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A note on Klein curve

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Abstract

It is known that the Jacobian variety of Klein curve is isomorphic to $E\times E\times E$, where E is an elliptic curve. In this paper we compute the defining equation of E in Weierstrass normal form and show that the Klein curve covers E doubly and triply.

1. Introduction

Let C be the curve in $\boldsymbol{P}^2$, the projective plane over $\boldsymbol{C}$, defined by the equation

$$XY^3+YZ^3+ZX^3=0.$$

C is the non-singular curve of genus three. C is called the Klein curve and has been studied for a long time.

C has a special property, that is, the number of automorphisms on C is equal to the Hurwitz upper bound (=168 for the curve of genus three). The explicit representation of the automorphisms on C is computed by A. Kuribayashi, K. Komiya and I. Kuribayashi in [2], [3] :

Proposition 1. *Put the real numbers*

$$b_1=\zeta+\zeta^6=2\cos\left(\frac{2\pi}{7}\right),\quad b_2=\zeta^2+\zeta^5=2\cos\left(\frac{4\pi}{7}\right),\quad b_3=\zeta^3+\zeta^4=2\cos\left(\frac{6\pi}{7}\right)$$

$$a_1=b_3+b_1,\quad a_2=b_1+b_2,\quad a_3=b_2+b_3,\quad \textit{where}\ \zeta=\exp\left(\frac{2\pi i}{7}\right).$$

We define the projective transformations in $\boldsymbol{P}^2$

$$\sigma:\begin{pmatrix}X\\Y\\Z\end{pmatrix}\longmapsto\begin{pmatrix}\zeta&0&0\\0&\zeta^3&0\\0&0&1\end{pmatrix}\begin{pmatrix}X\\Y\\Z\end{pmatrix}=\begin{pmatrix}\zeta X\\\zeta^3Y\\Z\end{pmatrix},$$

$$\tau:\begin{pmatrix}X\\Y\\Z\end{pmatrix}\longmapsto\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}X\\Y\\Z\end{pmatrix}=\begin{pmatrix}Y\\Z\\X\end{pmatrix},$$

$$\lambda_i:\begin{pmatrix}X\\Y\\Z\end{pmatrix}\longmapsto\begin{pmatrix}a_i&b_i&1\\b_i&1&a_i\\1&a_i&b_i\end{pmatrix}\begin{pmatrix}X\\Y\\Z\end{pmatrix}=\begin{pmatrix}a_iX+b_iY+Z\\b_iX+Y+a_iZ\\X+a_iY+b_iZ\end{pmatrix},(i=1,2,3).$$

Then the automorphism group of C is generated by these projective transformations. (*The order of* σ, τ *and* λ_i *are* 7, 3 *and* 2.)

Remark 1. *The points* $\begin{pmatrix}X\\Y\\Z\end{pmatrix}=\begin{pmatrix}\zeta^j a_i\\\zeta^{3j}b_i\\1\end{pmatrix}$,$(j=1,\cdots,7\ i=1,2,3)$ *are the Weierstrass points of C.*

Remark 2. *We can easily verify that* a_i *and* b_i *satisfy* $a_i+b_i+a_ib_i=0$, $a_i=1-b_i^2$, $(i=1,2,3)$. *From these equalities we find that* b_i *satisfies*

$$b_i^3+b_i^2-2b_i-1=0,\quad \text{for } i=1,2,3.$$

We use this equality to reduce the degree of b_i.

C is the compact Riemann surface and the orbit space $C/\langle\rho\rangle$ is also the compact Riemann surface, where $\langle\rho\rangle$ is a cyclic group generated by an automorphism ρ on C.

In this paper we show some interesting properties related to C.

Theorem 1. *Let* E *be the elliptic curve in* $\mathbf{C}^2$ *defined by the Weierstrass normal form*

$$y^2=4x^3-35x+49.$$

Then $C/\langle\tau\rangle$ *and* $C/\langle\lambda_i\rangle$, $(i=1,2,3)$ *are isomorphic to* E. *In other words,* C *can be regarded as a triple and double covering of* E.

Let (I,T) *be a normalized period matrix of* C, *where* I *is the* 3×3 *identity matrix and* T *is a theta matrix of* C. *We can choose* T *such that*

$$T=\begin{pmatrix}t&0&0\\0&t&0\\0&0&t\end{pmatrix},\quad \text{where } t=\frac{(-1+\sqrt{-7})}{2}.$$

Moreover E *is isomorphic to the elliptic curve* $\mathbf{C}/(\mathbf{Z}+\mathbf{Z}\cdot t)$.

2. The orbit space $C/\langle\tau\rangle$

Since we can find that the genus of the compact Riemann surface $C/\langle\tau\rangle$ is 1 by the Riemann-Hurwitz formula, we can describe the defining equation of $C/\langle\tau\rangle$ in the Weierstrass normal form. To describe the defining equation, we find the generators of $\langle\tau\rangle$-invariant ring and rewrite C by them.

We change variables in order to simplify the computation. Let ω be a primitive cubic root of unity. Let A be the representation matrix of τ and $\tilde{A}$ be the diagonalized matrix of A. Then

$$\tilde{A}=U^{-1}AU=\begin{pmatrix}1 & 0 & 0\\ 0 & \omega & 0\\ 0 & 0 & \omega^2\end{pmatrix},\quad \text{where } U=\begin{pmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{pmatrix}.$$

Now if we change variables

$$\begin{pmatrix}\tilde{X}\\ \tilde{Y}\\ \tilde{Z}\end{pmatrix}=U^{-1}\begin{pmatrix}X\\ Y\\ Z\end{pmatrix},$$

C and τ are changed into

$$\tilde{C}: \tilde{X}^4+3\tilde{X}^2\tilde{Y}\tilde{Z}-3\tilde{Y}^2\tilde{Z}^2-\left(\frac{1}{2}+\frac{3\sqrt{3}i}{2}\right)\tilde{X}\tilde{Y}^3-\left(\frac{1}{2}-\frac{3\sqrt{3}i}{2}\right)\tilde{X}\tilde{Z}^3=0$$

and

$$\tilde{\tau}:\begin{pmatrix}\tilde{X}\\ \tilde{Y}\\ \tilde{Z}\end{pmatrix}\longmapsto\tilde{A}\begin{pmatrix}\tilde{X}\\ \tilde{Y}\\ \tilde{Z}\end{pmatrix}=\begin{pmatrix}\tilde{X}\\ \omega\tilde{Y}\\ \omega^2\tilde{Z}\end{pmatrix}.$$

Using the affine coordinate $(u, v)=\left(\frac{\tilde{Y}}{\tilde{X}}, \frac{\tilde{Z}}{\tilde{X}}\right)$, $\tilde{C}$ and $\tilde{\tau}$ are expressed by

$$\tilde{C}_a: 1+3uv-3u^2v^2-\left(\frac{1}{2}+\frac{3\sqrt{3}i}{2}\right)u^3-\left(\frac{1}{2}-\frac{3\sqrt{3}i}{2}\right)v^3=0$$

and

$$\tilde{\tau}_a: (u, v)\longmapsto(\omega u, \omega^2 v).$$

Then we can easily find that the generators of $\langle \bar{\tau}_a \rangle$-invariant ring are uv, u^3, v^3. If we set $(x, y)=(uv, u^3)$, we can express $\tilde{C}_a/\langle \bar{\tau}_a \rangle$ by

$$(1) \qquad y+3xy-3x^2y-\left(\frac{1}{2}+\frac{3\sqrt{3}i}{2}\right)y^2-\left(\frac{1}{2}-\frac{3\sqrt{3}i}{2}\right)x^3=0.$$

Letting

$$\begin{cases} x=x_1 \\ y=\dfrac{y_1-3x_1^2+3x_1+1}{1+3\sqrt{3}i}, \end{cases}$$

we can transform (1) into

$$(2) \qquad y_1^2=9x_1^4-46x_1^3+3x_1^2+6x_1+1.$$

Successively letting

$$\begin{cases} x_1=\dfrac{36x_2+6\sqrt{6}y_2-98}{36x_2^2-12x_2-35} \\ y_1=\dfrac{1296x_2^4+432x_2^3-30240x_2^2+67284x_2+648\sqrt{6}x_2^2y_2-3528\sqrt{6}x_2y_2+1218\sqrt{6}y_2-21609}{(36x_2^2-12x_2-35)^2}, \end{cases}$$

we can transform (2) into

$$E: y_2^2=4x_2^3-35x_2+49.$$

Since these transformations are birational, $\tilde{C}_a/\langle \bar{\tau}_a \rangle$ is isomorphic to E. Therefore C can be regarded as a triple covering of E and j-invariant of E is $-3375=-3^3\cdot 5^3$.

3. The orbit space $C/\langle \lambda_i \rangle$

The genus of the compact Riemann surface $C/\langle \lambda_i \rangle$ is 1, and we follow the previous method to describe the defining equation. Since the discussions for $i=1, 2, 3$ are parallel, we remove the suffix i. Let A be the representation matrix of λ. Let $d=\sqrt{a^2+b^2+1}$ and $B=\frac{1}{d}A$. Let $\bar{B}$ be the diagonalized matrix of B. Then

$$\bar{B}=U^{-1}BU=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \text{where } U=\begin{pmatrix} -a & 0 & b-1 \\ 1 & 1 & -2a-b \\ 0 & -b & 2a+1 \end{pmatrix}.$$

Now if we change variables

$$\begin{pmatrix} \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \end{pmatrix} = U^{-1} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

C and λ are changed into

$$\begin{aligned}\tilde{C}:(-a\tilde{X}+(b-1)\tilde{Z})(\tilde{X}+\tilde{Y}+(-2a-b)\tilde{Z})^3+(\tilde{X}+\tilde{Y}+(-2a-b)\tilde{Z})\\ \times(-b\tilde{Y}+(2a+1)\tilde{Z})^3+(-b\tilde{Y}+(2a+1)\tilde{Z})(-a\tilde{X}+(b-1)\tilde{Z})^3=0\end{aligned}$$

and

$$\tilde{\lambda}: \begin{pmatrix} \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \end{pmatrix} \longmapsto \tilde{A} \begin{pmatrix} \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \end{pmatrix} = \begin{pmatrix} \tilde{X} \\ \tilde{Y} \\ -\tilde{Z} \end{pmatrix} = \begin{pmatrix} -\tilde{X} \\ -\tilde{Y} \\ \tilde{Z} \end{pmatrix}.$$

If we expand the expression of $\tilde{C}$ and apply the equalities in Remark 2, then

$$\begin{aligned}\tilde{C}:(b^2-1)\tilde{X}^4+(8b^2-4b-6)\tilde{X}^3\tilde{Y}+(3b^2-3)\tilde{X}^2\tilde{Y}^2+(90b^2-69b-54)\tilde{X}^2\tilde{Z}^2\\ +(2b^2-2b-2)\tilde{X}\tilde{Y}^3+(72b^2-66b-30)\tilde{X}\tilde{Y}\tilde{Z}^2+(b^2-2b-1)\tilde{Y}^4\\ +(-54b^2+36b+39)\tilde{Y}^2\tilde{Z}^2+(-204b^2+179b+94)\tilde{Z}^4=0.\end{aligned}$$

Using the affine coordinate $(u, v)=\left(\frac{\tilde{X}}{\tilde{Y}}, \frac{\tilde{Z}}{\tilde{Y}}\right)$, $\tilde{\tau}$ is expressed by

$$\tilde{\tau}_a:(u, v)\longmapsto(u, -v).$$

Then we can find that the generators of $\langle\tilde{\tau}_a\rangle$ -invariant ring are u, v^2. If we set $(x, y)=(u, v^2)$, we can express $\tilde{C}_a/\langle\tilde{\tau}_a\rangle$ by

$$\begin{aligned}(-204b^2+179b+94)y^2+\{(90b^2-69b-54)x^2+(72b^2-66b-30)x\\ +(-54b^2+36b+39)\}y+(b^2-1)x^4+(8b^2-4b-6)x^3+(3b^2-3)x^2\\ +(2b^2-2b-2)x+(b^2-2b-1)=0.\end{aligned} \tag{3}$$

In the following computation we often use the reduction formula for b in Remark 2.

Letting

$$\begin{cases} x=x_1 \\ y=y_1-\dfrac{(90b^2-69b-54)x_1^2+(72b^2-66b-30)x_1+(-54b^2+36b+39)}{2(-204b^2+179b+94)}, \end{cases}$$

we can transform (3) into

$$(766276b^2-616112b-423248)y_1^2+(-33733b^2+27020b+18760)x_1^4$$
$$+(-80164b^2+64036b+44800)x_1^3+(4200b^2-3486b-2184)x_1^2$$
$$+(24528b^2-19516b-13804)x_1+(-14756b^2+11928b+8071)=0. \tag{4}$$

Dividing (4) by the coefficient of y_1^2, we obtain

$$y_1^2+\left(-150b^2-337b-\frac{481}{4}\right)x_1^4+(-847b^2-1903b-679)x_1^3$$
$$+\left(-\frac{3195}{2}b^2-\frac{7179}{2}b-1281\right)x_1^2+(-445b^2-1000b-357)x_1$$
$$+\left(\frac{229}{2}b^2+\frac{1029}{4}b+\frac{367}{4}\right)=0. \tag{5}$$

Letting

$$\begin{cases} x_1=\dfrac{1}{x_2}+(1-b^2) \\ y_1=\dfrac{y_2}{x_2^2}, \end{cases}$$

we can transform (5) into

$$y_2^2+(648b^2+1456b+520)x_2^3+(-465b^2-1044b-372)x_2^2$$
$$+(-514b^2-1155b-412)x_2+\left(-150b^2-337b-\frac{481}{4}\right)=0. \tag{6}$$

Finally letting

$$\begin{cases} x_2=\dfrac{1}{8}\left(\dfrac{-x_3+155b^2+348b+124}{81b^2+182b+65}\right) \\ y_2=\dfrac{1}{16}\left(\dfrac{y_3}{81b^2+182b+65}\right), \end{cases}$$

we obtain

$$y_3^2=4x_3^3-(8358180b^2+18780660b+6702740)x_3$$
$$+(12987852690b^2+29183440104b+10415449184). \tag{7}$$

Since these transformations are birational, $\bar{C}_a/\langle\tilde{\tau}_a\rangle$ is isomorphic to (7) and j-invariant of (7) is $-3^3\cdot5^3$. This means that the elliptic curve (7) is isomorphic to E; hence, C can be regarded as a double covering of E.

4. The theta matrix of C

We can transform the affine Klein curve

$$xy^3+y+x^3=0$$

by the birational transformation

$$\begin{cases} x=-s \\ y=-\dfrac{s^3}{(t-1)} \end{cases}$$

into

$$C_a : s^7=t(t-1)^2.$$

We compute a normalized period matrix of C_a. The differential 1-forms

$$\omega_1=\frac{(t-1)}{s^6}dt, \quad \omega_2=\frac{(t-1)}{s^5}dt, \quad \omega_3=\frac{1}{s^3}dt$$

form a basis of the vector space of holomorphic 1-forms. (cf. [1], pp. 265-272)

We can regard C_a as a seven sheeted covering of $\hat{C}=C\cup\{\infty\}$ and let $X_1, \cdots, X_7$ be these $\hat{C}$'s. Cut X_1 along the real axis from 0 to 1 and from 1 to ∞. Then s does not change to another branch by analytic continuation along all paths lying completely interior to X_1. Let s_1 be a branch of s on X_1. We cut $X_2, \cdots, X_7$ like X_1 and assign another branch $s_i=\zeta^{i-1}s_1$ to X_i $(i=2, \cdots, 7)$.

The union of two paths from 0 to 1 on the real axis of X_i and from 1 to 0 on the real axis of X_{i+1} is a closed path on C_a for $i=1, \cdots, 6$. We denote it by a_i.

The integrals of ω_i along a_j are as follows.

$$\begin{aligned} \int_{a_j}\omega_1 &= \int_0^1 \frac{(t-1)}{(\zeta^{j-1}s_1)^6}dt+\int_1^0\frac{(t-1)}{(\zeta^j s_1)^6}dt \\ &=(\zeta^{j-1}-\zeta^j)\int_0^1\frac{(t-1)}{s_1^6}dt \\ &=\zeta^{j-1}(1-\zeta)\int_0^1\frac{(t-1)}{s_1^6}dt \end{aligned}$$

$$\int_{a_j}\omega_2=\int_0^1\frac{(t-1)}{(\zeta^{j-1}s_1)^5}dt+\int_1^0\frac{(t-1)}{(\zeta^j s_1)^5}dt$$

$$=(\zeta^{2j-2}-\zeta^{2j})\int_0^1\frac{(t-1)}{s_1^5}dt$$

$$=\zeta^{2j-2}(1-\zeta^2)\int_0^1\frac{(t-1)}{s_1^5}dt$$

$$\int_{a_j}\omega_3=\int_0^1\frac{1}{(\zeta^{j-1}s_1)^3}dt+\int_1^0\frac{1}{(\zeta^j s_1)^3}dt$$

$$=(\zeta^{4j-4}-\zeta^{4j})\int_0^1\frac{1}{s_1^3}dt$$

$$=\zeta^{4j-4}(1-\zeta^4)\int_0^1\frac{1}{s_1^3}dt$$

If we change the basis of the vector space of holomorphic 1-forms by

$$\tilde{\omega}_1=\frac{1}{(1-\zeta)\int_0^1\frac{(t-1)}{s_1^6}dt}\omega_1,\quad \tilde{\omega}_2=\frac{1}{(1-\zeta^2)\int_0^1\frac{(t-1)}{s_1^5}dt}\omega_2,\quad \tilde{\omega}_3=\frac{1}{(1-\zeta^4)\int_0^1\frac{1}{s_1^3}dt}\omega_3,$$

the period lattice of C_a relative to $\{\tilde{\omega}_i\}$ and $\{\alpha_j\}$ is

$$P=\begin{pmatrix}1&\zeta&\zeta^2&\zeta^3&\zeta^4&\zeta^5\\1&\zeta^2&\zeta^4&\zeta^6&\zeta&\zeta^3\\1&\zeta^4&\zeta&\zeta^5&\zeta^2&\zeta^6\end{pmatrix}.$$

Setting

$$N=\frac{1}{(\zeta^2-1)(\zeta-1)}\begin{pmatrix}\zeta^4+\zeta^3+1&-\zeta^2&-\zeta^4-\zeta\\\zeta^4&\zeta^5+\zeta&-\zeta^5-\zeta^4-\zeta\\\zeta^5+\zeta^4+\zeta&-\zeta^4&-\zeta^5-\zeta\end{pmatrix}$$

and

$$M=\begin{bmatrix}1&0&0&0&0&-1\\0&1&0&1&0&-1\\0&0&1&1&1&-1\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&0&0&1&-1\end{bmatrix},$$

we can equivalently transform P into the normalized period matrix

$$\tilde{P}=NPM=\begin{pmatrix} 1 & 0 & 0 & \zeta^4+\zeta^2+\zeta & 0 & 0 \\ 0 & 1 & 0 & 0 & \zeta^4+\zeta^2+\zeta & 0 \\ 0 & 0 & 1 & 0 & 0 & \zeta^4+\zeta^2+\zeta \end{pmatrix}.$$

Let $t=\zeta^4+\zeta^2+\zeta$, and we find t satisfies $t^2+t+2=0$. Now, if we put $t=\frac{-1+\sqrt{-7}}{2}$, the theta matrix of C_a has positive definite imaginary part.

Then the Jacobian variety of C_a is $J(C_a)=\boldsymbol{C}^3/L$, where L is the lattice over $\boldsymbol{Z}$ generated by the column vectors of $\tilde{P}$, and is obviously isomorphic to

$$\boldsymbol{C}/(\boldsymbol{Z}+\boldsymbol{Z}\cdot t)\times \boldsymbol{C}/(\boldsymbol{Z}+\boldsymbol{Z}\cdot t)\times \boldsymbol{C}/(\boldsymbol{Z}+\boldsymbol{Z}\cdot t).$$

It shows that E is isomorphic to $\boldsymbol{C}/(\boldsymbol{Z}+\boldsymbol{Z}\cdot t)$.

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