)-$(1.8)_c$ have nontrivial nonnegative solutions. This completes the proof of Theorem 1.

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