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(p,q,r)-Generations of the Conway group Co_1 for odd p*

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Abstract

In this paper we will find all the (p, q, r)-generations of the Conway group Co_1 where p, q, r are prime divisors of the order of Co_1 and $3 \leq p < q < r$.

1. Introduction

The motivation for this paper comes from [6], in which the authors considered the $(2, p, q)$-generations of the Conway group Co_1. In this paper we investigate the remaining (p, q, r)-generations for the Conway group Co_1, where p, q, r are prime numbers dividing the order of Co_1 and $3 \leq p < q < r$.

We encourage reader to consult [2], [6] and [7] for discussion and background material about the triangle groups. For details of the computational techniques which we use the reader is referred to [8]. For the description of the conjugacy classes and the character tables of various groups see [4]. For basic properties of Co_1 and informations on its maximal subgroups the reader is referred to [1], [5], [10], and [11].

The maximal subgroups of Co_1 are listed in Table I. Throughout this paper we use the same notation as in [6]. A group G is said to be (l, m, n)-generated if it can be generated by two elements x and y such that $O(x)=l$, $O(y)=m$, $O(xy)=n$ and by [2] if the simple group G is (l, m, n)-generated then $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}<1$. Hence for a non-abelian finite simple group G and divisors l, m, n of the order of G such that $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}<1$, it is natural to ask if G is a (l, m, n)-generated group. If x is an element of order l in a group G, then we say that x belongs to the conjugacy class lX for suitable X. If G is a finite group, then we set $\Delta(G)=\Delta_G(lX, mY, nZ)$ to denote the structure constant of the center of the group algebra CG, for the conjugacy classes lX, mY and nZ, whose value is the cardinality of the set $\Gamma=\{(x, y) \mid xy=z\}$, where $x \in lX$, $y \in mY$ and z is a fixed element of nZ. This number can be calculated from the character table of G and it is equal to;

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Table I : The maximal subgroups of Co_1

Group	*Order*	*Group*	*Order*
Co_2	$2^{18}.3^6.5^3.7.11.23$	$3 \cdot Suz \cdot 2$	$2^{14}.3^8.5^2.7.11.13$
$2^{11} : M_{24}$	$2^{21}.3^3.5.7.11.23$	Co_3	$2^{10}.3^7.5^3.7.11.23$
$2_+^{1+8}.O_8^+(2)$	$2^{21}.3^5.5^2.7$	$U_6(2).S_3$	$2^{16}.3^7.5.7.11$
$(A_4 \times G_2(4)) : 2$	$2^{15}.3^4.5^2.7.13$	$2^{2+12} : (A_8 \times S_3)$	$2^{21}.3^3.5.7$
$2^{4+12}.(S_3 \times 3S_6)$	$2^{21}.3^4.5$	$3^2.U_4(3).D_8$	$2^{10}.3^8.5.7$
$3^6 : 2M_{12}$	$2^7.3^9.5.11$	$(A_5 \times J_2) : 2$	$2^{10}.3^4.5^3.7$
$3^{1+4}.2U_4(2).2$	$2^8.3^9.5$	$(A_6 \times U_3(3)) : 2$	$2^9.3^5.5.7$
$3^{3+4} : 2(S_4 \times S_4)$	$2^7.3^9$	$A_9 \times S_3$	$2^7.3^5.5.7$
$(A_7 \times L_2(7)) : 2$	$2^7.3^3.5.7^2$	$(D_{10} \times (A_5 \times A_5).2).2$	$2^7.3^2.5^3$
$5^{1+2} : GL_2(5)$	$2^5.3.5^4$	$5^3 : (4 \times A_5).2$	$2^5.3.5^4$
$5^2 : 2A_5$	$2^3.3.5^3$	$7^2 : (3 \times 2A_4)$	$2^3.3^2.7^2$

$$\frac{|G|}{|C_G(x)||C_G(y)|} \sum_{\chi \in Irr(G)} \frac{\chi(x)\chi(y)\overline{\chi(z)}}{\chi(1)}.$$

In Table II, we list the values $\Delta_G(pX, qY, rZ)$, p, q, r distinct prime divisors of $|Co_1|$, using the character table Co_1 and computational facilities in [8]. Also, $\Delta^*(G)=\Delta_G^*(lX, mY, nZ)$ denotes the number of pairs $(x, y)\in\Gamma$ such that $G=\langle x, y\rangle$.

If H is a subgroup of G containing $z\in nZ$, then $\Sigma(H)$ denotes the number of pair $(x, y)\in \Gamma$ such that $\langle x, y\rangle\leq H$. Clearly, if $\Delta^*(G)>0$ then G is (lX, mY, nZ)-generated and (lX, mY, nZ) is called a generating triple for G and for any permutation $\pi\in S_3$, the group G is also $((l)\pi, (m)\pi, (n)\pi)$-generated. The number of conjugates of a given subgroup H of G containing a fixed element z, is given by $\chi_{N_G(H)}(z)$, where $\chi_{N_G(H)}$ is the permutation character of G with action on the conjugates of H [11]. In most cases we will calculate this value from the fusion map from $N_G(H)$ into G stored in *GAP* [8].

Now we discuss techniques that are useful in resolving generation type questions for finite groups. We begin with a theorem of Scott that, in certain situations, is very effective at establishing non-generations (see [9]).

Lemma 1.1. [9] Let x_1, x_2, $\cdots$, x_m be elements generating a group G with $x_1.x_2\cdots x_m=1$, and let V be an irreducible module for G of dimension n. Let $C_V(x_i)$ denote the fixed point space of $\langle x_i\rangle$ on V, and let d_i be the dimension of $V/C_V(x_i)$. Then $d_1+\cdots+d_m\geq 2n$.

Table II : Non-zero (3, p, q) Structure Constants of Co_1

pX	$\Delta(3A,5A,pX)$	$\Delta(3A,5B,pX)$	$\Delta(3A,5C,pX)$	$\Delta(3B,5A,pX)$
$7A$	-	-	-	1205400
$7B$	-	1764	-	235697
$11A$	-	484	198	77000
$13A$	-	234	156	118352
$23A$	-	46	184	19895
pX	$\Delta(3B,5B,pX)$	$\Delta(3B,5C,pX)$	$\Delta(3C,5A,pX)$	$\Delta(3C,5B,pX)$
$7A$	12908560	4568760	329280	27398840
$7B$	17357172	3891972	325066	12367992
$11A$	5842232	4344648	145332	10622788
$13A$	2959320	5179512	168792	13058656
$23A$	1929838	4716288	97980	9068808
pX	$\Delta(3C,5C,pX)$	$\Delta(3D,5A,pX)$	$\Delta(3D,5B,pX)$	$\Delta(3D,5C,pX)$
$7A$	33780600	5347566	5347566	369117000
$7B$	23095464	1599360	183670032	514280088
$11A$	22258896	2273062	200516228	512490066
$13A$	25020840	3478462	213945602	481031460
$23A$	21995820	2505781	212118627	509215998
pX	$\Delta(3A,7A,pX)$	$\Delta(3A,7B,pX)$	$\Delta(3B,7A,pX)$	$\Delta(3B,7B,pX)$
$11A$	330	11088	8055476	120492768
$13A$	3549	6084	12818988	80507856
$23A$	46	2369	3909885	60132120
pX	$\Delta(3C,7A,pX)$	$\Delta(3C,7B,pX)$	$\Delta(3D,7A,pX)$	$\Delta(3D,7B,pX)$
$11A$	23530342	310154064	418285428	6281516340
$13A$	25686284	340774200	493319905	6486527892
$23A$	18427876	280671760	433043494	6496166115
pX	$\Delta(3A,11A,pX)$	$\Delta(3B,11A,pX)$	$\Delta(3C,11A,pX)$	$\Delta(3D,11A,pX)$
$13A$	80028	1226256720	5344906840	115735162244
$23A$	52831	1075384320	5011865600	115734030855

Table II : (Continued)

pX	$\Delta(3A,13A,pX)$	$\Delta(3B,13A,pX)$	$\Delta(3C,13A,pX)$	$\Delta(3D,13A,pX)$
$23A$	19458	453342558	2116562396	48962275294
pX	$\Delta(5A,7A,pX)$	$\Delta(5A,7B,pX)$	$\Delta(5B,7A,pX)$	$\Delta(5B,7B,pX)$
$11A$	153182458	1981445928	9727662472	136990349496
$13A$	365158742	1924299936	11016985434	114784975032
$23A$	74364267	1168184858	6513704949	98206426674
pX	$\Delta(5C,7A,pX)$	$\Delta(5C,7B,pX)$	-	-
$11A$	15773894466	230416767504	-	-
$13A$	16299619908	243388351752	-	-
$23A$	15713281818	235703005848	-	-
pX	$\Delta(5A,11A,pX)$	$\Delta(5B,11A,pX)$	$\Delta(5C,11A,pX)$	-
$13A$	25559399840	1858852505432	4245451335528	-
$23A$	20969605402	1751187990762	4199775606936	-
pX	$\Delta(5A,13A,pX)$	$\Delta(5B,13A,pX)$	$\Delta(5C,13A,pX)$	-
$23A$	8813357810	740345494098	1776825805608	-
pX	$\Delta(7A,11A,pX)$	$\Delta(7B,11A,pX)$	-	-
$13A$	3961393756476	55259477523120	-	-
$23A$	3574899110415	53568552007800	-	-
pX	$\Delta(7A,13A,pX)$	$\Delta(7B,13A,pX)$	$(11A,13A,pX)$	-
$23A$	1510911910876	22663617611250	403824480386250	-

Lemma 1.2. [3] Let G be a finite centerless group and suppose lX, mY and nZ are G-conjugacy classes for which $\Delta^*(G)<|C_G(z)|$, $z\in nZ$. Then $\Delta^*(G)=0$ and therefore G is not (lX, mY, nZ)-generated.

Lemma 1.3. [13] Let G be a finite simple group and H a maximal subgroup of G containing a fixed element x. Then the number h of conjugates of H containing x is $\chi_H(x)$, where χ_H is the permutation character of G with action on the conjugates of H. In particular,

$$h=\sum_{i=1}^{m}\frac{|C_G(x)|}{|C_H(x_i)|},$$

where $x_1, x_2,\cdots, x_m$ are representatives of the H-conjugacy classes that fuse to the G-conjugacy class of x.

We calculated h for suitable triples in Table IV. In this paper we will prove the following theorem:

Theorem. The Conway group Co_1 is (p, q, r)-generated for all $p, q, r \in \{3, 5, 7, 11, 13, 23\}$ with $p<q<r$.

2. (3, p, q)-Generations for Co_1

The group Co_1 acts on a 24-dimensional vector space $\overline{\Lambda}$ over $GF(2)$ and has three orbits on the set of non-zero vectors. The stabilizers are the groups Co_2, Co_3 and $2^{11}: M_{24}$, and the permutation characters of Co_1 on $\overline{\Lambda}-\{0\}$ is $\chi = 1_{Co_2}\uparrow^{Co_1} + 1_{Co_3}\uparrow^{Co_1} + 1_{2^{11}:M_{24}}\uparrow^{Co_1}$, and using GAP[8], we find that

$$\chi = 3\cdot 1a + 2\cdot 299a + 3\cdot 17250a + 3\cdot 80730a + 376740a + 644644a + 2055625a + 2417415a + 2\cdot 5494125a$$

Now for $g \in Co_1$, the value of $\chi(g)$ is the number of non-zero vectors of $\overline{\Lambda}$ fixed by g from which we can find the dimension δ_{nX} of the fixed space of $g \in nX$. Using the character table of Co_1 we list in Table III the values of $d_{pX} = 24 - \delta_{pX}$, for all conjugacy classes with prime order representatives. Finally, in Table IV, we list the partial fusion maps of the maximal subgroups into Co_1 that we will use later. In this table h denotes the number of conjugates of the maximal subgroup H containing a fixed element z (see Lemma 1.3).

It is a well known fact that Co_1 has exactly 22 conjugacy classes of maximal subgroups, as listed in Table I. We will use the maximal subgroups of Co_1 listed in the ATLAS extensively, especially those with order divisible by $p, q, 3 \leq p < q$. We listed in Table IV, the fusion maps of these maximal subgroups into Co_1 (obtained from GAP) that will enable us to evaluate $\Delta^*(Co_1)$.

We first begin with $(3, 5, p)$-generations of Co_1. We obtain all the $(3, 5, p)$-generations of the Conway group Co_1, with the exception of $(3B, 5A, 7B)$, $(3B, 5B, 11A)$ and $(3A, 5B, 13A)$. In Table IV, we list the partial fusion maps of maximal subgroups into Co_1 that we will use later.

Table III : The codimensions $d_{nX} = \dim(V/C_V(nX))$

d_{2A}	d_{2B}	d_{2C}	d_{3A}	d_{3B}	d_{3C}	d_{3D}	d_{5A}
8	12	12	24	12	18	16	24
d_{5B}	d_{5C}	d_{7A}	d_{7B}	d_{11A}	d_{13A}	d_{23A}	d_{23B}
16	20	24	18	20	24	22	22

Table IV : Partial Fusion Maps into Co_1

Co_2-class	2a	2b	2c	3a	3b	5a	5b	
$\rightarrow Co_1$	2A	2A	2C	3C	3B	5C	5B	
Co_2-class	7a	11a	23a	23b				
$\rightarrow Co_1$	7B	11A	23A	23B				
h	21	6	1	1				
$3 \cdot Suz \cdot 2$-classes	2a	2b	2c	2d	3a	3b	3c	3d
$\rightarrow Co_1$	2A	2B	2B	2C	3A	3A	3B	3B
$3 \cdot Suz \cdot 2$-classes	3e	3f	5a	5b	7a	11a	13a	
$\rightarrow Co_1$	3C	3D	5A	5B	7A	11A	13A	
h					35	1	4	
$2^{11} : M_{24}$-classes	2a	2b	2c	2d	2e	2f	3a	3b
$\rightarrow Co_1$	2A	2C	2A	2C	2C	2B	3B	3D
$2^{11} : M_{24}$-classes	5a	7a	7b	11a	23a	23b		
$\rightarrow Co_1$	5B	7B	7B	11A	23A	23B		
h		14	14	3	1	1		
Co_3-classes	2a	2b	3a	3b	3c	5a	5b	
$\rightarrow Co_1$	2A	2C	3C	3B	3D	5C	5B	
Co_3-classes	7a	11a	11b	23a	23b			
$\rightarrow Co_1$	7B	11A	11A	23A	23B			
h	28	6	6	1	1			
$U_6(2).S_3$-classes	2a	2b	2c	2d	2e	3a	3b	3c
$\rightarrow Co_1$	2A	2A	2C	2A	2C	3B	3C	3B
$U_6(2).S_3$-classes	3d	3e	3f	3g	5a	7a	11a	
$\rightarrow Co_1$	3A	3B	3C	3D	5B	7B	11A	
h						28	2	
$(A_4 \times G_2(4)) : 2$-classes	2a	2b	2c	2d	2e	2f	3a	3b
$\rightarrow Co_1$	2A	2B	2B	2B	2B	2C	3A	3D
$(A_4 \times G_2(4)) : 2$-classes	3c	3d	3e	5a	5b	7a	13a	
$\rightarrow Co_1$	3A	3B	3D	5B	5A	7A	13A	
h						35	1	

Table IV : (Continued)

$2_+^{1+8}.O_8^+(2)$-class	2a	2b	2c	2d	2e	2f	2g	2h	3a	3b	3c
$\rightarrow Co_1$	2A	2A	2B	2A	2C	2C	2C	2A	3B	3C	3A
$2_+^{1+8}.O_8^+(2)$-class	3d	3e	5a	5b	5c	7a					
$\rightarrow Co_1$	3B	3C	5A	5B	5C	7B					
h						21					
$2^{2+12} : (A_8 \times S_3)$-class	2a	2b	2c	2d	2e	2f	2g	2h	2i	2j	2k
$\rightarrow Co_1$	2A	2A	2B	2C	2A	2C	2C	2A	2C	2C	2B
$2^{2+12} : (A_8 \times S_3)$-class	2l	2m	3a	3b	3c	3d	3e	5a	7a	7b	
$\rightarrow Co_1$	2C	2B	3D	3A	3D	3B	3D	5A	7B	7B	
h										420	
$A_9 \times S_3$-class	2a	2b	2c	2d	2e	3a	3b	3c	3d	3e	3f
$\rightarrow Co_1$	2A	2B	2B	2C	2C	3D	3A	3D	3C	3D	3B
$A_9 \times S_3$-class	3g	5a	7a								
$\rightarrow Co_1$	3D	5A	7B								
h			420								
$(A_7 \times L_2(7)) : 2$-class	2a	2b	2c	2d	2e	3a	3b	3c	3d	3e	5a
$\rightarrow Co_1$	2A	2B	2B	2B	2C	3D	3A	3D	3B	3D	5A
$(A_7 \times L_2(7)) : 2$-class	7a	7b	7c	7d							
$\rightarrow Co_1$	7A	7B	7A	7B							
h			722	750							
$(A_5 \times J_2) : 2$-class	2a	2b	2c	2d	2e	2f	3a	3b	3c	3d	3e
$\rightarrow Co_1$	2A	2B	2B	2B	2C	2B	3A	3D	3A	3B	3D
$(A_5 \times J_2) : 2$-class	5a	5b	5c	5d	5e	5f	5g	7a			
$\rightarrow Co_1$	5B	5A	5A	5A	5B	5C	5B	7A			
h								21			
$3^2.U_4(3).D_8$-class	2a	2b	2c	2d	2e	2f	3a	3b	3c	3d	3e
$\rightarrow Co_1$	2A	2B	2C	2A	2C	2B	3B	3A	3B	3C	3A
$3^2.U_4(3).D_8$-class	3f	3g	3h	5a	7a						
$\rightarrow Co_1$	3C	3B	3D	5B	7A						
h					70						

Using character table of Co_1, stored in GAP, we can see that Co_1 has exactly two conjugacy classes of elements of order 23, i.e., $23A$ and $23B$, and that $23A^{-1}=23B$. Therefore, for investigation of $(3X, pY, 23Z)$-generation of Co_1, we can assume that $Z=A$.

In the following lemma we compute all the (3, 5, 7)-generations of the Conway group Co_1, with the exception of $(3B, 5A, 7B)$.

Lemma 2.1 Suppose $(X, Y, Z)\neq(B, A, B)$. The group Co_1 is $(3X, 5Y, 7Z)$-generated if and only if $X\in\{C, D\}$ or $(X, Y, Z)\in\{(A, B, B), (B, A, A), (B, A, B), (B, B, A), B, C, A), (B, C, B)\}$.

Proof. Set, $R=\{C, D\}\times\{A, B, C\}\times\{A, B\}$ and $S=\{(A, B, B), (B, A, A), (B, A, B), (B, B, A), (B, C, A), (B, C, B)\}$. We consider the triple $(3B, 5B, 7B)$, using Table III, we have,

$$d_{3B}+d_{5B}+d_{7B}=12+16+18=46<48,$$

hence, by Scott's theorem $(3B, 5B, 7B)$ is a non-generating triple of Co_1. Therefore, we can assume that $(X, Y, Z)\neq(B, B, B)$. If $(X, Y, Z)\notin R\cup S$, then $\Delta_{Co_1}(3X, 5Y, 7Z)=0$ and therefore Co_1 is not $(3X, 5Y, 7Z)$-generated. Our main proof will consider a number of cases,

Case $(3A, 5B, 7B)$. In this case, $\Delta(Co_1)=1764$ and maximal subgroups of Co_1 with non-empty intersection with all the conjugacy classes in this triple are, up to isomorphisms, $U_6(2)\cdot 3\cdot 2$ and $2_+^{1+8}\cdot O_8^+(2)$. Our calculations give, $\Sigma(U_6(2)\cdot 3\cdot 2)=0$ and $\Sigma(2_+^{1+8}\cdot O_8^+(2))=28$. This implies that, $\Delta^*(Co_1)\geq\Delta(Co_1)-28\cdot 21>0$, and whence $(3A, 5B, 7B)$ is a generating triple for Co_1.

Case $(3B, 5A, 7A)$. The maximal subgroups of Co_1 that may contain $(3B, 5A, 7A)$-generated proper subgroups are isomorphic to $3\cdot Suz\cdot 2, (A_4\times G_2(4)):2, (A_5\times J_2):2, (A_6\times U_3(3)):2$ and $(A_7\times L_2(7)):2$. We calculate that $\Delta(Co_1)=1205400$, $\Sigma(3\cdot Suz\cdot 2)=11760$ and $\Sigma((A_4\times G_2(4)):2)=\Sigma((A_5\times J_2):2)=\Sigma((A_6\times U_3(3)):2)=\Sigma((A_7\times L_2(7)):2)=0$. Therefore, $\Delta^*(Co_1)\geq\Delta(Co_1)-35\cdot 11760>0$, and whence $(3B, 5A, 7A)$ is a generating triple for Co_1.

Case $(3B, 5B, 7A)$. From the list of maximal subgroups of Co_1 we observe that, up to isomorphisms, $3\cdot Suz\cdot 2, (A_4\times G_2(4)):2, 3^2\cdot U_4(3)\cdot D_8$ and $(A_5\times J_2):2$ are the only maximal subgroups of Co_1 with non-empty intersection with all the conjugacy classes in this triple. We calculate $\Delta(Co_1)=235697$, $\Sigma(3\cdot Suz\cdot 2)=29176$, $\Sigma(3^2\cdot U_4(3)\cdot D_8)=37240$ and $\Sigma((A_5\times J_2):2)=\Sigma((A_4\times G_2(4)):2)=0$. Also, a fixed element of order 7 is contained in 35 conjugates of $3\cdot Suz\cdot 2$ and 70 conjugates of $3^2\cdot U_4(3)\cdot D_8$. Therefore, $\Delta^*(Co_1)\geq\Delta(Co_1)-29176\cdot 35-532\cdot 70>0$, which proves the group Co_1 is $(3B, 5B, 7A)$-generated.

Case $(3B, 5C, 7A)$. The only maximal subgroups of Co_1 that may contain $(3B, 5C, 7A)$-generated proper subgroups are isomorphic to $(A_5\times J_2):2$. Moreover, $\Delta(Co_1)=4568760$

and $\Sigma((A_5\times J_2):2)=0$. Therefore, Co_1 is $(3B, 5C, 7A)$-generated.

Case $(3B, 5C, 7B)$. Amongst the maximal subgroups of Co_1 with order divisible by $3\times5\times7$, the only maximal subgroups with non-empty intersection with any conjugacy class in this triple are isomorphic to Co_2, Co_3 and $2_+^{1+8}\cdot O_8^+(2)$. We can see that $\Delta(Co_1)=3891972$, $\Sigma(Co_2)=27048$, $\Sigma(Co_3)=4095$ and $\Sigma(2_+^{1+8}\cdot O_8^+(2))=6244$. Our calculations give, $\Delta^*(Co_1)>0$, and so Co_1 is $(3B, 5C, 7B)$-generated.

Case $(3C, 5A, 7A)$. The maximal subgroups of Co_1 that may contain $(3C, 5A, 7A)$-generated proper subgroups are isomorphic to $3\cdot Suz\cdot2$ and $(A_6\times U_3(3)):2$. We calculate that $\Delta(Co_1)=329280$, $\Sigma(3\cdot Suz\cdot2)=4368$, $\Sigma((A_6\times U_3(3)):2)=0$. Therefore, $\Delta^*(Co_1)\geq\Delta(Co_1)-35\cdot4368>0$, and whence $(3C, 5A, 7A)$ is a generating triple for Co_1.

Case $(3C, 5A, 7B)$. From the list of maximal subgroups of Co_1 we observe that, up to isomorphisms, $2_+^{1+8}\cdot O_8^+(2)$ and $A_9\times S_3$ are the only maximal subgroups of Co_1 with non-empty intersection with all the conjugacy classes in this triple. We calculate $\Delta(Co_1)=325066$, $\Sigma(2_+^{1+8}\cdot O_8^+(2))=938$ and $\Sigma(A_9\times S_3)=28$. Also, a fixed element of order 7 is contained in 21 conjugates of $2_+^{1+8}\cdot O_8^+(2)$ and 420 conjugates of $A_9\times S_3$. Therefore, $\Delta^*(Co_1)\geq\Delta(Co_1)-19698-11760>0$, which proves the group Co_1 is $(3C, 5A, 7B)$-generated.

Case $(3C, 5B, 7A)$. The only maximal subgroups of Co_1 that may contain $(3C, 5B, 7A)$-generated proper subgroups are isomorphic to $3\cdot Suz\cdot2$ and $3^2\cdot U_4(3)\cdot D_8$. Moreover, $\Delta(Co_1)=27398840$, $\Sigma(3\cdot Suz\cdot2)=32144$, and $\Sigma(3^2\cdot U_4(3)\cdot D_8))=896$. Therefore, Co_1 is $(3C, 5B, 7A)$-generated.

Case $(3C, 5B, 7B)$. Amongst the maximal subgroups of Co_1 with order divisible by $3\times5\times7$, the only maximal subgroups with non-empty intersection with any conjugacy class in this triple are isomorphic to Co_2, Co_3, $2_+^{1+8}\cdot O_8^+(2)$ and $U_6(2)\cdot S_3$. We can see that $\Delta(Co_1)=12367992$, $\Sigma(Co_2)=44296$, $\Sigma(Co_3)=1680$, $\Sigma(2_+^{1+8}\cdot O_8^+(2))=11144$ and $\Sigma(U_6(2)\cdot S_3)=3591$. Our calculations give, $\Delta^*(Co_1)\geq\Delta(Co_1)-930216-47040-100548>0$. Therefore, Co_1 is $(3C, 5B, 7B)$-generated.

Case $(3C, 5C, 7A)$. In this case, $\Delta(Co_1)=33780600$ and there is no maximal subgroups with non-empty intersection with any conjugacy class in this triple. Therefore, Co_1 is $(3C, 5C, 7A)$-generated.

Case $(3C, 5C, 7B)$. The maximal subgroups of Co_1 that may contain $(3C, 5C, 7B)$-generated proper subgroups are isomorphic to Co_3 and $2_+^{1+8}\cdot O_8^+(2)$. We calculate that $\Delta(Co_1)=23095464$, $\Sigma(Co_3)=1680$ and $\Sigma(2_+^{1+8}\cdot O_8^+(2)=10024$. Therefore, $\Delta^*(Co_1)\geq\Delta(Co_1)-47040-210504>0$, and whence $(3C, 5C, 7B)$ is a generating triple for Co_1.

Case $(3D, 5A, 7A)$. From the list of maximal subgroups of Co_1 we observe that, up to isomorphisms, $3\cdot Suz\cdot2$, $(A_4\times G_2(2)):2$, $(A_5\times J_2):2$, $(A_6\times U_3(3)):2$ and $(A_7\times L_2(7)):2$ are the only maximal subgroups of Co_1 with non-empty intersection with all the conjugacy classes in this triple. We calculate $\Delta(Co_1)=5347566$, $\Sigma(3\cdot Suz\cdot2)=75852$, $\Sigma((A_4\times G_2(2)):2)=7938$, $\Sigma((A_5\times J_2):2)=686$ and $\Sigma((A_6\times U_3(3)):2)=\Sigma((A_7\times L_2(7)):2)=0$. Also, a fixed element of order 7 is contained in 35 conjugates of $3 \cdot Suz \cdot 2$, 35 conjugates of $(A_4 \times G_4(2)):2)$ and 21 conjugates of $(A_5 \times J_2):2$. Therefore, $\Delta^*(Co_1) \geq \Delta(Co_1) - 2654820 - 277830 - 14406 > 0$, which proves the group Co_1 is $(3D, 5A, 7A)$-generated.

Case $(3D, 5A, 7B)$. Amongst the maximal subgroups of Co_1 with order divisible by $3 \times 5 \times 7$, the only maximal subgroups with non-empty intersection with any conjugacy class in this triple are isomorphic to $2^{2+12} \cdot (A_8 \times S_3)$, $A_9 \times S_3$ and $(A_7 \times L_2(7)):2$. We can see that $\Delta(Co_1) = 1599360$ and $\Sigma(2^{2+12} \cdot (A_8 \times S_3) = \Sigma(A_9 \times S_3) = \Sigma((A_7 \times L_2(7)):2) = 0$. Our calculations give, $\Delta^*(Co_1) = \Delta(Co_1) > 0$ and therefore, Co_1 is $(3D, 5A, 7B)$-generated.

Case $(3D, 5B, 7A)$. The only maximal subgroups of Co_1 that may contain $(3D, 5B, 7A)$-generated proper subgroups are isomorphic to $3 \cdot Suz \cdot 2$, $(A_4 \times G_4(2)):2$, $3^2 \cdot U_4(3) \cdot D_8$ and $(A_5 \times J_2):2$. Moreover, $\Delta(Co_1) = 195701982$, $\Sigma(3 \cdot Suz \cdot 2) = 438186$, $\Sigma((A_4 \times G_4(2)):2) = 9282$, $\Sigma(3^2 \cdot U_4(3) \cdot D_8)) = 8064$, and $\Sigma((A_5 \times J_2):2 = 112$. Therefore, Co_1 is $(3D, 5B, 7A)$-generated.

Case $(3D, 5B, 7B)$. In this case, $\Delta(Co_1) = 183670032$ and maximal subgroups of Co_1 with non-empty intersection with all the conjugacy classes in this triple are, up to isomorphisms, $2^{11}:M_{24}$, Co_3 and $U_6(2) \cdot 3 \cdot 2$. Our calculations give, $\Sigma(2^{11}:M_{24}) = 381696$, $\Sigma(Co_3) = 269136$ and $\Sigma(U_6(2) \cdot 3 \cdot 2) = 0$. This implies that, $\Delta^*(Co_1) \geq \Delta(Co_1) - 7535808 - 10687488 > 0$, and whence $(3D, 5B, 7B)$ is a generating triple for Co_1.

Case $(3D, 5C, 7A)$. In this case, $\Delta(Co_1) = 369117000$ and $(A_5 \times J_2):2$ is the unique maximal subgroup with non-empty intersection with any conjugacy class in this triple. But, $\Sigma((A_5 \times J_2):2) = 0$ and therefore, Co_1 is $(3D, 5C, 7A)$-generated.

Case $(3D, 5C, 7B)$. In this case, $\Delta(Co_1) = 514280088$ and Co_3 is the unique maximal subgroup with non-empty intersection with any conjugacy class in this triple. But, $\Sigma(Co_3) = 85428$ and $\Delta^*(Co_1) \geq \Delta(Co_1) - 2391984 > 0$. Therefore, Co_1 is $(3D, 5C, 7B)$-generated. This completes the proof. □

In the following lemma we compute all the $(3, 5, 11)$-generations of the Conway group Co_1, with the exception of $(3B, 5B, 11A)$.

Lemma 2.2. Suppose $(X, Y) \neq (B, B)$. The group Co_1 is $(3X, 5Y, 11A)$-generated if and only if $X \in \{C, D\}$ or $(X, Y) \in \{(A, B), (A, C), (B, A), (B, C)\}$.

Proof. Set $R = \{C, D\} \times \{A, B, C\} \times \{A\}$ and $S = \{(A, B), (A, C), (B, A), (B, C)\}$. If $(X, Y) \not\subseteq R \cup S$, then $\Delta_{Co_1}(3X, 5Y, 11A) = 0$ and so $(3X, 5Y, 11A)$ is not a generating triple. Now, our main proof will consider a number of cases.

Case $(3A, 5B, 11A)$. The only maximal subgroups of Co_1 that may contain $(3A, 5B, 11A)$-generated proper subgroups are isomorphic to $3 \cdot Suz \cdot 2$, $U_6(2) \cdot S_3$. Moreover, $\Delta(Co_1) = 484$, $\Sigma(3 \cdot Suz \cdot 2) = 88$ and $\Sigma(U_6(2) \cdot S_3) = 0$. Therefore, Co_1 is $(3A, 5B, 11A)$-generated.

Case $(3A, 5C, 11A)$. In this case, $\Delta(Co_1) = 198$ and there is no maximal subgroup with non-empty intersection with any conjugacy classes in this triple. Therefore, Co_1 is $(3A, 5C, 11A)$-generated.

Case $(3B, 5A, 11A)$. The only maximal subgroups of Co_1 that may contain $(3B, 5A, 11A)$-generated proper subgroups is isomorphic to $3\cdot Suz\cdot 2$. Moreover, $\Delta(Co_1)=77000$ and $\Sigma(3\cdot Suz\cdot 2)=3410$. Therefore, $\Delta^*(Co_1)>0$ and Co_1 is $(3B, 5A, 11A)$-generated.

Case $(3B, 5C, 11A)$. From the list of maximal subgroups of Co_1 we observe that, up to isomorphisms, Co_2 and Co_3 are the only maximal subgroups of Co_1 with non-empty intersection with all the conjugacy classes in this triple. We calculate $\Delta(Co_1)=4344648$, $\Sigma(Co_2)=51513$ and $\Sigma(Co_3)=6380$. Also, a fixed element of order 11 is contained in 6 conjugates of Co_2 and 12 conjugates of Co_3. Therefore, $\Delta^*(Co_1)\geq\Delta(Co_1)-309078-153120>0$, which proves the group Co_1 is $(3B, 5C, 11A)$-generated.

Case $(3C, 5A, 11A)$. In this case, $\Delta(Co_1)=145332$ and $3\cdot Suz\cdot 2$ is the unique maximal subgroup with non-empty intersection with any conjugacy classes in this triple. But, $\Sigma(3\cdot Suz\cdot 2)=3762$ and therefore, Co_1 is $(3C, 5A, 11A)$-generated.

Case $(3C, 5B, 11A)$. Amongst the maximal subgroups of Co_1 with order divisible by $3\times 5\times 11$, the only maximal subgroups with non-empty intersection with any conjugacy classes in this triple are isomorphic to Co_2, $3\cdot Suz\cdot 2$, Co_3 and $U_6(2)\cdot S_3$. We can see that $\Delta(Co_1)=10622788$, $\Sigma(Co_2)=77055$, $\Sigma(3\cdot Suz\cdot 2)=23452$, $\Sigma(Co_3)=4664$ and $\Sigma(U_6(2)\cdot S_3)=6105$. Our calculations give, $\Delta^*(Co_1)\geq\Delta(Co_1)-462330-23452-55968-12210>0$ and therefore, Co_1 is $(3C, 5B, 11A)$-generated.

Case $(3C, 5C, 11A)$. In this case, $\Delta(Co_1)=22258896$. Using Table I, the only maximal subgroups with non-empty intersection with any conjugacy classes in this triple are Co_2 and Co_3. But, $\Sigma(Co_2)=40755$, $\Sigma(Co_3)=2992$ and $\Delta^*(Co_1)\geq\Delta(Co_1)-244530-35904>0$. Therefore, Co_1 is $(3C, 5C, 11A)$-generated.

Case $(3D, 5A, 11A)$. The only maximal subgroups of Co_1 that may contain $(3D, 5A, 11A)$-generated proper subgroups is isomorphic to $3\cdot Suz\cdot 2$. Moreover, $\Delta(Co_1)=2273062$ and $\Sigma(3\cdot Suz\cdot 2)=70642$. Therefore, $\Delta^*(Co_1)>0$ and Co_1 is $(3D, 5A, 11A)$-generated.

Case $(3D, 5B, 11A)$. Amongst the maximal subgroups of Co_1 with order divisible by $3\times 5\times 11$, the only maximal subgroups with non-empty intersection with any conjugacy classes in this triple are isomorphic to $3\cdot Suz\cdot 2$, $2^{11}:M_{24}$, Co_3 and $U_6(2)\cdot S_3$. We can see that $\Delta(Co_1)=200516228$, $\Sigma(3\cdot Suz\cdot 2)=481547$, $\Sigma(2^{11}:M_{24})=226688$, $\Sigma(Co_3)=7753944$ and $\Sigma(U_6(2)\cdot S_3)=0$. Our calculations give, $\Delta^*(Co_1)\geq\Delta(Co_1)-481547-226688-93047328>0$ and therefore, Co_1 is $(3D, 5B, 11A)$-generated.

Case $(3D, 5C, 11A)$. In this case, $\Delta(Co_1)=5124900066$ and Co_3 is the unique maximal subgroup with non-empty intersection with any conjugacy classes in this triple. But, $\Sigma(Co_3)=152944$ and therefore, Co_1 is $(3D, 5C, 11A)$-generated, proving the result. □

Lemma 2.3. Suppose $(X, Y) \neq (A, B)$. The Conway group Co_1 is $(3X, 5Y, 13A)$-generated if and only if $X \in \{B, C, D\}$ or $(X, Y) = (A, C)$.

Proof. Set, $R = \{(A, C), (B, C), (C, C), (D, C)\}$ and $S = \{B, C, D\} \times \{A, B, C\} \times \{A\}$. Suppose $(X, Y) \in R$, then there is no maximal subgroup of Co_1 with non-empty intersection with this triple. Therefore, Co_1 is $(3X, 5Y, 13A)$-generated. We assume that $(X, Y) \in S - R$. Now, our main proof will consider a number of cases.

Case $(3B, 5A, 13A)$. Amongst the maximal subgroups of Co_1 with order divisible by $3 \times 5 \times 13$, the only maximal subgroups with non-empty intersection with any conjugacy classes in this triple are isomorphic to $3 \cdot Suz \cdot 2$ and $(A_4 \times G_2(4)) : 2$. We can see that $\Delta(Co_1) = 118352$, $\Sigma(3 \cdot Suz \cdot 2) = 3575$ and $\Sigma((A_4 \times G_2(4)) : 2) = 0$. Our calculations give, $\Delta^*(Co_1) \geq \Delta(Co_1) - 14300 > 0$ and therefore, Co_1 is $(3B, 5A, 13A)$-generated.

Case $(3B, 5B, 13A)$. The only maximal subgroups of Co_1 that may contain $(3B, 5B, 13A)$-generated proper subgroups are isomorphic to $3 \cdot Suz \cdot 2$ and $(A_4 \times G_2(4)) : 2$. Moreover, $\Delta(Co_1) = 2959320$, $\Sigma(3 \cdot Suz \cdot 2) = 15210$ and $\Sigma(A_4 \times G_2(4)) : 2) = 0$. Therefore, $\Delta^*(Co_1) > 0$ and Co_1 is $(3B, 5B, 13A)$-generated.

Case $(3C, 5A, 13A)$. In this case, $\Delta(Co_1) = 168792$ and $3 \cdot Suz \cdot 2$ is the unique maximal subgroup with non-empty intersection with any conjugacy classes in this triple. But, $\Sigma(3 \cdot Suz \cdot 2) = 3510$ and therefore, Co_1 is $(3C, 5A, 13A)$-generated.

Case $(3C, 5B, 13A)$. The only maximal subgroups of Co_1 that may contain $(3C, 5B, 13A)$-generated proper subgroups is isomorphic to $3 \cdot Suz \cdot 2$. Moreover, $\Delta(Co_1) = 13058656$ and $\Sigma(3 \cdot Suz \cdot 2) = 26572$. Therefore, $\Delta^*(Co_1) > 0$ and Co_1 is $(3C, 5B, 13A)$-generated.

Case $(3D, 5A, 13A)$. Amongst the maximal subgroups of Co_1 with order divisible by $3 \times 5 \times 13$, the only maximal subgroups with non-empty intersection with any conjugacy classes in this triple are isomorphic to $3 \cdot Suz \cdot 2$ and $(A_4 \times G_2(4)) : 2$. We can see that $\Delta(Co_1) = 3478462$, $\Sigma(3 \cdot Suz \cdot 2) = 76843$ and $\Sigma((A_4 \times G_2(4)) : 2) = 9282$. Our calculations give, $\Delta^*(Co_1) \geq \Delta(Co_1) - 307372 - 9282 > 0$ and therefore, Co_1 is $(3D, 5A, 13A)$-generated.

Case $(3D, 5B, 13A)$. The only maximal subgroups of Co_1 that may contain $(3D, 5B, 13A)$-generated proper subgroups are isomorphic to $3 \cdot Suz \cdot 2$ and $(A_4 \times G_2(4)) : 2$. Moreover, $\Delta(Co_1) = 213945602$, $\Sigma(3 \cdot Suz \cdot 2) = 461006$ and $\Sigma((A_4 \times G_2(4)) : 2) = 9282$. Therefore, $\Delta^*(Co_1) > 0$ and Co_1 is $(3D, 5B, 13A)$-generated. This completes the proof. □

Lemma 2.4. The Conway group Co_1 is $(3X, 5Y, 23A)$-generated if and only if $X \in \{B, C, D\}$ or $(X, Y) \in \{(A, B), (A, C)\}$.

Proof. Set, $R = \{(A, B), (A, C), (B, A), (C, A), (D, A)\}$ and $S = \{B, C, D\} \times \{A, B, C\} \times \{A\}$. Suppose $(X, Y) \in R$. Then there is no maximal subgroup of Co_1 with non-empty intersection with this triple. Therefore, Co_1 is $(3X, 5Y, 23A)$-generated. Assume that $(X, Y) \in S-R$. Now, our main proof will consider a number of cases.

Case $(3B, 5B, 23A)$. The only maximal subgroups of Co_1 that may contain $(3B, 5B, 23A)$-generated proper subgroups are isomorphic to Co_2, $2^{11}:M_{24}$ and Co_3. Moreover, $\Delta(Co_1) = 1929838$, $\Sigma(Co_2)=344701$, $\Sigma(2^{11}:M_{24})=27232$, $\Sigma(Co_3)=44160$. Therefore, $\Delta^*(Co_1)>0$ and Co_1 is $(3B, 5B, 23A)$-generated.

Case $(3B, 5C, 23A)$. Amongst the maximal subgroups of Co_1 with order divisible by $3\times 5 \times 23$, the only maximal subgroups with non-empty intersection with any conjugacy classes in this triple are isomorphic to Co_2 and Co_3. We can see that $\Delta(Co_1)=4716288$, $\Sigma(Co_2)=90137$ and $\Sigma(Co_3)=10166$. Our calculations give, $\Delta^*(Co_1) \geq \Delta(Co_1)-90137-10166>0$ and therefore, Co_1 is $(3B, 5C, 23A)$-generated.

Case $(3C, 5B, 23A)$. The only maximal subgroups of Co_1 that may contain $(3C, 5B, 23A)$-generated proper subgroups are isomorphic to Co_2 and Co_3. Moreover, $\Delta(Co_1)=9068808$, $\Sigma(Co_2)=137471$ and $\Sigma(Co_3)=3818$. Therefore, $\Delta^*(Co_1)>0$ and Co_1 is $(3C, 5B, 23A)$-generated.

Case $(3C, 5C, 23A)$. From the list of maximal subgroups of Co_1 we observe that, up to isomorphisms, Co_2 and Co_3 are the only maximal subgroups of Co_1 with non-empty intersection with all the conjugacy classes in this triple. We calculate $\Delta(Co_1)=21995820$, $\Sigma(Co_2)=31027$ and $\Sigma(Co_3)=1380$. Also, a fixed element of order 23 is contained in 1 conjugate of Co_2 and 1 conjugate of Co_3. Therefore, $\Delta^*(Co_1) \geq \Delta(Co_1)-31027-1380>0$, which proves the group Co_1 is $(3C, 5C, 23A)$-generated.

Case $(3D, 5B, 23A)$. Amongst the maximal subgroups of Co_1 with order divisible by $3\times 5 \times 23$, the only maximal subgroups with non-empty intersection with any conjugacy classes in this triple are isomorphic to $2^{11}:M_{24}$ and Co_3. We can see that $\Delta(Co_1)=212118627$, $\Sigma(2^{11}:M_{24}) =259072$ and $\Sigma(Co_3)=376372$. Our calculations give, $\Delta^*(Co_1) \geq \Delta(Co_1)-259072-376372>0$ and therefore, Co_1 is $(3D, 5B, 23A)$-generated.

Case $(3D, 5C, 23A)$. In this case, $\Delta(Co_1)=509215998$ and Co_3 is the unique maximal subgroup with non-empty intersection with any conjugacy classes in this triple. But $\Sigma(Co_3)= 70219$ and therefore, Co_1 is $(3D, 5C, 23A)$-generated, proving the lemma. □

In the following lemma we investigate the $(3X, pY, qZ)$-generations of the Conway group Co_1, $7 \leq p<q$.

Lemma 2.5 The Conway group Co_1 is $(3X, pY, qZ)$-generated, in which $7 \leq p<q$.

Proof. Set, $R=\{(3A, 7A, 23A), (3A, 7B, 13A), (3A, 7B, 23A), (3B, 7A, 23A), (3B, 7B, 13A), (3C, 7A, 23A), (3C, 7B, 13A), (3D, 7A, 23A), (3D, 7B, 13A), (3A, 11A, 23A), (3A, 13A, 23A), (3B, 13A, 23A), (3C, 13A, 23A), (3D, 13A, 23A)\}$. Suppose $7 \leq p<q$ and $(3X, pY, qZ) \in R$.

Then there is no maximal subgroup of Co_1 with non-empty intersection with this triple. Therefore, Co_1 is $(3X, pY, qZ)$-generated. Now, our main proof will consider a number of cases.

Case $(3A, 7A, 11A)$. The only maximal subgroups of Co_1 that may contain $(3A, 7A, 11A)$-generated proper subgroups is isomorphic to $3{\cdot}Suz{\cdot}2$. Moreover, $\Delta(Co_1)=330$ and $\Sigma(3{\cdot}Suz{\cdot}2)=132$. Therefore, $\Delta^*(Co_1)>0$ and Co_1 is $(3A, 7A, 11A)$-generated.

Case $(3A, 7A, 13A)$. Amongst the maximal subgroups of Co_1 with order divisible by $3\times 7\times 13$, the only maximal subgroups with non-empty intersection with any conjugacy classes in this triple are isomorphic to $3{\cdot}Suz{\cdot}2$ and $(A_4\times G_2(4)):2$. We can see that $\Delta(Co_1)=3549$, $\Sigma((A_4\times G_2(4)):2)=221$ and $\Sigma(3{\cdot}Suz{\cdot}2)=546$. Our calculations give, $\Delta^*(Co_1)\geq\Delta(Co_1)-2184-221>0$ and therefore, Co_1 is $(3A, 7A, 13A)$-generated.

Case $(3A, 7B, 11A)$. In this case, $\Delta(Co_1)=11088$ and $U_6(2){\cdot}S_3$ is the unique maximal subgroup with non-empty intersection with any conjugacy classes in this triple. But, $\Sigma(U_6(2){\cdot}S_3)=0$ and therefore, Co_1 is $(3A, 7B, 11A)$-generated.

Case $(3B, 7A, 11A)$. The only maximal subgroups of Co_1 that may contain $(3B, 7A, 11A)$-generated proper subgroups is isomorphic to $3{\cdot}Suz{\cdot}2$. Moreover, $\Delta(Co_1)=8055476$ and $\Sigma(3{\cdot}Suz{\cdot}2)=58190$. Therefore, $\Delta^*(Co_1)>0$ and Co_1 is $(3B, 7A, 11A)$-generated.

Case $(3B, 7A, 13A)$. Amongst the maximal subgroups of Co_1 with order divisible by $3\times 7\times 13$, the only maximal subgroups with non-empty intersection with any conjugacy classes in this triple are isomorphic to $3{\cdot}Suz{\cdot}2$ and $(A_4\times G_2(4)):2$. We can see that $\Delta(Co_1)=12818988$, $\Sigma(3{\cdot}Suz{\cdot}2)=65442$ and $\Sigma((A_4\times G_2(4)):2)=0$. Our calculations give, $\Delta^*(Co_1)\geq\Delta(Co_1)-4{\cdot}65442>0$ and therefore, Co_1 is $(3B, 7A, 13A)$-generated.

Case $(3B, 7B, 11A)$. From the list of maximal subgroups of Co_1 we observe that, up to isomorphisms, Co_2, $2^{11}:M_{24}$, Co_3 and $U_6(2){\cdot}S_3$ are the only maximal subgroups of Co_1 with non-empty intersection with all the conjugacy classes in this triple. We calculate $\Delta(Co_1)=120492768$, $\Sigma(Co_2)=6972416$, $\Sigma(2^{11}:M_{24})=304128$, $\Sigma(Co_3)=1161600$ and $\Sigma(U_6(2){\cdot}S_3)=6772604$. Also, a fixed element of order 11 is contained in 6 conjugates of Co_2, 3 conjugates of $2^{11}:M_{24}$, 12 conjugates of Co_3, and 2 conjugates of $U_6(2){\cdot}S_3$. Therefore, $\Delta^*(Co_1)\geq\Delta(Co_1)-41834496-912384-13939200-1385208>0$, which proves the group Co_1 is $(3B, 7B, 11A)$-generated.

Case $(3B, 7B, 23A)$. The only maximal subgroups of Co_1 that may contain $(3B, 7B, 23A)$-generated proper subgroups are isomorphic to Co_2, $2^{11}:M_{24}$ and Co_3. Moreover, $\Delta(Co_1)=60132120$, $\Sigma(Co_2)=4363008$, $\Sigma(2^{11}:M_{24})=172224$ and $\Sigma(Co_3)=361284$. Therefore, $\Delta^*(Co_1)>0$ and Co_1 is $(3B, 7B, 23A)$-generated.

Case $(3C, 7A, 11A)$. In this case, $\Delta(Co_1)=23530342$ and $3{\cdot}Suz{\cdot}2$ is the unique maximal subgroup with non-empty intersection with any conjugacy classes in this triple. But, $\Sigma(3{\cdot}Suz{\cdot}2)=96316$ and $\Delta^*(Co_1)\geq\Delta(Co_1)-96316>0$. Therefore, Co_1 is $(3C, 7A, 11A)$-generated.

Case $(3C, 7A, 13A)$. The only maximal subgroups of Co_1 that may contain $(3C, 7A,$

$13A$)-generated proper subgroups is isomorphic to $3\cdot Suz\cdot 2$. Moreover, $\Delta(Co_1)=25686284$ and $\Sigma(3\cdot Suz\cdot 2)=89180$. Therefore, $\Delta^*(Co_1)>0$ and Co_1 is $(3C, 7A, 13A)$-generated.

Case $(3C, 7B, 11A)$. From the list of maximal subgroups of Co_1 we observe that, up to isomorphisms, Co_2, Co_3 and $U_6(2)\cdot S_3$ are the only maximal subgroups of Co_1 with non-empty intersection with all the conjugacy classes in this triple. We calculate $\Delta(Co_1)=310154064$, $\Sigma(Co_2)=1063040$, $\Sigma(Co_3)=44000$ and $\Sigma(U_6(2)\cdot S_3)=28215$. Also, a fixed element of order 11 is contained in 6 conjugates of Co_2, 12 conjugates of Co_3, and 2 conjugates of $U_6(2)\cdot S_3$. Therefore, $\Delta^*(Co_1)\geq\Delta(Co_1)-6378240-528000-56450>0$, which proves the group Co_1 is $(3C, 7B, 11A)$-generated.

Case $(3C, 7B, 23A)$. The only maximal subgroups of Co_1 that may contain $(3C, 7B, 23A)$-generated proper subgroups are isomorphic to Co_2 and Co_3. Moreover, $\Delta(Co_1)=280671760$, $\Sigma(Co_2)=1560320$ and $\Sigma(Co_3)=33350$. Therefore, $\Delta^*(Co_1)>0$ and Co_1 is $(3C, 7B, 23A)$-generated.

Case $(3D, 7A, 11A)$. In this case, $\Delta(Co_1)=418285428$ and $3\cdot Suz\cdot 2$ is the unique maximal subgroup with non-empty intersection with any conjugacy classes in this triple. But, $\Sigma(3\cdot Suz\cdot 2)=1606968$ and therefore, Co_1 is $(3D, 7A, 11A)$-generated.

Case $(3D, 7A, 13A)$. Amongst the maximal subgroups of Co_1 with order divisible by $3\times 7\times 13$, the only maximal subgroups with non-empty intersection with any conjugacy classes in this triple are isomorphic to $3\cdot Suz\cdot 2$ and $(A_4\times G_2(4)):2$. We can see that $\Delta(Co_1)=493319905$, $\Sigma(3\cdot Suz\cdot 2)=1647685$ and $\Sigma((A_4\times G_2(4)):2)=66625$. Our calculations give, $\Delta^*(Co_1)\geq\Delta(Co_1)-4\cdot 1647685-66625>0$ and therefore, Co_1 is $(3D, 7A, 13A)$-generated.

Case $(3D, 7B, 11A)$. The only maximal subgroups of Co_1 that may contain $(3D, 7B, 11A)$-generated proper subgroups are isomorphic to $2^{11}:M_{24}$, Co_3 and $U_6(2)\cdot S_3$. Moreover, $\Delta(Co_1)=6281516340$, $\Sigma(2^{11}:M_{24})=1250304$, $\Sigma(Co_3)=4749228$ and $\Sigma(U_6(2)\cdot S_3)=0$. Therefore, $\Delta^*(Co_1)>0$ and Co_1 is $(3D, 7B, 11A)$-generated.

Case $(3D, 7B, 23A)$. The only maximal subgroups of Co_1 that may contain $(3D, 7B, 23A)$-generated proper subgroups are isomorphic to $2^{11}:M_{24}$ and Co_3. Moreover, $\Delta(Co_1)=6496166115$, $\Sigma(2^{11}:M_{24})=1577984$ and $\Sigma(Co_3)=2635317$. Therefore, $\Delta^*(Co_1)\geq\Delta(Co_1)-1577984-2635317>0$ and Co_1 is $(3D, 7B, 23A)$-generated.

Case $(3A, 11A, 13A)$. In this case, $\Delta(Co_1)=80028$ and $3\cdot Suz\cdot 2$ is the unique maximal subgroup with non-empty intersection with any conjugacy classes in this triple. But, $\Sigma(3\cdot Suz\cdot 2)=1560$ and therefore, Co_1 is $(3A, 11A, 13A)$-generated.

Case $(3B, 11A, 13A)$. In this case, $\Delta(Co_1)=1226256720$ and $3\cdot Suz\cdot 2$ is the unique maximal subgroup with non-empty intersection with any conjugacy classes in this triple. But, $\Sigma(3\cdot Suz\cdot 2)=391560$ and therefore, Co_1 is $(3B, 11A, 13A)$-generated.

Case $(3B, 11A, 23A)$. From the list of maximal subgroups of Co_1 we observe that, up to isomorphisms, Co_2, $2^{11}:M_{24}$ and Co_3 are the only maximal subgroups of Co_1 with non-empty intersection with all the conjugacy classes in this triple. We calculate $\Delta(Co_1)=1075384320$, $\Sigma(Co_2)=24727875$, $\Sigma(2^{11}:M_{24})=683008$ and $\Sigma(Co_3)=1438700$. Also, a fixed element of order 23 is contained in 1 conjugate of Co_2, 1 conjugate of $2^{11}:M_{24}$ and 1 conjugate of Co_3. Therefore, $\Delta^*(Co_1)\geq\Delta(Co_1)-24727875-683008-1438700>0$, which proves the group Co_1 is $(3B, 11A, 23A)$-generated.

Case $(3C, 11A, 13A)$. In this case, $\Delta(Co_1)=5344906840$ and $3\cdot Suz\cdot 2$ is the unique maximal subgroup with non-empty intersection with any conjugacy classes in this triple. But, $\Sigma(3\cdot Suz\cdot 2)=775840$ and therefore, Co_1 is $(3C, 11A, 13A)$-generated.

Case $(3C, 11A, 23A)$. Amongst the maximal subgroups of Co_1 with order divisible by $3\times 11\times 23$, the only maximal subgroups with non-empty intersection with any conjugacy classes in this triple are isomorphic to Co_2 and Co_3. We can see that $\Delta(Co_1)=5011865600$, $\Sigma(Co_2)=8242625$ and $\Sigma(Co_3)=133400$. Our calculations give, $\Delta^*(Co_1)\geq\Delta(Co_1)-8242625-133400>0$ and therefore, Co_1 is $(3C, 11A, 23A)$-generated.

Case $(3D, 11A, 13A)$. The only maximal subgroups of Co_1 that may contain $(3D, 11A, 13A)$-generated proper subgroups is isomorphic to $3\cdot Suz\cdot 2$. Moreover, $\Delta(Co_1)=115735162244$ and $\Sigma(3\cdot Suz\cdot 2)=12580295$. Therefore, $\Delta^*(Co_1)>0$ and Co_1 is $(3D, 11A, 13A)$-generated.

Case $(3D, 11A, 23A)$. From the list of maximal subgroups of Co_1 we observe that, up to isomorphisms, $2^{11}:M_{24}$ and Co_3 are the only maximal subgroups of Co_1 with non-empty intersection with all the conjugacy classes in this triple. We calculate $\Delta(Co_1)=115734030855$, $\Sigma(2^{11}:M_{24})=4898816$ and $\Sigma(Co_3)=9852556$. Also, a fixed element of order 23 is contained in 1 conjugate of $2^{11}:M_{24}$ and 1 conjugate of Co_3. Therefore, $\Delta^*(Co_1)\geq\Delta(Co_1)-4898816-9852556>0$, which proves the group Co_1 is $(3D, 11A, 23A)$-generated. This completes the proof. □

3. (p, q, r)-Generations of Co_1, $5\leq p<q<r$

We first consider $p=7$ then $q\in\{11, 13, 23\}$. We deal separately with each case as follows.

Lemma 3.1. The group Co_1 is $(5X, 7Y, 11A)$-generated if and only if $X\in\{A, B, C\}$ and $Y\in\{A, B\}$.

Proof. By [4] Co_1 has 3 classes of elements of order 5 denoted by $5A$, $5B$ and $5C$, and two classes of elements of order 7 denoted by $7A$ and $7B$ and one class of elements of order 11 denoted by $11A$. If we calculate $\Delta_{Co_1}(5X, 7Y, 11A)=\{(x, y)\mid x\in 5X, y\in 7Y, xy=z\in 11A\}$, we see that $\Delta_{Co_1}(5X, 7Y, 11A)\neq 0$ for all $X\in\{A, B, C\}$ and $Y\in\{A, B\}$. Therefore we consider the following cases:

Case $(5A, 7A, 11A)$. The only maximal subgroup of Co_1, up to isomorphism, that contains this triple is $3\cdot Suz\cdot 2$. We calculate $\Delta(Co_1)=153182458$ and $\Sigma(3\cdot Suz\cdot 2)=1025200$. Then Δ

$^*(Co_1)>0$, because $\Delta(Co_1)-\Sigma(3\cdot Suz\cdot 2)>0$. Therefore Co_1 is $(5A, 7A, 11A)$-generated.

Case $(5A, 7B, 11A)$. In this case there is no maximal subgroup of Co_1 containing this triple, so $\Delta^*(Co_1)=\Delta(Co_1)>0$ and we are done.

Case $(5B, 7A, 11A)$. The only maximal subgroup of Co_1 that contains this triple is $3\cdot Suz\cdot 2$. We calculate $\Delta(Co_1)=9727662472$ and $\Sigma(3\cdot Suz\cdot 2)=6386842$ and therefore $\Delta(Co_1)-\Sigma(3\cdot Suz\cdot 2)>0$. Hence $\Delta^*(Co_1)>0$.

Case $(5B, 7B, 11A)$. The maximal subgroups of Co_1 that contain $(5B, 7B, 11A)$ are Co_2, $2^{12}: M_{24}$, Co_3 and $U_6(2)\cdot S_3$. We calculate $\Delta(Co_1)=136990349496$, $\Sigma(Co_2)=1587536896$, $\Sigma(2^{11}: M_{24})=17290240$, $\Sigma(Co_3)=99237512$ and $\Sigma(U_6(2)\cdot S_3)=43797105$. Now by Table III $\Delta(Co_1)-6\Sigma(Co_2)-3\Sigma(2^{11}: M_{24})-12\Sigma(Co_3)-2\Sigma(U_6(2)\cdot S_3)>0$.

Case $(5C, 7A, 11A)$. There is no maximal subgroup containing this triple, so $\Delta^*(Co_1)=\Delta(Co_1)>0$.

Case $(5C, 7B, 11A)$. The maximal subgroups of Co_1 that contain this triple are Co_2 and Co_3. Now $\Delta(Co_1)=2304167677504$, $\Sigma(Co_2)=208023552$, and $\Sigma(Co_3)=12997424$. By Table III, $\Delta(Co_1)-6\Sigma(Co_2)-12\Sigma(Co_3)>0$. Therefore, $\Delta^*(Co_1)>0$. The proof is completed. □

Lemma 3.2. The group Co_1 is $(5X, 7Y, 13A)$-generated if and only if $X\in\{A, B, C\}$ and $Y\in\{A, B\}$.

Proof. By [4] Co_1 has three classes of elements of order 5 denoted by $5A$, $5B$ and $5C$, two classes of elements of order 7 denoted by $7A$ and $7B$ and one class of elements of order 13 denoted by $13A$.

For the following cases $(5A, 7B, 13A)$, $(5B, 7B, 13A)$, $(5C, 7A, 13A)$ and $(5C, 7B, 13A)$ there are no maximal subgroup of Co_1 containing these triples, so $\Delta^*(Co_1)=\Delta(Co_1)>0$. Hence, it is enough to investigate the following cases:

Case $(5A, 7A, 13A)$. The maximal subgroups of Co_1 containing $(5A, 7A, 13A)$ are $3\cdot Suz\cdot 2$ and $(A_4\times G_2(4)):2$. We calculated $\Delta(Co_1)=365158742$, $\Sigma(3\cdot Suz\cdot 2)=1528202$ and $\Sigma((A_4\times G_2(4)):2)=79798$. Then by Table III, $\Delta(Co_1)-4\Sigma(3\cdot Suz\cdot 2)-\Sigma((A_4\times G_2(4)):2)>0$ and hence $\Delta^*(G)>0$.

Case $(5B, 7A, 13A)$. The maximal subgroups of Co_1 that contain $(5B, 7A, 13A)$ are $3\cdot Suz\cdot 2$ and $(A_4\times G_2(4)):2$. We calculate $\Delta(Co_1)=11016985434$, $\Sigma(3\cdot Suz\cdot 2)=6484998$ and $\Sigma((A_4\times G_2(4)):2)=79794$ then $\Delta(Co_1)-4\Sigma(3\cdot Suz\cdot 2)-\Sigma((A_4\times G_2(4):2)>0$ so $\Delta^*(G)>0$. This completes the proof. □

Now we obtain all the $(5X, 7Y, 23Z)$-generation of the Conway group Co_1. Since $23A^{-1}=23B$ hence the group Co_1 is $(5X, 7Y, 23A)$-generated if and only if it is $(5X, 7Y, 23B)$-generated. Therefore it is enough to investigate $(5X, 7Y, 23A)$-generations of Co_1.

Lemma 3.3. The group Co_1 is $(5X, 7Y, 23A)$-generated if and only if $X \in \{A, B, C\}$ and $Y \in \{A, B\}$.

Proof: For the following cases $(5A, 7A, 23A)$, $(5A, 7B, 23A)$, $(5B, 7A, 23A)$ and $(5C, 7A, 23A)$ there are no maximal subgroups of Co_1 containing these triples, then $\Delta^*(Co_1)=\Delta(Co_1)>0$. Now we consider two cases:

Case $(5B, 7B, 23A)$. The maximal subgroups of Co_1 that contain $(5B, 7B, 23A)$ are Co_2, $2^{11}:M_{24}$ and Co_3. We calculate $\Delta(Co_1)=98206426674$, $\Sigma(Co_2)=4363008$, $\Sigma(2^{11}:M_{24})=12435456$ and $\Sigma(Co_3)=37913246$. Then $\Delta(G)-\Sigma(Co_2)-\Sigma(2^{11}:M_{24})-\Sigma(Co_3)>0$ and therefore $\Delta^*(Co_1)>0$.

Case $(5C, 7B, 23A)$. The maximal subgroups of Co_1 that contain $(5C, 7B, 23A)$ are Co_2 and Co_3. We calculate $\Delta(Co_1)=235703005848$, $\Sigma(Co_2)=251817984$, $\Sigma(Co_3)=7893692$. Then $\Delta(Co_1)-\Sigma(Co_2)-\Sigma(Co_3)>0$ hence $\Delta^*(Co_1)>0$ and the proof is completed. □

Lemma 3.4. The group Co_1 is $(5X, 11A, 13A)$-generated if and only if $X \in \{A, B, C\}$.

Proof. By [4] Co_1 has three classes of order 5 denoted by $5A$, $5B$ and $5C$. Now we have:

Case $(5A, 11A, 13A)$. The only maximal subgroup of Co_1 containing this triple is $3 \cdot Suz \cdot 2$. We calculate $\Delta(Co_1)=25559399840$, $\Sigma(3 \cdot Suz \cdot 2)=7572968$ then $\Delta(Co_1)-4\Sigma(3 \cdot Suz \cdot 2)>0$ and $\Delta^*(Co_1)>0$.

Case $(5B, 11A, 13A)$. The only maximal subgroup of Co_1 contain this triple is $3 \cdot Suz \cdot 2$. $\Delta(Co_1)=1858852505432$, $\Sigma(3 \cdot Suz \cdot 2)=45234722$ and $\Delta(Co_1)-\Sigma(3 \cdot Suz \cdot 2)>0$ so $\Delta^*(Co_1)>0$.

Case $(5C, 11A, 13A)$. Since there is no maximal subgroup containing this triple, so $\Delta^*(Co_1)=\Delta(Co_1)>0$ and the proof is completed. □

Lemma 3.5. The group Co_1 is $(5X, 11A, 23A)$-generated if and only if $X \in \{A, B, C\}$.

Proof. Case $(5A, 11A, 23A)$. There is no maximal subgroup of Co_1 which intersects this triple, hence $\Delta^*(Co_1)=\Delta(Co_1)>0$.

Case $(5B, 11A, 23A)$. The maximal subgroups of Co_1 that contain $(5B, 11A, 23A)$ are Co_1, $2^{11}:M_{24}$ and Co_3. We calculate $\Delta(Co_1)=1751187990762$, $\Sigma(Co_2)=6409940265$, $\Sigma(2^{11}:M_{24})=47857664$, $\Sigma(Co_3)=150404820$, $\Delta(Co_1)-\Sigma(Co_2)-\Sigma(2^{11}:M_{24})-\Sigma(Co_3)>0$ so $\Delta^*(Co_1)>0$.

Case $(5C, 11A, 23A)$. The maximal subgroups of Co_1 that contain $(5C, 11A, 23A)$ are Co_2 and Co_3. We calculate $\Delta(Co_1)=4199775606936$, $\Sigma(Co_2)=1281988053$ and $\Sigma(Co_3)=49908464$. $\Delta(Co_1)-\Sigma(Co_2)-\Sigma(Co_3)>0$ hence $\Delta^*(Co_1)>0$. The proof is completed. □

Lemma 3.6. The group Co_1 is $(5X, 13A, 23A)$-generated if and only if $X \in \{A, B, C\}$.

Proof. For each cases there are no maximal subgroups of Co_1 which intersect the triple and $\Delta^*(Co_1)=\Delta(Co_1)>0$ therefore the proof is completed. □

Lemma 3.7. The group Co_1 is $(7, 11, p)$-generated for $p\in\{13, 23\}$.

Proof. For the following cases $(7A, 11A, 23A)$ and $(7B, 11A, 13A)$ there are no maximal subgroup of Co_1 which intersect these triple so $\Delta^*(Co_1)=\Delta(Co_1)>0$. Our main proof will consider a number of cases,

Case $(7A, 11A, 13A)$. The unique maximal subgroup of Co_1 contain this triple is $3\cdot Suz\cdot 2$. We calculate $\Delta(Co_1)=3961393756476$ and $\Sigma(3\cdot Suz\cdot 2)=161738850$ then $\Delta(Co_1)-4\Sigma(3\cdot Suz\cdot 2)>0$ hence $\Delta^*(Co_1)>0$.

Case $(7B, 11A, 23A)$. The maximal subgroup of Co_1 that contain $(7B, 11A, 23A)$ are Co_2, $2^{11}:M_{24}$ and Co_3. We calculate $\Delta(Co_1)=53568552007800$, $\Sigma(Co_2)=68679139328$, $\Sigma(2^{11}:M_{24})=277348352$ and $\Sigma(Co_3)=536538388$. $\Delta(Co_1)-\Sigma(Co_2)-\Sigma(2^{11}:M_{24})-\Sigma(Co_3)>0$ so $\Delta^*(Co_1)>0$, proving the lemma. □

Lemma 3.8. The group Co_1 is $(7X, 13A, 23A)$- and $(11A, 13A, 23A)$-generated group for $X\in\{A, B\}$.

Proof. There are no maximal subgroups of Co_1 containing $(7A, 13A, 23A)$, $(7B, 13A, 23A)$ and $(11A, 13A, 23A)$. Thus $\Delta^*(Co_1)=\Delta(Co_1)>0$ and the proof is completed. □

We now summarize the results of two sections 2 and 3 in the following theorem:

Theorem. The Conway group Co_1 is (p, q, r)-generated for all $p, q, r\in\{3, 5, 7, 11, 13, 23\}$ with $p<q<r$.

Proof. The proof follows from the lemmas proved in section 2 and 3. □

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