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On formulae for correlation coefficients between normally scattered functions II

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1. Introduction

In the previous paper [1], we derived the formula

$$\rho_I(N^{\theta}_{\alpha,\beta}, N^{\zeta}_{\alpha,\beta})=\cos \pi(\theta-\zeta)$$

for correlation coefficients between normally scattered functions $N^{\theta}_{\alpha,\beta}(t)$ and $N^{\zeta}_{\alpha,\beta}(t)$, where $0<\alpha\neq\beta<1$, $-\infty<\theta$, $\zeta<\infty$, and I is an interval.

In this paper, we prove the following formula

$$\rho_I(N^{\theta}_{\alpha,\beta}, N^{\zeta}_{\gamma,\beta})=\frac{1}{2}\left(\int_0^1 \sqrt{-2\log u}\, du\right)^2 \cos \pi(\theta-\zeta)$$
$$\approx 0.785 \cos \pi(\theta-\zeta)$$

when α, β, γ are mutually different.

Additionally we prove formulae in all other cases, which were announced in [2], and published in the internet homepage [3].

From there formulae, we can control covariant matrices of 2- or 3-dimensional normally scattered functions [1], which may be useful for studying 2- or 3-dimensional Brownian motion in the future. Brownian motion was found in the early 18th century, and its mathematical theory was studied by P. Levy and N. Wiener, and the theory of stochastic integrals was completed by K. Ito, which derived integral formulae for Brownian motions and was applied to the theory of Black-Sholes model of mathematical finance. Computational models were based on random numbers, which are difficult to treat theoretically. These theories are based on stochastic processes and are difficult to understand, but using scattered functions we can study Brownian motions on the frame of integrals of simple real functions.

Numbers used in this paper and the previous paper [1] are rational numbers in computers. Computer experiments were done by 64 bits floating point numbers. We cannot reach to the exact mathematical theory, which is the future problem.

2. Numerical experiments of $\rho_I(N^{\theta}_{\alpha,\beta}, N^{\xi}_{\gamma,\delta})$

We divide the square $[0,1]\times[0,1]$ in (θ,ζ)-plane into 50×50 equally spaced meshes, and calculate correlation coefficients ρ for 10,000 samples at each mesh point. We draw the perspective view of 51×51 points (θ,ζ,ρ) in $[0,1]\times[0,1]\times[-1,1]$.

Figure 1 is for a case when $\beta=\delta$ and α, β, γ are mutually different.

From our careful examination, we conclude that the following formula is valid.

$$\rho_I(N^{\theta}_{\alpha,\beta}, N^{\xi}_{\gamma,\beta})\approx 0.785\cos\pi(\theta-\zeta)$$

For other cases, graphs of experimental results are figures such as Figure 2, and we conclude that the following formula is valid.

$$\rho_I(N^{\theta}_{\alpha,\beta}, N^{\xi}_{\gamma,\delta})\approx 0$$

To show typical cases, we draw points $(N^{\theta}_{\alpha,\beta}(t), N^{\xi}_{\gamma,\delta}(t))$ for 10,000 numbers of t and the calculated correlation coefficient ρ in the following figures 3~14.

3. Integral formulae for the composition of a scattered function and a continuous function

Let $S(t)$ be a scattered function with the density function $\sigma(u)$.

Let a Riemannian sum be

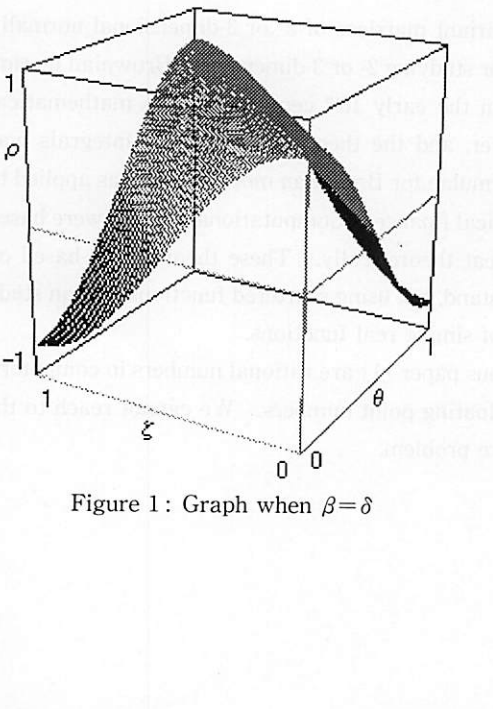

Figure 1 : Graph when $\beta=\delta$

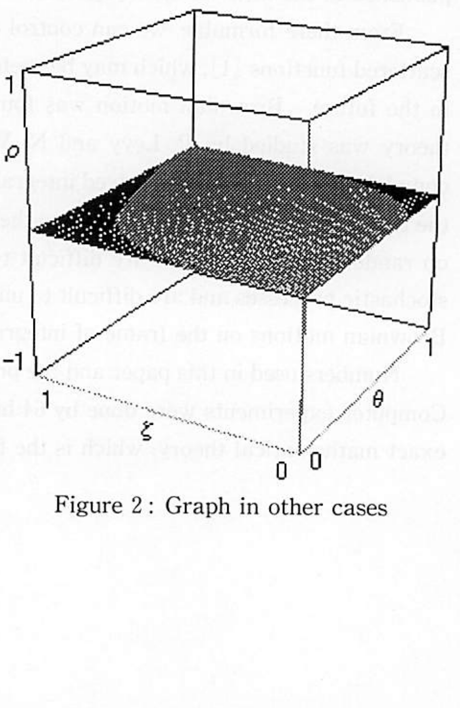

Figure 2 : Graph in other cases

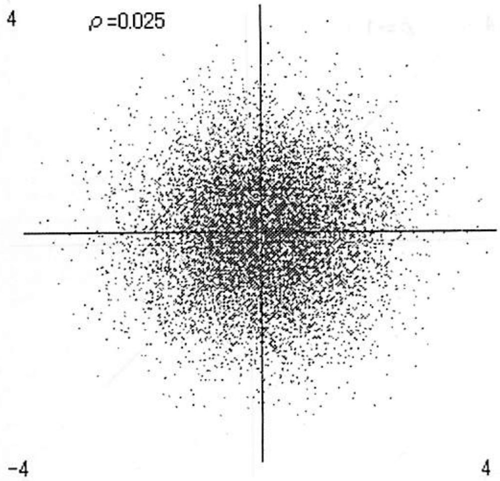

Figure 3 : $(N^{0}_{0.2,\,0.4}(t), N^{0}_{0.6,\,0.8}(t))$

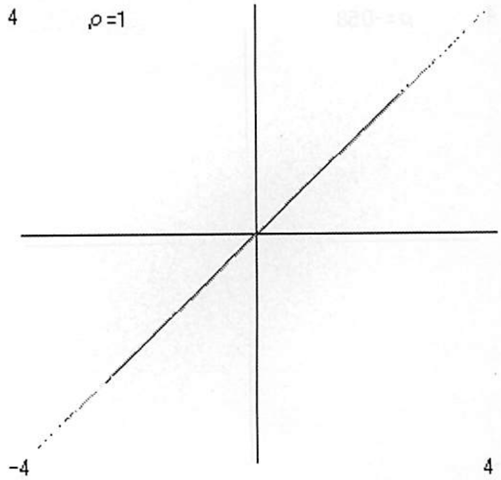

Figure 4 : $(N^{0}_{0.2,\,0.4}(t), N^{0}_{0.2,\,0.4}(t))$

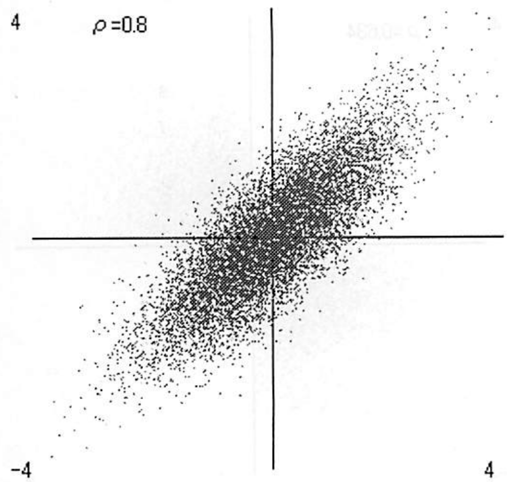

Figure 5 : $(N^{0}_{0.2,\,0.4}(t), N^{0.2}_{0.2,\,0.4}(t))$

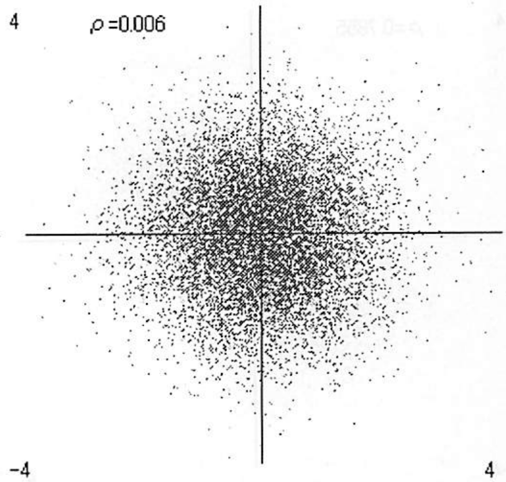

Figure 6 : $(N^{0}_{0.2,\,0.4}(t), N^{0.5}_{0.2,\,0.4}(t))$

$$\sum_{k=1}^{n} S(\xi_k)(t_k - t_{k-1}),$$

where $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ and $t_{k-1} \leq \xi_k \leq t_k$.

When we calculate this Riemannian sum for many divisions, and calculate density distributions of these values, their mean approach to a fixed value, and their variances approach to 0. We call this fixed value as the stochastic Riemannian integral of $S(t)$ and write this value as

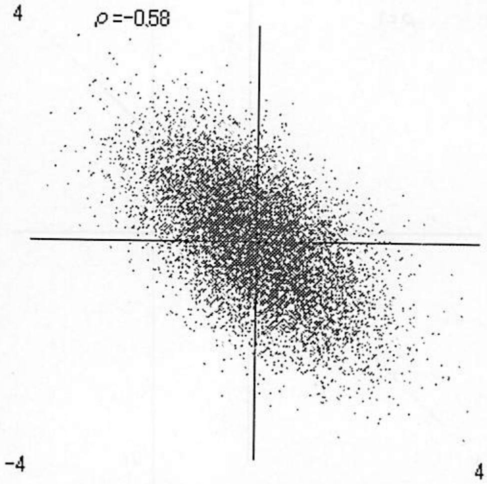

Figure 7 : $(N^{0}_{0.2,\,0.4}(t), N^{0.7}_{0.2,\,0.4}(t))$

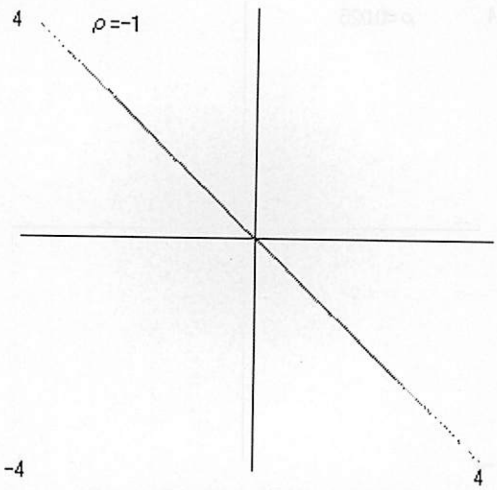

Figure 8 : $(N^{0}_{0.2,\,0.4}(t), N^{1}_{0.2,\,0.4}(t))$

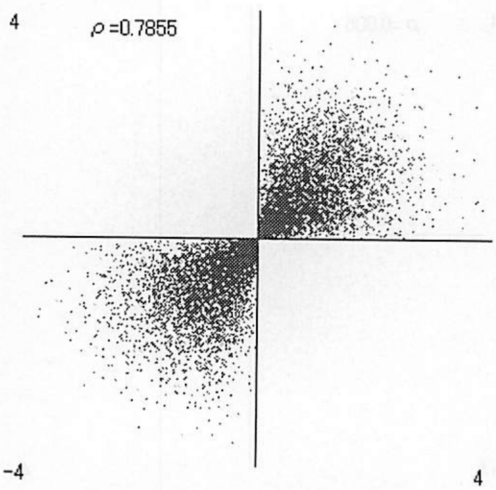

Figure 9 : $(N^{0}_{0.2,\,0.4}(t), N^{0}_{0.6,\,0.4}(t))$

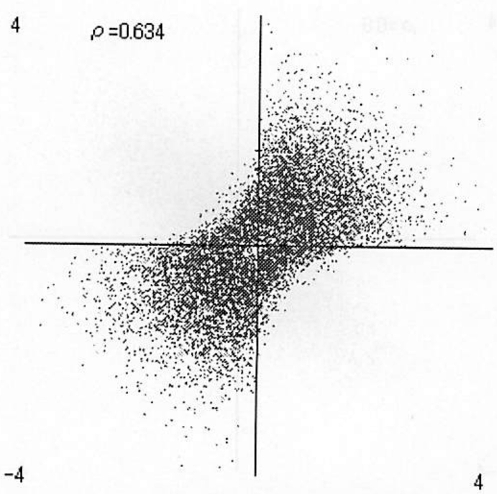

Figure 10 : $(N^{0}_{0.2,\,0.4}(t), N^{0.2}_{0.6,\,0.4}(t))$

$\int_a^b S(t)dt$.

This can be verified using stochastic variables [3].

We will use the following two theorems to prove our main theorem.

Theorem 1. Let $S(t)$ be a scattered function with the density function $\sigma(u)$, and $f(u)$ be a bounded continuous function. Then the following formula is valid.

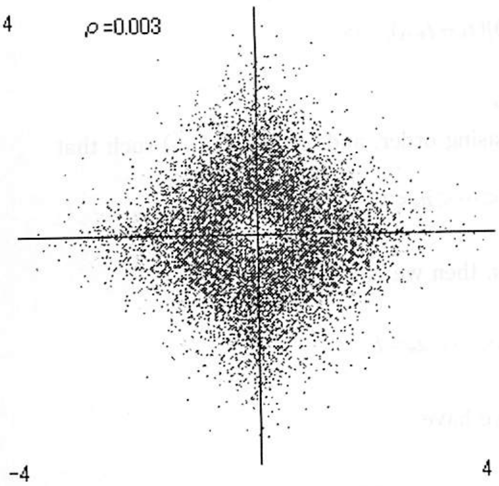

Figure 11 : $(N^{0}_{0.2,\,0.4}(t), N^{0.5}_{0.6,\,0.4}(t))$

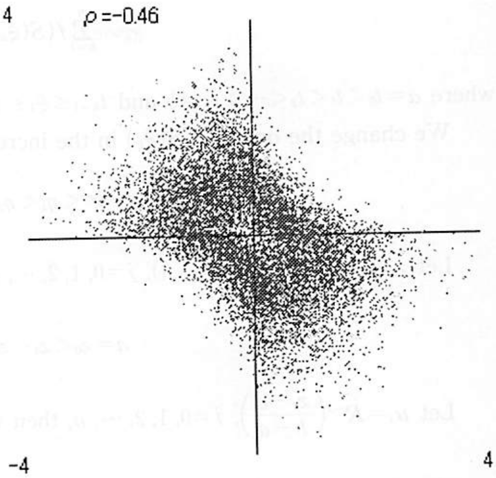

Figure 12 : $(N^{0}_{0.2,\,0.4}(t), N^{0.7}_{0.6,\,0.4}(t))$

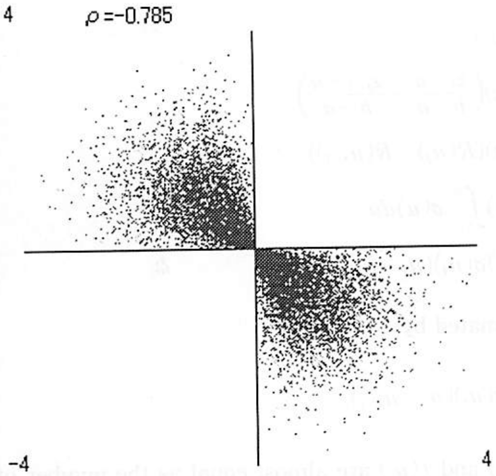

Figure 13 : $(N^{0}_{0.2,\,0.4}(t), N^{1}_{0.6,\,0.4}(t))$

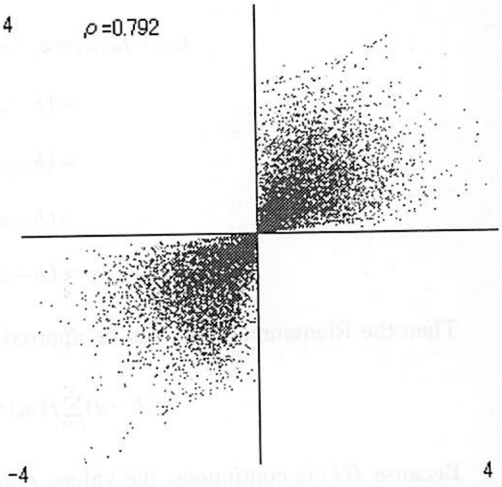

Figure 14 : $(N^{0}_{0.2,\,0.4}(t), N^{0}_{0.3,\,0.4}(t))$

$$\int_a^b f(S(t))dt=(b-a)\int_{-\infty}^{\infty} f(u)\sigma(u)du$$

Proof. Let $R(u)=\int_{-\infty}^{u}\sigma(u)du$ be the distribution function of $S(t)$, then values of this function increase monotonously from 0 to 1.

Let a Riemannian sum be

$$\sum_{k=1}^{n} f(S(\xi_k))(t_k - t_{k-1}),$$

where $a=t_0<t_1<t_2<\cdots<t_n=b$ and $t_{k-1}\leq\xi_k\leq t_k$.

We change the order of $S(\xi_k)$ in the increasing order, and let $\eta_j=S(\xi_{k(j)})$ such that

$$-\infty<\eta_1<\eta_2<\cdots<\eta_n<\infty.$$

Let $z_j=a+\sum_{m=1}^{j}(t_{k(m)}-t_{k(m)-1})$, $j=0, 1, 2, \cdots, n$, then we have

$$a=z_0<z_1<z_2<\cdots<z_n=b.$$

Let $u_j=R^{-1}\left(\frac{z_j-a}{b-a}\right)$, $j=0, 1, 2, \cdots, n$, then we have

$$-\infty<u_0<u_1<u_2<\cdots<u_n<\infty.$$

The length of each interval is calculated as follows.

$$\begin{aligned} t_{k(j)}-t_{k(j)-1} &= z_j - z_{j-1} \\ &= (b-a)\left(\frac{z_j-a}{b-a}-\frac{z_{j-1}-a}{b-a}\right) \\ &= (b-a)(R(u_j)-R(u_{j-1})) \\ &= (b-a)\int_{u_{j-1}}^{u_j}\sigma(u)du \\ &\approx (b-a)\sigma(u_j)(u_j-u_{j-1}) \end{aligned}$$

Then the Riemannian sum can be approximated by

$$(b-a)\sum_{j=1}^{n} f(\eta_j)\sigma(u_j)(u_j-u_{j-1}).$$

Because $f(t)$ is continuous, the values $f(\eta_j)$ and $f(u_j)$ are almost equal as the number of division increases.

Because $f(t)$ is bounded and the integral of $\sigma(u)$ on $(-\infty, \infty)$ is equal to 1, the above sum approaches to the right side of the integral formula.

From this theorem we can calculate integral formulae in [1]. The boundeness condition of $f(t)$ is not satisfied in some examples, but the convergence of the infinite integrals is clear if functions are given explicitly.

Theorem 2. Let $S(t)$ be a scattered function with the density function $\sigma(u)$, and $T(t)$ be a scattered function with the density function $\tau(u)$. Let $f(u)$ and $g(u)$ be bounded continuous functions. If the values of $S(t)$ and $T(t)$ are independent as stochastic variables, then we have the following integral formula.

$$\int_a^b f(S(t))g(T(t))dt=(b-a)\int_{-\infty}^{\infty} f(u)\sigma(u)du\int_{-\infty}^{\infty} g(u)\tau(u)du$$

Moreover, if $R(t)$ is a scattered function with the density function $\nu(u)$ and $h(t)$ is a bounded continuous function and the values of S and T and R are mutually independent, we have the following similar formula.

$$\int_a^b f(S(t))g(T(t))h(R(t))dt=(b-a)\int_{-\infty}^{\infty} f(u)\sigma(u)du\int_{-\infty}^{\infty} g(u)\tau(u)du\int_{-\infty}^{\infty} h(u)\nu(u)du$$

Proof. Let $\sum_{k=1}^{n} f(S(\xi_k))g(T(\xi_k))(t_k-t_{k-1})$ be a Riemannian sum for the left integral of the above formula. Let $y_j=S(\xi_j)$ and $z_j=T(\xi_j)$, then y_j distribute on y-axis with the density function $\sigma(u)$, and z_j distribute independently on z-axis with the density function $\tau(u)$.

Divide the square $[0,1]\times[0,1]$ into n regions with the area proportional to t_k-t_{k-1}, and transform them to plane regions in $\mathbb{R}^2$ using distribution functions, then the Rimannian sum approaches to a double integral in $\mathbb{R}^2$.

Then by arguing in y-z plane as in theorem 1, we can prove the above theorem.

The boundedness condition of $f(t)$ is not satisfied in the following examples, but convergence of the infinite integrals is clear if functions are given explicitly.

Example 1. For uniformly scattered functions $U_\alpha(t)$ with the density function $\sigma(u)=\begin{cases}1, & \textit{for } 0\le u\le 1\\ 0, & \textit{elsewhere}\end{cases}$ [1], let us calculate

$$\int_a^b (U_\alpha(t)-0.5)(U_\beta(t)-0.5)dt$$

where $0<\alpha,\ \beta<1$.

(1) If $\alpha=\beta$, then by theorem 1,

$$\int_a^b (U_\alpha(t)-0.5)^2dt=(b-a)\int_0^1 (u-0.5)^2du=\frac{b-a}{12}.$$

(2) If $\alpha\neq\beta$, then by theorem 2,

$$\int_a^b (U_\alpha(t)-0.5)(U_\beta(t)-0.5)dt=(b-a)\int_0^1 (u-0.5)du\int_0^1 (u-0.5)du=0$$

When we calculate this integral using numbers with finite precision, there arise some problem, which will be discussed in section 5.

4. The main theorem

Theorem 3. For $0<\alpha, \beta, \gamma, \delta<1$ and $-\infty<\theta, \zeta<\infty$, the correlation coefficients ρ between $N^{\theta}_{\alpha,\beta}(t)$ and $N^{\zeta}_{\gamma,\delta}(t)$ are given as follows.

(1) If $\alpha=\gamma\neq\beta=\delta$, then

$$\rho_I(N^{\theta}_{\alpha,\beta}, N^{\zeta}_{\alpha,\beta})=\cos \pi(\theta-\zeta).$$

(2) if $\beta=\delta$ and $\alpha\neq\beta$, $\beta\neq\gamma$ and $\gamma\neq\alpha$, then

$$\rho_I(N^{\theta}_{\alpha,\beta}, N^{\zeta}_{\gamma,\beta})=\frac{1}{2}\left(\int_0^1\sqrt{-2\log u}\,du\right)^2\cos \pi(\theta-\zeta)$$

(3) For other cases, if $\alpha\neq\beta$, $\beta\neq\delta$ and $\delta\neq\gamma$, then

$$\rho_I(N^{\theta}_{\alpha,\beta}, N^{\zeta}_{\gamma,\delta})=0$$

Proof. (1) is proved in [1].

(2) The numerator of ρ is calculated as follows.

$$\int_a^b N^{\theta}_{\alpha,\beta}(t)N^{\zeta}_{\gamma,\beta}(t)dt$$
$$=-\int_a^b\sqrt{-\log U_\alpha(t)}\sqrt{-\log U_\gamma(t)}\frac{1}{2}\{\cos \pi(4U_\beta(t)+\theta+\zeta)-\cos \pi(\theta-\zeta)\}dt$$

Using Theorem 2, the first term is equal to

$$(b-a)\int_0^1\sqrt{-\log u}\,du\int_0^1\sqrt{-\log u}\,du\frac{1}{2}\int_0^1 \cos \pi(4u+\theta+\zeta)du=0.$$

The remaining term can be calculated similarly and the result is

$$\int_a^b N^{\theta}_{\alpha,\beta}(t)N^{\zeta}_{\gamma,\beta}(t)dt=\frac{b-a}{2}\left(\int_0^1\sqrt{-2\log u}\,du\right)^2\cos \pi(\theta-\zeta).$$

The denominator is equal to $b-a$ as is calculated in [1].

(3) The numerator is

$$\int_a^b N^{\theta}_{\alpha,\beta}(t)N^{\zeta}_{\gamma,\delta}(t)dt$$

$$=-\int_a^b\sqrt{-\log U_\alpha(t)}\sqrt{-\log U_\gamma(t)}\cos\pi(2U_\beta(t)+\theta)\cos\pi(2U_\delta(t)+\zeta)dt$$

Using Theorem 2, if α, β, γ, δ are mutually different, the above integral is equal to

$$=-\int_0^1\sqrt{-\log u}\,du\int_0^1\sqrt{-\log u}\,du\int_0^1\cos\pi(2u+\theta)du\int_0^1\cos\pi(2u+\zeta)du=0.$$

If $\alpha=\gamma$,

$$=-\int_0^1(-\log u)du\int_0^1\cos\pi(2u+\theta)du\int_0^1\cos\pi(2u+\zeta)du=0.$$

If $\alpha=\delta$,

$$=-\int_0^1\sqrt{-\log u}\,du\int_0^1\cos\pi(2u+\theta)du\int_0^1\sqrt{-\log u}\cos\pi(2u+\zeta)du=0.$$

If $\beta=\gamma$, then we can calculate similarly, and this completes the proof.

We calculated $\frac{1}{2}\left(\int_0^1\sqrt{-2\log u}\,du\right)^2$ by numerical integration and its value is approximately equal to 0.785.

5. The evaluation of accuracy

To check the orthogonality of $U_\alpha(t)-0.5$ and $U_{0.5}(t)-0.5$, we draw a graph of Riemannian sum of $(U_\alpha(t)-0.5)(U_{0.5}(t)-0.5)$ as a function of α. The number of divided points is 7,000,000 in Figure 15 and calculated numbers are represented by 64 bits.

From this experiment, we should keep $|\alpha-\beta|\geqq 0.2$ to guarantee the independence of $U_\alpha(t)-0.5$ and $U_\beta(t)-0.5$.

To overcome this restriction, we should use numbers of longer precision.

For numbers of finite precision, formulae of the main theorem in cases (1), (2) and (3) changes continuously as γ approaches to α. The main theorem 3 is described for numbers of infinite precision, and three cases are separated discontinuously, whose behaviour may be important in the future study such as phase transition.

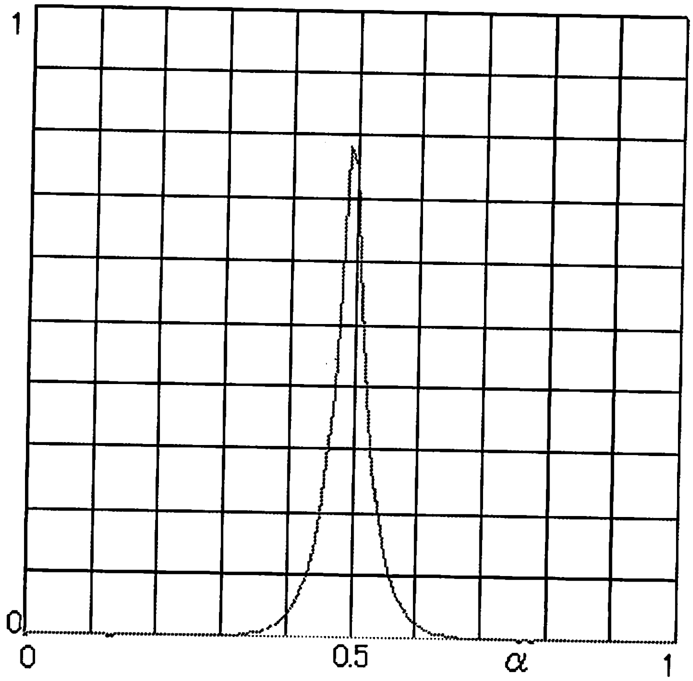

Figure 15 : graph of alpha-Riemannian sum

References

[1] S. Ohwaki and K. Matsuda, *On a formula for correlation coefficients between normally scattered functions*, Kumamoto J. Math., Vol. 12, (1998), 73-79.

[2] S. Ohwaki and K. Matsuda, *Proof of formulae for correlation coefficients between normally scattered functions using stochastic Riemannian integrals*, Proceedings of 1999 spring annual conference of Japan Mathematical Society, (1999), 90-93 (in Japanese)

[3] http://www.sci.kumamoto-u.ac.jp/~ohwaki/VisualPDE98/SFindex.html (in Japanese)

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