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Classification of Pfaffian systems of Fuchs type of a particular class

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1. Introduction

1.1. Summary.

In their 1972 paper [4], R. Gerard and A. H. M. Levelt gave a classification of completely integrable Pfaffian systems of Fuchs type of particular class on two dimensional complex projective space. In particular they characterized the Appell's system (F_1) (see **section 2.1** for the definition) among such systems. However, their proof is not sufficient because their Proposition 2 [4, p.210] is not correct. (Namely, their proof of $b_1 \neq 0$ case of case 2^* is false.) This caused their omission of the $\alpha=0$ case of the system (F_1) in their classification (see Remark 2.12).

The purpose of this paper is twofold. First, correcting Gérard-Levelt's proof, we give a complete classification under their condition (R) (Theorem 2.10), which characterizes the Appell's system (F_1). Our conclusion is different from Gérard-Levelt's one in that we include the case of $\alpha=0$ of the system (F_1), whereas this case is omitted from their classification. Secondly we give the classification in which the condition (R) is not satisfied. The proof is a rather tedious case-by-case calculation. The systems in this case are almost decomposable (Definition 2.3) or can be reduced to the systems (G_1), (G_2) which are obtained from the Gaussian hypergeometric system (Theorem 3.1, Theorem 3.2).

1.2. Definitions.

Let $P_n(C)$ be a complex projective space of dimension n, and $\pi:(C^{n+1})^*:=C^{n+1}(x_0,\cdots,x_n)-\{0\}\rightarrow P_n(C)$ be the canonical projection. We denote an algebraic hypersurface in $P_n(C)$ by $\mathfrak{A}:=\cup_{i=1}^{q}\mathfrak{A}_i$, where $\mathfrak{A}_i$ $(i=1,\cdots,q)$ is an irreducible component with the defining equation

$$P_i(x_0,\cdots,x_n)=0.$$

Let $\Omega^{p\times p}(P_n(C),\mathfrak{A})$ denote the set of differential forms ω of degree 1 with matricial values, holomorphic on $P_n(C)-\mathfrak{A}$ such that

$$\pi^*\omega=\sum_{i=1}^{q}A_i\frac{dP_i}{P_i},$$

where A_i $(1\leq i\leq q)$ are $p\times p$ complex matrices with the relation

$$\sum_{i=1}^{q}(degree P_i)A_i=0.$$

The class $F^{p\times p}(P_n(C),\mathfrak{A})$ of Pfaffian systems of Fuchs type (or Fuchsian systems, for short) with $\mathfrak{A}$ for the singular set is the set of Pfaffian systems of the form

$$df=\omega f,\quad \omega\in\Omega^{p\times p}(P_n(C),\mathfrak{A}). \tag{1.1}$$

Remark 1.1. The system (1.1) is completely integrable if and only if $d\omega=\omega\wedge\omega$, that is $\omega\wedge\omega=0$.

Solutions of a completely integrable Pfaffian system of Fuchs type in $F^{p\times p}(P_n(C),\mathfrak{A})$ generate a p-dimensional vector subspace of the vector space $H^{p\times 1}(R(P_n(C)-\mathfrak{A}))$, C^p-valued functions holomorphic on the universal covering space $R(P_n(C)-\mathfrak{A})$ of $P_n(C)-\mathfrak{A}$.

Definition 1.2. A holomorphic function on $R(P_n(C)-\mathfrak{A})$ with values in C^p is called *elementary* if its components are polynomials with constant coefficients of

(1) $(\log P_i)_{1\leq i\leq q}$,

(2) $\prod_{i=1}^{q}P_i^{a_i}$ $(a_i\in C)$.

Definition 1.3. *A* p-dimensional vector subspace E of $H^{p\times 1}(R(P_n(C)-\mathfrak{A}))$ is called *elementary* if E is generated by elementary functions.

Definition 1.4. *A* holomorphic matrix H on $R(P_n(C)-\mathfrak{A})$ is called *elementary* if the vector space over C generated by its columns is elementary.

Definition 1.5. *A* divisor $\mathfrak{A}'$ of $P_n(C)$ is called *simpler* than $\mathfrak{A}$ if

(1) $\mathfrak{A}'\neq\mathfrak{A}$;

(2) $\mathfrak{A}'$ is a union if certain number of irreducible components of $\mathfrak{A}$.

Definition 1.6. For a Pfaffian system of Fuchs type (1.1) $df=\omega f$ we say the *singular set* $\mathfrak{A}$ *is redundant* if by a change of solution of type $f=Hg$ where H is an invertible elementary matrix on $R(P_n(C)-\mathfrak{A})$, (1.1) is transformed to a system whose singular set $\mathfrak{A}'$ is simpler than $\mathfrak{A}$. In particular, if $\mathfrak{A}\backslash\mathfrak{A}'=\cup_{l=1}^{t}\mathfrak{A}_{i_l}$, then we say the singular set is redundant in $\mathfrak{A}_{i_1},\cdots,\mathfrak{A}_{i_t}$ or in $A_{i_1},\cdots,A_{i_t}$. We also say $\mathfrak{A}_{i_1},\cdots,\mathfrak{A}_{i_t}$ (or $A_{i_1},\cdots,A_{i_t}$) are redundant.

2. The classification of the case that the condition (R) is satisfied

2.1. The problem.

Appell's hypergeometric function $F_1(\alpha, \beta, \beta', \gamma, x, y)$ is defined by

$$F_1(\alpha, \beta, \beta', \gamma, x, y)=\sum_{m,n=0}^{\infty}\frac{(\alpha, m+n)(\beta, m)(\beta', n)}{(\gamma, m+n)(1, m)(1, n)}x^m y^n,$$

$$(\alpha, n):=\begin{cases}\alpha(\alpha+1)\cdots(\alpha+n-1), & (n\geq 1)\\ 1, \quad (n=0)\end{cases}$$

where $\alpha, \beta, \beta', \gamma$ are complex parameters and $\gamma\neq 0, -1, -2, \cdots$(*cf.*[1], [2]). F_1 satisfies the following system of partial differential equations:

$$x(1-x)(x-y)\frac{\partial^2 z}{\partial x^2}+[\gamma(x-y)-(\alpha+\beta+1)x^2+(\alpha+\beta-\beta'+1)xy+\beta' y]\frac{\partial z}{\partial x}$$
$$+\beta y(1-y)\frac{\partial z}{\partial y}-\alpha\beta(x-y)z=0,$$

$$y(1-y)(y-x)\frac{\partial^2 z}{\partial y^2}+[\gamma(y-x)-(\alpha+\beta'+1)y^2+(\alpha+\beta'-\beta+1)xy+\beta x]\frac{\partial z}{\partial y}$$
$$-\beta' x(1-x)\frac{\partial z}{\partial x}-\alpha\beta'(y-x)z=0,$$

$$(x-y)\frac{\partial^2 z}{\partial x\partial y}-\beta'\frac{\partial z}{\partial x}+\beta\frac{\partial z}{\partial y}=0.$$

The application

$$z\rightsquigarrow\tilde{z}=\begin{pmatrix} z\\ x\dfrac{\partial z}{\partial x}\\ y\dfrac{\partial z}{\partial y}\end{pmatrix}$$

transforms this system to a completely integrable Fuchsian system $df=\omega f$ where

$$\pi^*\omega=\sum_{i=1}^{3}A_i\frac{dx_i}{x_i}+\sum_{i=1}^{3}B_i\frac{du_i}{u_i},$$

(x_1, x_2, x_3) being a homogeneous coordinate on $P_2(C)$ with $u_i=x_j-x_k$ for any even permutation (i, j, k) of $\{1, 2, 3\}$, and

$$A_1=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1-\gamma+\beta' & 0 \\ 0 & -\beta' & 0 \end{pmatrix}, \quad B_1=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\alpha\beta' & -\beta' & \gamma-\alpha-\beta'-1 \end{pmatrix},$$
$$A_2=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -\beta \\ 0 & 0 & 1-\gamma+\beta \end{pmatrix}, \quad B_2=\begin{pmatrix} 0 & 0 & 0 \\ -\alpha\beta & \gamma-\alpha-\beta-1 & -\beta \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.1}$$
$$A_3=\begin{pmatrix} 0 & -1 & -1 \\ \alpha\beta & \alpha+\beta & \beta \\ \alpha\beta' & \beta' & \alpha+\beta' \end{pmatrix}, \quad B_3=\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\beta' & \beta \\ 0 & \beta' & -\beta \end{pmatrix}.$$

Denote by

$$\mathfrak{A}=\bigcup_{i=1}^{3}(\mathfrak{A}_i\cup\mathfrak{B}_i)$$

an algebraic subset of $P_2(C)$ defined by

$$\mathfrak{A}_i : x_i=0, \quad \mathfrak{B}_i : u_i=0 \; (i=1, 2, 3).$$

We want to solve the classification problem of 3-dimensional completely integrable Fuchsian system with the singular set $\mathfrak{A}$. Let us consider the following system;

$$df=\omega f=\left(\sum_{i=1}^{3}A_i\frac{dx_i}{x_i}+\sum_{i=1}^{3}B_i\frac{du_i}{u_i}\right)f, \quad \sum_{i=1}^{3}(A_i+B_i)=0,$$

where A_i, B_i $(i=1, 2, 3)$ are complex matrices of degree 3. Put

$$B=B_1+B_2+B_3, \quad C_i=A_j+A_h+B_i,$$

where $(i, j\ h)$ is a permutation of $\{1, 2, 3\}$. The condition of completely integrability is

$$(I)\begin{cases} (I_1) & [A_i, B_i]=0 \; (i=1, 2, 3), \\ (I_2) & [C_i, A_j]=[C_i, A_h]=[C_i, B_i]=0 \\ & \text{for any permutation } (i, j, h) \text{ of } \{1, 2, 3\}, \\ (I_3) & [B, B_i]=0 \; (i=1, 2, 3), \end{cases}$$

$$(S) \quad \sum_{i=1}^{3}(A_i+B_i)=0.$$

Our classification problem is equivalent to the classification of ordered 6-tuples $(A_1, A_2, A_3, B_1, B_2, B_3)$ satisfying (I) and (S) above modulo simultaneous similitude. In other words, this problem is equivalent to the classification of 6-tuples $(A_1, A_2, A_3, B_1, B_2, B_3)$ of linear transformations of 3-dimensional vector space V over C.

2.2. The symmetry of problem.

We introduce some operations on 6-tuples. For any permutation σ of $\{1, 2, 3\}$, we set

$$\sigma(A_1, A_2, A_3, B_1, B_2, B_3) := (A_{\sigma(1)}, A_{\sigma(2)}, A_{\sigma(3)}, B_{\sigma(1)}, B_{\sigma(2)}, B_{\sigma(3)}).$$

Define transformations $\tau_{12}, \tau_{23}, \tau_{31}, \nu$ by

$$\tau_{12}(A_1, A_2, A_3, B_1, B_2, B_3) := (B_1, B_2, A_3, A_1, A_2, B_3),$$
$$\tau_{23}(A_1, A_2, A_3, B_1, B_2, B_3) := (A_1, B_2, B_3, B_1, A_2, A_3),$$
$$\tau_{31}(A_1, A_2, A_3, B_1, B_2, B_3) := (B_1, A_2, B_3, A_1, B_2, A_3),$$
$$\nu(A_1, A_2, A_3, B_1, B_2, B_3) := (C_1, C_2, C_3, B_1, B_2, B_3).$$

The transformations $\sigma, \tau_{12}, \tau_{23}, \tau_{31}, \nu$ generate the symmetric group S_5 of degree 5. We denote this group by G. The group G corresponds to transformations of our system induced by holomorphic automorphisms on $P_2(C)-\mathfrak{A}$.

Proposition 2.1. *If a* 6*-tuple* $H := (A_1, A_2, A_3, B_1, B_2, B_3)$ *satisfies the relation* (I) *and* (S), *then the transformed* 6*-tuple by an element of* G *also satisfies* (I) *and* (S).

Definition 2.2. *A* 6-tuple $H=(A_1, A_2, A_3, B_1, B_2, B_3)$ is called *decomposable* if V is the direct sum of non trivial two invariant subspaces V_1, V_2 under the linear transformations $A_1, A_2, A_3, B_1, B_2, B_3$.

Definition 2.3. *A* 6-tuple H is called *almost decomposable* if there exist linear transformations $N_1, N_2, N_3, M_1, M_2, M_3$ of V satisfying

(1) For any i, N_i, M_i are nilpotent, and $\sum_{i=1}^{3}(N_i+M_i)=0$,

(2) N_i, M_j are commutative and commute with every element of H,

(3) V is the direct sum of non trivial invariant subspaces under the endomorphisms A_i-N_i, B_j-M_j $(i, j=1, 2, 3)$.

Definition 2.4. *A* 6-tuple H is called *elementary* if:

(1) it is almost decomposable;

(2) V is the direct sum of 1-dimensional invariant subspaces under the endomorphisms A_i-N_i, B_j-M_j $(i, j=1, 2, 3)$.

Remark 2.5. *A* 6-tuple H is decomposable (*resp.* almost decomposable, elementary) if and only if the transformed system by an element of G is decomposable (*resp.* almost decomposable, elementary).

Remark 2.6. The 6-tuple $(A_1-N_1, A_2-N_2, A_3-N_3, B_1-M_1, B_2-M_2, B_3-M_3)$ in Definition 2.3 satisfies (I) and (S).

Remark 2.7. *A* 6-tuple H is elementary if and only if every element of H commutes with each other.

Consider a 6-tuple $H=(A_1, A_2, A_3, B_1, B_2, B_3)$ of a Pfaffian system of Fuchs type

$$df=\left(\sum_{i=1}^{3}A_i\frac{dx_i}{x_i}+\sum_{i=1}^{3}B_i\frac{du_i}{u_i}\right)f. \tag{2.2}$$

Then:

(1) the 6-tuple H is decomposable if and only if the system (2.2) is decomposable.

(2) the 6-tuple H is almost decomposable if and only if $f=(\prod_{i=1}^{3}x_i^{N_i})\times(\prod_{i=1}^{3}u_i^{M_i})g$ transforms the system (2.2) to a decomposable system.

(3) If the 6-tuple H is elementary, then the solutions of the system (2.2) are elementary.

Let V^* be the dual vector space of V and T^* the dual transformation of T. Then

(1) a 6-tuple $H=(A_1, A_2, A_3, B_1, B_2, B_3)$ satisfies (I) and (S) if and only if the dual transformation $H^*=(A_1^*, A_2^*, A_3^*, B_1^*, B_2^*, B_3^*)$ of H satisfies (I) and (S).

(2) H is decomposable (*resp.* almost decomposable, elementary) if and only if H^* is decomposable (*resp.* almost decomposable, elementary).

From now on we assume $\dim V=3$ throughout. Under this hypothesis, we call an endomorphism X of V satisfies (R) if

(R) There exists $\xi\in C$ such that $rank(X-\xi I)=1$.

For any endomorphism X satisfying (R), we denote the kernel and image of $X-\xi I$ by N_x, I_x respectively. Use will be made of the following lemmas [4, p.202]:

Lemma 2.8. *If an endomorphism X satisfies (R), $N_x=\{(x_1, x_2, x_3)\,|\,p_1x_1+p_2x_2+p_3x_3=0\}$, $(q_1, q_2, q_3)\neq 0\in I_x$, then there exists a base of V such that X is represented by*

$$\lambda\begin{pmatrix} p_1q_1+\xi & p_2q_1 & p_3q_1 \\ p_1q_2 & p_2q_2+\xi & p_3q_2 \\ p_1q_3 & p_2q_3 & p_3q_3+\xi \end{pmatrix} \quad (\lambda\in \boldsymbol{C}).$$

Lemma 2.9. *If endomorphisms X, Y of V satisfying (R) and $[X, Y]=0$, then :*

(1) $X(N_Y)\subset N_Y$, $Y(N_X)\subset N_X$.

(2) $N_X\neq N_Y \Longrightarrow I_X\subset N_Y$, $I_Y\subset N_X$.

We denote by $\boldsymbol{D}$ the set of transformations A_i, B_i, B, C_i $(i=1, 2, 3)$ satisfying (I) and (S).

2.3. The case that any element of D satisfies (R).

The following theorem is the corrected version of the main theorem of Gérard-Levelt [4, Theorem 1]. We include the case of (2.6), (2.7), which corresponds to the $\alpha=0$ case of the system (F_1). As stated in **Introduction**, it will suffice to correct [4, Proposition 2], i.e., to prove Proposition 2.17 below.

Theorem 2.10. If a 6-*tuple* $H=(A_1, A_2, A_3, B_1, B_2, B_3)$ *is not elementary and satisfies* (I), (S) *and* (R), *then* H *is decomposable, or modulo transformations of* G, H *is given by*

$$\left[\begin{array}{ll} A_1=\begin{pmatrix} \alpha_1' & 0 & 0 \\ -b_2 & \alpha_1 & 0 \\ -b_3 & 0 & \alpha_1 \end{pmatrix}, & B_1=\begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & b_2+\beta_1' & b_2 \\ 0 & b_3 & b_3+\beta_1' \end{pmatrix}, \\ A_2=\begin{pmatrix} \alpha_2 & -b_1 & 0 \\ 0 & \alpha_2' & 0 \\ 0 & -b_3 & \alpha_2 \end{pmatrix}, & B_2=\begin{pmatrix} b_1+\beta_2' & 0 & b_1 \\ 0 & \beta_2 & 0 \\ b_3 & 0 & b_3+\beta_2' \end{pmatrix}, \\ A_3=\begin{pmatrix} \alpha_3 & 0 & -b_1 \\ 0 & \alpha_3 & -b_2 \\ 0 & 0 & \alpha_3' \end{pmatrix}, & B_3=\begin{pmatrix} b_1+\beta_3' & b_1 & 0 \\ b_2 & b_2+\beta_3' & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}, \end{array}\right. \tag{2.3}$$

where

$$\begin{cases} b_1=\frac{1}{2}(-\beta_1+\beta_1'+\beta_2-\beta_2'+\beta_3-\beta_3'), \\ b_2=\frac{1}{2}(\beta_1-\beta_1'-\beta_2+\beta_2'+\beta_3-\beta_3'), \\ b_3=\frac{1}{2}(\beta_1-\beta_1'+\beta_2-\beta_2'-\beta_3+\beta_3'), \end{cases} \tag{2.4}$$

$$\begin{cases} \alpha_1'+\alpha_2+\alpha_3+\beta_1'+\beta_2+\beta_3=0, \\ \alpha_1+\alpha_2'+\alpha_3+\beta_1+\beta_2'+\beta_3=0, \\ \alpha_1+\alpha_2+\alpha_3'+\beta_1+\beta_2+\beta_3'=0, \end{cases} \tag{2.5}$$

or

$$\begin{cases} A_1=\begin{pmatrix} \alpha_1' & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_1 \end{pmatrix}, & B_1=\begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_1-b_3 & b_2 \\ 0 & b_3 & \beta_1-b_2 \end{pmatrix}, \\ A_2=\begin{pmatrix} \alpha_2 & -b_1 & 0 \\ 0 & \alpha_2' & 0 \\ 0 & -b_3 & \alpha_2 \end{pmatrix}, & B_2=\begin{pmatrix} \beta_2' & 0 & b_1 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_2 \end{pmatrix}, \\ A_3=\begin{pmatrix} \alpha_3 & 0 & -b_1 \\ 0 & \alpha_3 & -b_2 \\ 0 & 0 & \alpha_3' \end{pmatrix}, & B_3=\begin{pmatrix} \beta_3' & b_1 & 0 \\ 0 & \beta_3 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}, \end{cases} \tag{2.6}$$

where the constants $\alpha_i, \beta_i, \alpha_i', \beta_i'$ $(i=1,2,3)$ *satisfy the relation* (2.5) *and*

$$\begin{cases} \beta_1-\beta_1'=\beta_2-\beta_2'+\beta_3-\beta_3', \\ b_2=\beta_3-\beta_3', \\ b_3=\beta_2-\beta_2'. \end{cases} \tag{2.7}$$

Conversely, if the constants $\alpha_i, \beta_i, \alpha_i', \beta_i'$ $(i=1,2,3)$ *satisfy the relation* (2.5), *then the 6-tuple defined by* (2.3), (2.4) *or* (2.6), (2.7) *satisfy* (I) *and* (S).

Remark 2.11. For $\alpha, \beta, \beta', \gamma$ $(\alpha\neq 0)$, put

$$\begin{cases} \alpha_1=0,\ \alpha_1'=1-\gamma+\beta',\ \beta_1=0,\ \beta_1'=\gamma-\alpha-\beta'-1, \\ \alpha_2=0,\ \alpha_2'=1-\gamma+\beta,\ \beta_2=0,\ \beta_2'=\gamma-\alpha-\beta-1, \\ \alpha_3=\alpha,\ \alpha_3'=\beta+\beta',\quad \beta_3=0,\ \beta_3'=-(\beta+\beta'), \end{cases} \tag{2.8}$$

then one can easily check that these satisfy (2.5) in Theorem 2.10. Let

$$T=\begin{pmatrix} 1 & 1 & 1 \\ -\alpha & 0 & 0 \\ 0 & -\alpha & 0 \end{pmatrix},$$

A_i, B_i $(i=1,2,3)$ be defined by (2.3), (2.4) in Theorem 2.10 with (2.8). Then TA_iT^{-1}, TB_iT^{-1} $(i=1,2,3)$ are nothing but (2.1) of **section 2.1.** On the other hand, the 6-tuple of Theorem 2.10 is no more general than the one associated with the system (F_1): one can reduce the general case to the (F_1) case as follows (*cf.* [4, p.204]). substituting

$$f=x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3-\alpha}u_1^{\beta_1}u_2^{\beta_2}u_3^{\beta_3}g \quad (\alpha=\alpha_1+\alpha_2+\alpha_3+\beta_1+\beta_2+\beta_3) \tag{2.9}$$

in

$$df=\left(\sum_{i=1}^{3}A_i\frac{dx_i}{x_i}+\sum_{i=1}^{3}B_i\frac{du_i}{u_i}\right)f,$$

we have

$$df=\left(\sum_{i=1}^{3}A_i\frac{dx_i}{x_i}+\sum_{i=1}^{3}B_i\frac{du_i}{u_i}\right)x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3-\alpha}u_1^{\beta_1}u_2^{\beta_2}u_3^{\beta_3}g.$$

On the other hand, differentiating (2.9) gives

$$\begin{aligned} df=&\left(\frac{\alpha_1}{x_1}dx_1+\frac{\alpha_2}{x_2}dx_2+\frac{\alpha_3-\alpha}{x_3}dx_3+\frac{\beta_1}{u_1}du_1+\frac{\beta_2}{u_2}du_2+\frac{\beta_3}{u_3}du_3\right)\\ &\times x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3-\alpha}u_1^{\beta_1}u_2^{\beta_2}u_3^{\beta_3}g+x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3-\alpha}u_1^{\beta_1}u_2^{\beta_2}u_3^{\beta_3}dg. \end{aligned}$$

Thus we obtain

$$\begin{aligned} dg=\Big[&(A_1-\alpha_1I)\frac{dx_1}{x_1}+(A_2-\alpha_2I)\frac{dx_2}{x_2}+(A_3-(\alpha_3-\alpha)I)\frac{dx_3}{x_3}\\ &+(B_1-\beta_1I)\frac{du_1}{u_1}+(B_2-\beta_2I)\frac{du_2}{u_2}+(B_3-\beta_3I)\frac{du_3}{u_3}\Big]g. \end{aligned}$$

The coefficients of the transformed system satisfy (I) and (S).

Let $\boldsymbol{\alpha}_i$, $\boldsymbol{\beta}_i$, $\boldsymbol{\alpha}_i'$, $\boldsymbol{\beta}_i'$ $(i=1,2,3)$ be the numbers analogous to α_i, β_i, α_i', β_i' $(i=1,2,3)$ associated with the transformed system, then we see

$$\alpha_1=0,\quad \alpha_1'=\alpha_1'-\alpha_1,\qquad \beta_1=0,\quad \beta_1'=\beta_1'-\beta_1,$$
$$\alpha_2=0,\quad \alpha_2'=\alpha_2'-\alpha_2,\qquad \beta_2=0,\quad \beta_2'=\beta_2'-\beta_2,$$
$$\alpha_3=\alpha,\quad \alpha_3'=\alpha_3'-\alpha_3+\alpha,\qquad \beta_3=0,\quad \beta_3'=\beta_3'-\beta_3,$$

where

$$\alpha+\alpha_1'+\beta_1'=0,\quad \alpha+\alpha_2'+\beta_2'=0,\quad \alpha_3'+\beta_3'=0.$$

Hence by the linear transformation T, one obtains Appell's system (F_1).

Remark 2.12. The 6-tuple (2.6) in Theorem 2.10 reduces to that of Appell's system (F_1) with $\alpha=0$. (It is easy to see that the system of differential equations given by (2.6) can be integrated by using the Gaussian hypergeometric function.) In fact the 6-tuple (2.6) reduces to

$$\begin{aligned}
&A_1=\begin{pmatrix}0&0&0\\0&a_1-a_1'&0\\0&0&a_1-a_1'\end{pmatrix}, &&B_1=\begin{pmatrix}0&0&0\\0&-b_3&b_2\\0&b_3&-b_2\end{pmatrix},\\
&A_2=\begin{pmatrix}0&-b_1&0\\0&a_2'-a_2&0\\0&-b_3&0\end{pmatrix}, &&B_2=\begin{pmatrix}0&0&b_1\\0&b_3&0\\0&0&b_3\end{pmatrix},\\
&A_3=\begin{pmatrix}0&0&-b_1\\0&0&-b_2\\0&0&a_3'-a_3\end{pmatrix}, &&B_3=\begin{pmatrix}0&b_1&0\\0&b_2&0\\0&0&b_2\end{pmatrix},\\
&C_1=\begin{pmatrix}a_2-a_2'+b_3&-b_1&-b_1\\0&0&0\\0&0&0\end{pmatrix}, &&C_2=\begin{pmatrix}0&0&0\\0&a_3-a_3'&-b_2\\0&0&0\end{pmatrix},\\
&C_3=\begin{pmatrix}0&0&0\\0&0&0\\0&-b_3&a_2-a_2'\end{pmatrix}, &&B=\begin{pmatrix}0&b_1&b_1\\0&b_2&b_2\\0&b_3&b_3\end{pmatrix}.
\end{aligned} \tag{2.10}$$

We set

$$\begin{aligned}
(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{B}_1, \tilde{B}_2, \tilde{B}_3; \tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{B}) :&= \nu\sigma\tau_{23}\nu H\\
&= \nu\sigma\tau_{23}\nu(A_1, A_2, A_3, B_1, B_2, B_3, ; C_1, C_2, C_3, B)\\
&= (A_2, A_3, B, C_3, C_2, B_1; B_3, B_2, C_1, A_1),
\end{aligned}$$

where σ is the permutation $\sigma(1,2,3)=(3,2,1)$. Then by setting $b_1=-1$, $b_2=\beta$, $b_3=\beta'$, $\alpha_2'-\alpha_2=1-\gamma+\beta'$, $\alpha_3'-\alpha_3=1-\gamma+\beta$, we see that the 6-tuple $(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{B}_1, \tilde{B}_2, \tilde{B}_3)$ reduces to that of Appell's system (F_1) with $\alpha=0$ ((2.6) in Theorem 2.10), that is

$$\tilde{A}_1=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-\gamma+\beta' & 0 \\ 0 & -\beta' & 0 \end{pmatrix}, \quad \tilde{B}_1=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\beta' & \gamma-\beta'-1 \end{pmatrix},$$

$$\tilde{A}_2=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -\beta \\ 0 & 0 & 1-\gamma+\beta \end{pmatrix}, \quad \tilde{B}_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma-\beta-1 & -\beta \\ 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{A}_3=\begin{pmatrix} 0 & -1 & -1 \\ 0 & \beta & \beta \\ 0 & \beta' & \beta' \end{pmatrix}, \quad \tilde{B}_3=\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\beta' & \beta \\ 0 & \beta' & -\beta \end{pmatrix}.$$

Remark 2.13. Let us consider the case of $\alpha=0$ in Remark 2.11;

$$\begin{cases} \alpha_1=0,\ \alpha_1'=1-\gamma+\beta',\ \beta_1=0,\ \beta_1'=\gamma-\beta'-1, \\ \alpha_2=0,\ \alpha_2'=1-\gamma+\beta,\ \beta_2=0,\ \beta_2'=\gamma-\beta-1, \\ \alpha_3=0,\ \alpha_3'=\beta+\beta', \quad \beta_3=0,\ \beta_3'=-(\beta+\beta'). \end{cases} \tag{2.11}$$

We substitute this into (2.3) of Theorem 2.10:

$$\begin{cases} A_1=\begin{pmatrix} 1-\gamma+\beta' & 0 & 0 \\ -\beta' & 0 & 0 \\ \gamma-1 & 0 & 0 \end{pmatrix}, \quad B_1=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma-1 & \beta' \\ 0 & 1-\gamma & -\beta' \end{pmatrix}, \\ A_2=\begin{pmatrix} 0 & -\beta & 0 \\ 0 & 1-\gamma+\beta & 0 \\ 0 & \gamma-1 & 0 \end{pmatrix}, \quad B_2=\begin{pmatrix} \gamma-1 & 0 & \beta \\ 0 & 0 & 0 \\ 1-\gamma & 0 & -\beta \end{pmatrix}, \\ A_3=\begin{pmatrix} 0 & 0 & -\beta \\ 0 & 0 & -\beta' \\ 0 & 0 & \beta+\beta' \end{pmatrix}, \quad B_3=\begin{pmatrix} -\beta' & \beta & 0 \\ \beta' & -\beta & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{cases} \tag{2.12}$$

We assert that (2.12) can be reduced to the transposed matrices of Appell's system (F_1) with $\alpha=0$. Let

$$S=\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},$$

A_i, B_i ($i=1, 2, 3$) be defined by (2.12) in Theorem 2.10 with (2.11). Then SA_iS^{-1}, SB_iS^{-1} ($i=1, 2, 3$) are as follows:

$$SA_1S^{-1}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-\gamma+\beta' & 0 \\ 0 & -\beta' & 0 \end{pmatrix}, \qquad SA_2S^{-1}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\beta \\ 0 & 0 & 1-\gamma+\beta' \end{pmatrix},$$

$$SA_3S^{-1}=\begin{pmatrix} 0 & 0 & 0 \\ \beta & \beta & \beta \\ \beta' & \beta' & \beta' \end{pmatrix}, \qquad SB_1S^{-1}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\beta' & -\beta' & \gamma-\beta'-1 \end{pmatrix},$$

$$SB_2S^{-1}=\begin{pmatrix} 0 & 0 & 0 \\ -\beta & \gamma-\beta-1 & -\beta \\ 0 & 0 & 0 \end{pmatrix}, \quad SB_3S^{-1}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\beta' & \beta \\ 0 & \beta' & -\beta \end{pmatrix}.$$

Now we use A_i, B_i ($i=1, 2, 3$) instead of above SA_iS^{-1}, SB_iS^{-1} ($i=1, 2, 3$). Let σ be the transposition of 1 and 2. We have

$$\begin{aligned}(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{B}_1, \tilde{B}_2, \tilde{B}_3) &:= \sigma\tau_{12}H^* \\ &=(B_2^*, B_1^*, A_3^*, A_2^*, A_1^*, B_3^*)\end{aligned}$$

where

$$\left[\begin{array}{l} \tilde{A}_1=\begin{pmatrix} 0 & -\beta & 0 \\ 0 & \gamma-\beta-1 & 0 \\ 0 & -\beta & 0 \end{pmatrix}, \quad \tilde{B}_1=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\beta & 1-\gamma+\beta \end{pmatrix}, \\ \tilde{A}_2=\begin{pmatrix} 0 & 0 & -\beta' \\ 0 & 0 & -\beta' \\ 0 & 0 & \gamma-\beta'-1 \end{pmatrix}, \quad \tilde{B}_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-\gamma+\beta' & -\beta' \\ 0 & 0 & 0 \end{pmatrix}, \\ \tilde{A}_3=\begin{pmatrix} 0 & \beta & \beta' \\ 0 & \beta & \beta' \\ 0 & \beta & \beta' \end{pmatrix}, \quad \tilde{B}_3=\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\beta' & \beta' \\ 0 & \beta & -\beta \end{pmatrix}. \end{array}\right.$$

If we change the parameter γ to $2-\gamma+\beta+\beta'$, we have

$$\left[\begin{array}{ll}\tilde{A}_1=\begin{pmatrix}0 & -\beta & 0\\ 0 & 1-\gamma+\beta' & 0\\ 0 & -\beta & 0\end{pmatrix}, & \tilde{B}_1=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & -\beta & \gamma-\beta'-1\end{pmatrix},\\ \tilde{A}_2=\begin{pmatrix}0 & 0 & -\beta'\\ 0 & 0 & -\beta'\\ 0 & 0 & 1-\gamma+\beta\end{pmatrix}, & \tilde{B}_2=\begin{pmatrix}0 & 0 & 0\\ 0 & \gamma-\beta-1 & -\beta'\\ 0 & 0 & 0\end{pmatrix},\\ \tilde{A}_3=\begin{pmatrix}0 & \beta & \beta'\\ 0 & \beta & \beta'\\ 0 & \beta & \beta'\end{pmatrix}, & \tilde{B}_3=\begin{pmatrix}0 & 0 & 0\\ 0 & -\beta' & \beta'\\ 0 & \beta & -\beta\end{pmatrix}.\end{array}\right.$$

These are simultaneously similar to

$$\left[\begin{array}{ll}\tilde{A}_1=\begin{pmatrix}0 & 1 & 0\\ 0 & 1-\gamma+\beta' & 0\\ 0 & -\beta' & 0\end{pmatrix}, & \tilde{B}_1=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & -\beta' & \gamma-\beta'-1\end{pmatrix},\\ \tilde{A}_2=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & -\beta\\ 0 & 0 & 1-\gamma+\beta\end{pmatrix}, & \tilde{B}_2=\begin{pmatrix}0 & 0 & 0\\ 0 & \gamma-\beta-1 & -\beta\\ 0 & 0 & 0\end{pmatrix},\\ \tilde{A}_3=\begin{pmatrix}0 & -1 & -1\\ 0 & \beta & \beta\\ 0 & \beta' & \beta'\end{pmatrix}, & \tilde{B}_3=\begin{pmatrix}0 & 0 & 0\\ 0 & -\beta' & \beta\\ 0 & \beta' & -\beta\end{pmatrix}.\end{array}\right.$$

Thus we see that (2.12) reduces to the transposed matrices of Appell's system (F_1) with $\alpha=0$. That is

$$\left[\begin{array}{ll}\begin{pmatrix}0 & 0 & 0\\ 1 & 1-\gamma+\beta' & -\beta'\\ 0 & 0 & 0\end{pmatrix}, & \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -\beta'\\ 0 & 0 & \gamma-\beta'-1\end{pmatrix},\\ \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & -\beta & 1-\gamma+\beta\end{pmatrix}, & \begin{pmatrix}0 & 0 & 0\\ 0 & \gamma-\beta-1 & 0\\ 0 & -\beta & 0\end{pmatrix},\\ \begin{pmatrix}0 & 0 & 0\\ -1 & \beta & \beta'\\ -1 & \beta & \beta'\end{pmatrix}, & \begin{pmatrix}0 & 0 & 0\\ 0 & -\beta' & \beta'\\ 0 & \beta & -\beta\end{pmatrix}.\end{array}\right. \tag{2.13}$$

It is to see that the system of differential equations given by (2.13) can be integrated by using the Gaussian hypergeometric function.

We denote

$$\mathfrak{M}(resp.\ \mathfrak{N}) \quad := \text{the set of unordered } \{X, Y, Z\},$$
$$\mathfrak{M}^*(resp.\ \mathfrak{N}^*) := \text{the set of unordered } \{X^*, Y^*, Z^*\}$$

respectively, where $(X, Y, Z, \cdots)$ (*resp.* $(\cdots X, Y, Z)$) is a transformed 6-tuple of $H=(A_1, A_2, A_3, B_1, B_2, B_3)$ by G.

We prepare some lemmas [4, p.205-210].

Lemma 2.14. *If H is not elementary and $\{X, Y, Z\}\in\mathfrak{N}$ or $\{X, Y, Z\}\in\mathfrak{N}^*$, then:*

(1) $N_X\cap N_Y\cap N_Z\neq\{0\}$.

(2) *I_X, I_Y, I_Z are contained in a 2-dimensional vector subspace of V.*

Lemma 2.15. If $N_X\cap N_Y\cap N_Z\neq\{0\}$ *for any triple $\{X, Y, Z\}$ of $\mathfrak{M}$ and $\mathfrak{N}$, then:*

(1) *There exists $e\neq 0$ such that $e\in N_{A_i}\cap N_{B_i}\cap N_{C_i}\cap N_B$ $(i=1, 2, 3)$.*

(2) *There exists* a *2-dimensional vector subspace of V^* which contains $I_{A_i^*}, I_{B_i^*}, I_{C_i^*}, I_{B^*}$ $(i=1, 2, 3)$.*

Lemma 2.16. *If there exists a nonzero vector $e\in N_{A_i}\cap N_{B_i}\cap N_{C_i}\cap N_B$ $(i=1, 2, 3)$ and a linear form $l\neq 0$* on *$I_{A_i}, I_{B_i}, I_{C_i}, I_B$, then H is elementary or decomposable.*

The following Proposition 2.17 is proved in [4, Proposition 2]. We have corrected a mistake in their proof for Case II-2.

Proposition 2.17. *If $N_X\cap N_Y\cap N_Z\neq\{0\}$ for any $\{X, Y, Z\}\in\mathfrak{N}$ and there exists $\{X, Y, Z\}\in\mathfrak{M}$ such that $N_X\cap N_Y\cap N_Z=\{0\}$, and H is not elementary, then H is decomposable, or there exists a basis of V satisfying* (2.5) *and* (2.4), *or* (2.5) *and* (2.7) *such that H is of the form* (2.3) *or* (2.6) *respectively in Theorem* 2.10.

Proof. We may suppose that $N_{A_1}\cap N_{A_2}\cap N_{A_3}=\{0\}$. Take a basis (e_1, e_2, e_3) of V such that $e_1\in N_{A_2}\cap N_{A_3}$, $e_2\in N_{A_3}\cap N_{A_1}$, $e_3\in N_{A_1}\cap N_{A_2}$. With respect to this basis A_i $(i=1, 2, 3)$ is of the form;

$$A_1=\begin{pmatrix} \alpha_1' & 0 & 0 \\ * & \alpha_1 & 0 \\ * & 0 & \alpha_1 \end{pmatrix},\quad A_2=\begin{pmatrix} \alpha_2 & * & 0 \\ 0 & \alpha_2' & 0 \\ 0 & * & \alpha_2 \end{pmatrix},\quad A_3=\begin{pmatrix} \alpha_3 & 0 & * \\ 0 & \alpha_3 & * \\ 0 & 0 & \alpha_3' \end{pmatrix}.$$

By $\mathfrak{N}\ni\{A_1, A_2, A_3\}$, we have $e_3\in N_{B_3}$. In the same way, $e_1\in N_{B_1}$, $e_2\in N_{B_2}$. By Lemma 2.9, we see $B_1(N_{A_i})\subset N_{A_i}$ $(i=1, 2, 3)$. Hence with respect to the basis (e_1, e_2, e_3), B_i $(i=1, 2, 3)$ is of the form;

$$B_1=\begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix},\quad B_2=\begin{pmatrix} * & 0 & * \\ 0 & \beta_2 & 0 \\ * & 0 & * \end{pmatrix},\quad B_3=\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}.$$

Case I: If $e_1, e_2, e_3 \notin N_B$, then we may suppose that $N_B=\{(x_1, x_2, x_3)\,|\,x_1+x_2+x_3=0\}$. By Lemma 2.8, we have

$$B=\begin{pmatrix} b_1+\xi & b_1 & b_1 \\ b_2 & b_2+\xi & b_2 \\ b_3 & b_3 & b_3+\xi \end{pmatrix}.$$

Since $B=B_1+B_2+B_3$, we may put

$$B_1=\begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & a & b_2 \\ 0 & b_3 & b \end{pmatrix}.$$

In view of $[B, B_1]=0$, $N_{B_1}\neq N_B$ $(N_{B_1}\ni e_1)$, one can use Lemma 2.9 to conclude that $I_{B_1}\subset N_B$. On the other hand, we have $(0, a-\beta_1, b_3)\in I_{B_1}$, $(0, b_2, b-\beta_1)\in I_{B_1}$. By $I_{B_1}\subset N_B$, it follows that $a-\beta_1+b_3=0$, $b_2+b-\beta_1=0$. Hence

$$a=\beta_1-b_3,\ b=\beta_1-b_2. \tag{2.14}$$

$$trB_1=2\beta_1+\beta_1'=\beta_1+a+b. \tag{2.15}$$

Thus we get $\beta_1-\beta_1'=b_2+b_3$ by substituting (2.14) in (2.15). Therefore we obtain $a=b_2+\beta_1'$, $b=b_3+\beta_1'$, $b_2+b_3=\beta_1-\beta_1'$. Similarly, we may put

$$B_2=\begin{pmatrix} c & 0 & b_1 \\ 0 & \beta_2 & 0 \\ b_3 & 0 & d \end{pmatrix}.$$

Since $[B, B_2]=0$, $N_{B_2}\neq N_B$ $(N_{B_2}\ni e_2)$, we have $I_{B_2}\subset N_B$ by Lemma 2.9. On the other hand, we have $(c-\beta_2, 0, b_3)\in I_{B_2}$, $(b_1, 0, d-\beta_2)\in I_{B_2}$. By $I_{B_2}\subset N_B$, we see $c-\beta_2+b_3=0$, $b_1+d-\beta_2=0$. Hence

$$c=\beta_2-b_3,\ d=\beta_2-b_1. \tag{2.16}$$

$$trB_2=2\beta_2+\beta_2'=\beta_2+c+d. \tag{2.17}$$

Substituting (2.16) in (2.17) gives $\beta_2-\beta_2'=b_3+b_1$. Hence we get $c=b_1+\beta_2'$, $d=b_3+\beta_2'$, $b_3+b_1=\beta_2-\beta_2'$. In the same way, put

$$B_3=\begin{pmatrix} e & b_1 & 0 \\ b_2 & f & 0 \\ 0 & 0 & \beta_3 \end{pmatrix},$$

then we see

$$e=\beta_3-b_2,\ f=\beta_3-b_1. \tag{2.18}$$

$$trB_3=2\beta_3+\beta_3'=\beta_3+e+f. \tag{2.19}$$

Thus we get $\beta_3-\beta_3'=b_1+b_2$ by substituting (2.18) in (2.19). Therefore we obtain $e=b_1+\beta_3'$, $f=b_2+\beta_3'$, $b_1+b_2=\beta_3-\beta_3'$. We also have

$$b_2+b_3=\beta_1-\beta_1',$$
$$b_3+b_1=\beta_2-\beta_2',$$
$$b_1+b_2=\beta_3-\beta_3'.$$

Thus we get the relation (2.4)

$$\begin{cases} b_1=\dfrac{1}{2}(-\beta_1+\beta_1'+\beta_2-\beta_2'+\beta_3-\beta_3'), \\ b_2=\dfrac{1}{2}(\beta_1-\beta_1'-\beta_2+\beta_2'+\beta_3-\beta_3'), \\ b_3=\dfrac{1}{2}(\beta_1-\beta_1'+\beta_2-\beta_2'-\beta_3+\beta_3'). \end{cases}$$

By the relation (S), we obtain (2.3)

$$\left[\begin{array}{ll} A_1=\begin{pmatrix} \alpha_1' & 0 & 0 \\ -b_2 & \alpha_1 & 0 \\ -b_3 & 0 & \alpha_1 \end{pmatrix}, & B_1=\begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & b_2+\beta_1' & b_2 \\ 0 & b_3 & b_3+\beta_1' \end{pmatrix}, \\ A_2=\begin{pmatrix} \alpha_2 & -b_1 & 0 \\ 0 & \alpha_2' & 0 \\ 0 & -b_3 & \alpha_2 \end{pmatrix}, & B_2=\begin{pmatrix} b_1+\beta_2' & 0 & b_1 \\ 0 & \beta_2 & 0 \\ b_3 & 0 & b_3+\beta_2' \end{pmatrix}, \\ A_3=\begin{pmatrix} \alpha_3 & 0 & -b_1 \\ 0 & \alpha_3 & -b_2 \\ 0 & 0 & \alpha_3' \end{pmatrix}, & B_3=\begin{pmatrix} b_1+\beta_3' & b_1 & 0 \\ b_2 & b_2+\beta_3' & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}, \end{array}\right.$$

with the relation (2.5)

$$\left[\begin{array}{l} \alpha_1'+\alpha_2+\alpha_3+\beta_1'+\beta_2+\beta_3=0, \\ \alpha_1+\alpha_2'+\alpha_3+\beta_1+\beta_2'+\beta_3=0, \\ \alpha_1+\alpha_2+\alpha_3'+\beta_1+\beta_2+\beta_3'=0. \end{array}\right.$$

Case II: If N_B contains only one of e_1, e_2, e_3, then we may suppose that $e_1\in N_B$ and $N_B=\{(x_1, x_2, x_3)\,|\,x_2+x_3=0\}$. By Lemma 2.8, we see

$$B=\begin{pmatrix} \xi & b_1 & b_1 \\ 0 & b_2+\xi & b_2 \\ 0 & b_3 & b_3+\xi \end{pmatrix}.$$

Now we have $N_{B_2}\neq N_B$, $N_{B_3}\neq N_B$ ($N_{B_2}\ni e_2$, $N_{B_3}\ni e_3$). Since $[B, B_2]=[B, B_3]=0$, we can use Lemma 2.9 to conclude that $I_{B_2}\subset N_B$, $I_{B_3}\subset N_B$. Then since $B=B_1+B_2+B_3$, B_2 and B_3 are of the forms;

$$B_2=\begin{pmatrix} a & 0 & b_1 \\ 0 & \beta_2 & 0 \\ 0 & 0 & b \end{pmatrix}, \quad B_3=\begin{pmatrix} c & b_1 & 0 \\ 0 & d & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}.$$

By $(b_1, 0, b-\beta_2)\in I_{B_2}\subset N_B$, $(b_1, d-\beta_3, 0)\in I_{B_3}\subset N_B$, we have $0+b-\beta_2=0$, $d-\beta_3+0=0$. Hence $b=\beta_2$, $d=\beta_3$. Put $a=\beta_2'$, $c=\beta_3'$ so that

$$B_2=\begin{pmatrix} \beta_2' & 0 & b_1 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_2 \end{pmatrix}, \quad B_3=\begin{pmatrix} \beta_3' & b_1 & 0 \\ 0 & \beta_3 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}.$$

By $B=B_1+B_2+B_3$, we have

$$B_1=\begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & * & b_2 \\ 0 & b_3 & * \end{pmatrix}.$$

Case II-1: If $b_1=0$, then by the relation (S), we have

$$A_1=\begin{pmatrix} \alpha'_1 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_1 \end{pmatrix}, \quad B_1=\begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & * & b_2 \\ 0 & b_3 & * \end{pmatrix},$$

$$A_2=\begin{pmatrix} \alpha_2 & 0 & 0 \\ 0 & \alpha'_2 & 0 \\ 0 & -b_3 & \alpha_2 \end{pmatrix}, \quad B_2=\begin{pmatrix} \beta'_2 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_2 \end{pmatrix},$$

$$A_3=\begin{pmatrix} \alpha_3 & 0 & 0 \\ 0 & \alpha_3 & -b_2 \\ 0 & 0 & \alpha'_3 \end{pmatrix}, \quad B_3=\begin{pmatrix} \beta'_3 & 0 & 0 \\ 0 & \beta_3 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}.$$

Then H is decomposable.

Case II-2: We suppose $b_1\neq 0$. By $[B, B_i]=0\ (i=1, 2, 3)$, we get $b_3=\beta_2-\beta'_2$, $b_2=\beta_3-\beta'_3$. Thus we obtain

$$B_1=\begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_1-b_3 & b_2 \\ 0 & b_3 & \beta_1-b_2 \end{pmatrix}.$$

By $trB_1=\beta'_1+2\beta_1=3\beta_1-(b_2+b_3)$, we have

$$\beta_1-\beta'_1=b_2+b_3=\beta_2-\beta'_2+\beta_3-\beta'_3.$$

Therefore by conditions (I) and (S) we obtain

$$A_1=\begin{pmatrix} \alpha'_1 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_1 \end{pmatrix}, \quad B_1=\begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_1-b_3 & b_2 \\ 0 & b_3 & \beta_1-b_2 \end{pmatrix},$$

$$A_2=\begin{pmatrix} \alpha_2 & -b_1 & 0 \\ 0 & \alpha_2' & 0 \\ 0 & -b_3 & \alpha_2 \end{pmatrix},\quad B_2=\begin{pmatrix} \beta_2' & 0 & b_1 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_2 \end{pmatrix},$$

$$A_3=\begin{pmatrix} \alpha_3 & 0 & -b_1 \\ 0 & \alpha_3 & -b_2 \\ 0 & 0 & \alpha_3' \end{pmatrix},\quad B_3=\begin{pmatrix} \beta_3' & b_1 & 0 \\ 0 & \beta_3 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}.$$

It remains only to show that $N_X\cap N_Y\cap N_Z\neq\{0\}$ for any $\{X, Y, Z\}\in\mathfrak{N}$ of the above 6-tuple and this is straightforwardly checked.

Case III: If N_B contains two of e_1, e_2, e_3, then we may suppose that $e_1, e_2\in N_B$ and $N_B=\{(x_1, x_2, x_3)\mid x_3=0\}$. Thus B is of the form

$$B=\begin{pmatrix} \xi & 0 & b_1 \\ 0 & \xi & b_2 \\ 0 & 0 & b_3+\xi \end{pmatrix},$$

where $b_i\neq 0$ for some i. We see, relative to the basis (e_1, e_2, e_3), A_i, B_i $(i=1, 2, 3)$ are of the form;

$$A_1=\begin{pmatrix} \alpha_1' & 0 & 0 \\ * & \alpha_1 & 0 \\ * & 0 & \alpha_1 \end{pmatrix},\quad B_1=\begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix},$$

$$A_2=\begin{pmatrix} \alpha_2 & * & 0 \\ 0 & \alpha_2' & 0 \\ 0 & * & \alpha_2 \end{pmatrix},\quad B_2=\begin{pmatrix} * & 0 & * \\ 0 & \beta_2 & 0 \\ * & 0 & * \end{pmatrix},$$

$$A_3=\begin{pmatrix} \alpha_3 & 0 & * \\ 0 & \alpha_3 & * \\ 0 & 0 & \alpha_3' \end{pmatrix},\quad B_3=\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}.$$

Then by conditions (I) and (S), we have

$$A_1=\begin{pmatrix} \alpha_1' & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_1 \end{pmatrix},\quad B_1=\begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \lambda_1 & b_2 \\ 0 & 0 & \lambda_2 \end{pmatrix},$$

$$A_2=\begin{pmatrix} \alpha_2 & 0 & 0 \\ 0 & \alpha_2' & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix},\quad B_2=\begin{pmatrix} \mu_1 & 0 & b_1 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix},$$

$$A_3=\begin{pmatrix} \alpha_3 & 0 & -b_1 \\ 0 & \alpha_3 & -b_2 \\ 0 & 0 & \alpha_3' \end{pmatrix},\quad B_3=\begin{pmatrix} \nu_1 & 0 & 0 \\ 0 & \nu_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}.$$

By $[B, B_3]=0$, we have $\beta_3 b_1=\nu_1 b_1$, $\beta_3 b_2=\nu_2 b_2$. If $b_1 b_2\neq 0$, then we get $\nu_1=\nu_2=\beta_3$ and this is a contradiction. Hence we see $b_1 b_2=0$. If $b_1\neq 0$, $b_2=0$ or $b_1=0$, $b_2\neq 0$, then $\nu_1=\beta_3$ or $\nu_2=\beta_3$. Thus H is decomposable. If $b_1=b_2=0$, then H is elementary. □

3. The classification of the case that the condition (R) is not satisfied

3.1. The case that there exist elements of H satisfying (R′).

In this section we consider the case that the condition (R) is not satisfied in Theorem 2.10 of **section 2.3.** We say that an element X of H satisfies (R') if

(R') There exists $\xi\in C$ such that $rank(X-\xi I)=2$,

and X does not satisfy (R).

Theorem 3.1. *If a 6-tuple H satisfies (I) and (S), and at least one of the element of H satisfies (R'), then H is decomposable or elementary, or the singular set is redundant.*

Proof. We consider the case that there exist elements of H satisfying (R'). Then we may assume that B satisfy (R'). We have to consider 3 cases I~III of B.

I: The case of $B=\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ $a\neq b,\ b\neq c,\ c\neq a$

Then it is easy to see H is elementary or decomposable by the relations (I) and (S).

II: The case of $B=\begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}$ $a\neq b$

By the relation $[B, B_i]=0\ (i=1, 2, 3)$, we have

$$B_1=\begin{pmatrix}\beta_1 & m_1 & 0\\ 0 & \beta_1 & 0\\ 0 & 0 & \beta_1'\end{pmatrix},\quad B_2=\begin{pmatrix}\beta_2 & m_2 & 0\\ 0 & \beta_2 & 0\\ 0 & 0 & \beta_2'\end{pmatrix},\quad B_3=\begin{pmatrix}\beta_3 & m_3 & 0\\ 0 & \beta_3 & 0\\ 0 & 0 & \beta_3'\end{pmatrix}.$$

Case II-1. $m_1m_2m_3\neq 0$: If $\beta_i\neq\beta_i'$ $(i=1, 2, 3)$, or $\beta_1=\beta_1'$, $\beta_i\neq\beta_i'$ $(i=2, 3)$, then H is decomposable by the relations (I) and (S).

If $\beta_3\neq\beta_3'$, $\beta_i=\beta_i'$ $(i=1, 2)$, then by $[A_i, B_i]=0$ $(i=1, 2, 3)$, we get

$$A_1=\begin{pmatrix}\alpha_1 & n_1 & x_1\\ 0 & \alpha_1 & 0\\ 0 & x_2 & \alpha_1'\end{pmatrix},\quad A_2=\begin{pmatrix}\alpha_2 & n_2 & x_3\\ 0 & \alpha_2 & 0\\ 0 & x_4 & \alpha_2'\end{pmatrix},\quad A_3=\begin{pmatrix}\alpha_3 & n_3 & 0\\ 0 & \alpha_3 & 0\\ 0 & 0 & \alpha_3'\end{pmatrix}.$$

Then put

$$N_i=\begin{pmatrix}0 & n_i & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad M_j=\begin{pmatrix}0 & m_j & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\quad (i, j=1, 2, 3).$$

These are nilpotent and by (S) we see $\sum_{i=1}^{3}(N_i+M_i)=0$. We note N_i, M_j $(i=1, 2, 3)$ are commutative and commute with every element of H, so that the singular set is redundant.

Case II-2. $m_1=0$, $m_2m_3\neq 0$: If $\beta_1=\beta_1'$, then the singular set is redundant. Hence we may suppose $\beta_1\neq\beta_1'$.

If $\beta_i\neq\beta_i'$ $(i=2, 3)$, or $\beta_2=\beta_2'$, $\beta_3\neq\beta_3'$, then H is decomposable by the relations (I) and (S).

If $\beta_2=\beta_2'$, $\beta_3=\beta_3'$, then we see the singular set is redundant in the same way as in Case II-1.

Case II-3. $m_1=m_2=0$, $m_3=1$: If $\beta_1=\beta_1'$ (*resp.* $\beta_2=\beta_2'$), then the singular set is redundant. Hence we may suppose $\beta_i\neq\beta_i'$ $(i=1, 2)$. Then H is decomposable by the relations (I) and (S).

III: The case of $B=\begin{pmatrix}a & 1 & 0\\ 0 & a & 1\\ 0 & 0 & a\end{pmatrix}$

By the relation (I), we have

$$B_1=\begin{pmatrix}\beta_1 & m_1 & m_1'\\ 0 & \beta_1 & m_1\\ 0 & 0 & \beta_1\end{pmatrix},\quad B_2=\begin{pmatrix}\beta_2 & m_2 & m_2'\\ 0 & \beta_2 & m_2\\ 0 & 0 & \beta_2\end{pmatrix},\quad B_3=\begin{pmatrix}\beta_3 & m_3 & m_3'\\ 0 & \beta_3 & m_3\\ 0 & 0 & \beta_3\end{pmatrix}.$$

Case III-1. $m_1m_2m_3\neq 0$: H is elementary by the relations (I) and (S).

Case III-2. $m_1=0$, $m_2m_3\neq 0$: By the relations (I) and (S), we see this case reduces to Case III-1.

Case III-3. $m_1=m_2=0$, $m_3=1$: Then the singular set is redundant in the same way as in case

II-1. □

3.2. The case that the singular set is redundant.

We consider the case that the singular set is redundant. We may suppose

$$B_1=\begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_1 & 0 \\ 0 & 0 & \beta_1 \end{pmatrix}.$$

Hence by a change of solution (Remark 2.11), we may assume $B_1=0$. Now it suffices to classify 5-tuple $(A_1, A_2, A_3, B_2, B_3)$. By the definition of B, we see $B=B_1+B_2+B_3=B_2+B_3$. By the relation (I), we have

$$[A_2, C_1]=[A_2, A_2+A_3+B_1]=[A_2, A_2+A_3]=[A_2, A_3]=0.$$

$$[B_2, B]=[B_2, B_1+B_2+B_3]=[B_2, B_2+B_3]=[B_2, B_3]=0.$$

Hence

$$B=B_2+B_3, \tag{3.1}$$

$$[A_2, A_3]=0, [B_2, B_3]=0. \tag{3.2}$$

Let us consider the following system of partial differential equations:

$$x(1-x)u''+\{\gamma-(\alpha+1)x\}u'=0, \tag{3.3}$$

$$y(1-y)v''+\{\gamma'-(\alpha'+1)y\}v'=0. \tag{3.4}$$

We put $z=xu'$ in (3.3). Then we have

$$\begin{aligned} x(1-x)z'&=x(1-x)(xu''+u') \\ &=x[x(1-x)u'']+(1-x)xu' \\ &=x[\{-\gamma+(\alpha+1)x\}u']+(1-x)xu' \\ &=[1-\gamma+\alpha x]xu'. \end{aligned}$$

Therefore

$$z'=\left(\frac{1-\gamma}{x}+\frac{\gamma-\alpha-1}{x-1}\right)z.$$

Thus the equation (3.3) is transformed to

$$d\begin{pmatrix} u \\ xu_x \end{pmatrix}=\left(\frac{A_2'}{x}dx+\frac{B_3'}{x-1}dx\right)\begin{pmatrix} u \\ xu_x \end{pmatrix}.$$

In the same way, the equation (3.4) is transformed to

$$d\begin{pmatrix} v \\ yv_y \end{pmatrix}=\left(\frac{A_3'}{y}dy+\frac{B_2'}{y-1}dy\right)\begin{pmatrix} v \\ yv_y \end{pmatrix},$$

where

$$\left[\begin{array}{l} A_2'=\begin{pmatrix} 0 & 1 \\ 0 & 1-\gamma \end{pmatrix}, \quad B_2'=\begin{pmatrix} 0 & 0 \\ 0 & \gamma'-\alpha'-1 \end{pmatrix}, \\ A_3'=\begin{pmatrix} 0 & 1 \\ 0 & 1-\gamma' \end{pmatrix}, \quad B_3'=\begin{pmatrix} 0 & 0 \\ 0 & \gamma-\alpha-1 \end{pmatrix}. \end{array}\right.$$

Let $c(x)$, $d(y)$ be defined by

$$c'xu'=\frac{1}{x}, \quad d'yv'=\frac{1}{y}.$$

We construct the following two systems (G_1), (G_2) with the fundamental solutions:

$$\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u(x) \\ xu_x \\ 0 \end{pmatrix}, \begin{pmatrix} v(y) \\ 0 \\ yv_y \end{pmatrix}\right\}, \left\{\begin{pmatrix} 1 \\ c(x)u(x) \\ d(y)v(y) \end{pmatrix}, \begin{pmatrix} 0 \\ xu_x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ yv_y \end{pmatrix}\right\}$$

respectively. Namely,

$$d\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}=\left(\frac{A_2}{x}dx+\frac{A_3}{y}dy+\frac{B_2}{y-1}dy+\frac{B_3}{x-1}dx\right)\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \tag{3.5}$$

where

$$(G_1)\left[\begin{array}{l} A_2=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1-\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma'-\alpha'-1 \end{pmatrix}, \\ A_3=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1-\gamma' \end{pmatrix}, \quad B_3=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma-\alpha-1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{array}\right.$$

$$(G_2)\quad \left[\begin{array}{l} A_2=\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1-\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma'-\alpha'-1 \end{pmatrix}, \\ A_3=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1-\gamma' \end{pmatrix}, \quad B_3=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma-\alpha-1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{array}\right.$$

Theorem 3.2. *Assume that the* 6*-tuple H satisfies (I) and (S). If the singular set is redundant, then H is almost decomposable or elementary or reduces to the system (G_1) or (G_2).*

Proof. We have to check 6 cases I~VI of B.

I: The case of $B=\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}$ $a\neq b$.

By $[B, B_2]=0, [B, B_3]=0$ and (3.1) we may assume

$$B_2=\begin{pmatrix} \beta_2 & 0 & 0 \\ 0 & \beta_2' & c \\ 0 & 0 & \beta_2'' \end{pmatrix}, \quad B_3=\begin{pmatrix} \beta_3 & 0 & 0 \\ 0 & \beta_3' & -c \\ 0 & 0 & \beta_3'' \end{pmatrix}$$

where $\beta_2'+\beta_3'=\beta_2''+\beta_3''$.

Case I-1: We suppose that the 5-tuple satisfy (R).

(i) The case of $c=0$: If $\beta_2\neq\beta_2'=\beta_2''$, $\beta_3\neq\beta_3'=\beta_3''$, then by $[A_i, B_i]=0$ $(i=2, 3)$ we see

$$A_2=\begin{pmatrix} \alpha_2 & 0 & 0 \\ 0 & \alpha_2' & * \\ 0 & * & \alpha_2'' \end{pmatrix}, \quad A_3=\begin{pmatrix} \alpha_3 & 0 & 0 \\ 0 & \alpha_3' & * \\ 0 & * & \alpha_3'' \end{pmatrix}.$$

Then $[A_2, B_3]=0, [A_3, B_2]=0$ and hence H is elementary.

Now we consider the case in which we assume:

$$B_2=\begin{pmatrix} \beta_2 & 0 & 0 \\ 0 & \beta_2' & 0 \\ 0 & 0 & \beta_2 \end{pmatrix}, \quad B_3=\begin{pmatrix} \beta_3 & 0 & 0 \\ 0 & \beta_3 & 0 \\ 0 & 0 & \beta_3' \end{pmatrix}.$$

Since $[A_i, B_i]=0$ $(i=2, 3)$ we see

$$A_2=\begin{pmatrix} \alpha_2 & 0 & e \\ 0 & \alpha_2' & 0 \\ f & 0 & \alpha_2'' \end{pmatrix},\quad A_3=\begin{pmatrix} \alpha_3 & g & 0 \\ h & \alpha_3' & 0 \\ 0 & 0 & \alpha_3'' \end{pmatrix}.$$

By $[A_2, A_3]=0$ we get

$$\alpha_2 g=\alpha_2' g,\quad \alpha_2 h=\alpha_2' h,$$
$$\alpha_3 e=\alpha_3'' e,\quad \alpha_3 f=\alpha_3'' f,$$
$$eh=0,\qquad fg=0.$$

There are 4 cases $(\mathrm{i}_1)\sim(\mathrm{i}_4)$ to check.

(i_1) $e=f=g=h=0$: Then H is elementary.

(i_2) $e\neq 0, f=g=h=0$ (*resp.* $f\neq 0, e=g=h=0$ or $g\neq 0, e=f=h=0$ or $h\neq 0, e=f=g=0$): Then we see

$$A_2=\begin{pmatrix} \alpha_2 & 0 & e \\ 0 & \alpha_2' & 0 \\ 0 & 0 & \alpha_2'' \end{pmatrix},\quad A_3=\begin{pmatrix} \alpha_3 & 0 & 0 \\ 0 & \alpha_3' & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}.$$

By the assumption, we have $\alpha_3\neq\alpha_3'$. If $\alpha_2=\alpha_2'\neq\alpha_2''$ (*resp.* $\alpha_2'=\alpha_2''\neq\alpha_2$), then $A_i, B_i\,(i=2,3)$ reduce to

$$A_2=\begin{pmatrix} 0 & 0 & e \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_2'' \end{pmatrix},\quad A_3=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_3' & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta_2' & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad B_3=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta_3' \end{pmatrix}.$$

These are simultaneously similar to

$$A_2=\begin{pmatrix} 0 & e & 0 \\ 0 & \alpha_2'' & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad A_3=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_3' \end{pmatrix},$$

$$B_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta_2' \end{pmatrix},\quad B_3=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta_3' & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus H is decomposable.

If $\alpha_2=\alpha_2'=\alpha_2''$, then one can show that H is decomposable in a similar way.

(i_3) $eg\neq 0, f=h=0$: Then we see

$$A_2=\begin{pmatrix} \alpha_2 & 0 & e \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_2'' \end{pmatrix},\quad A_3=\begin{pmatrix} \alpha_3 & g & 0 \\ 0 & \alpha_3' & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}.$$

These reduce to

$$A_2=\begin{pmatrix} 0 & 0 & e \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_2''-\alpha_2 \end{pmatrix},\quad A_3=\begin{pmatrix} 0 & g & 0 \\ 0 & \alpha_3'-\alpha_3 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta_2'-\beta_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad B_3=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta_3'-\beta_3 \end{pmatrix}.$$

These are simultaneously similar to

$$A_2=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_2''-\alpha_2 \end{pmatrix},\quad A_3=\begin{pmatrix} 0 & 1 & 0 \\ 0 & \alpha_3'-\alpha_3 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta_2'-\beta_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad B_3=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta_3'-\beta_3 \end{pmatrix}.$$

We have $\sigma H=(A_1, A_3, A_2, B_1, B_3, B_2)$. Thus these reduce to the (G_1) case.

(i_4) $fh\neq 0, e=g=0$: In the same way as above, this case reduces to the (G_2) case.

(ii) The case of $c\neq 0$: Since B_2, B_3 satisfy (R), if we have $\beta_2'\neq\beta_2''$ and $\beta_3'\neq\beta_3''$, then the case reduces to the case (i). The case $\beta_2'=\beta_2''$ and $\beta_3'=\beta_3''$ does not occur because B_2 and B_3 satisfy (R).

Case I-2: We suppose that B_2 satisfies (R'). Then we have H is elementary or decomposable by (3.1), (3.2) and (I).

II: The case of $B=\begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$

By $[B, B_2]=0, [B, B_3]=0$ and (3.1), we have

$$B_2=\begin{pmatrix}\beta_2 & b & c\\ 0 & \beta_2 & 0\\ 0 & d & \beta_2'\end{pmatrix},\quad B_3=\begin{pmatrix}\beta_3 & 1-b & -c\\ 0 & \beta_3 & 0\\ 0 & -d & \beta_3'\end{pmatrix}.$$

Case II-1 : We suppose that B_2 and B_3 satisfy (R). Then

$$(\beta_2'-\beta_2)b=cd,\quad (\beta_3'-\beta_3)(1-b)=cd.$$

By (3.1), we get $\beta_2+\beta_3=\beta_2'+\beta_3'$. Hence $\beta_2'=\beta_2$, $\beta_3'=\beta_3$, $cd=0$. We see

$$B_2=\begin{pmatrix}\beta_2 & b & c\\ 0 & \beta_2 & 0\\ 0 & d & \beta_2\end{pmatrix},\quad B_3=\begin{pmatrix}\beta_3 & 1-b & -c\\ 0 & \beta_3 & 0\\ 0 & -d & \beta_3\end{pmatrix}.$$

Put

$$T=\begin{pmatrix}t & s_1 & s_2\\ 0 & t & 0\\ 0 & s_3 & t\end{pmatrix}\quad (t\neq 0),$$

so that

$$TB_2T^{-1}=\begin{pmatrix}\beta_3 & b-\frac{s_2}{t}d-\frac{s_3}{t}c & c\\ 0 & \beta_3 & 0\\ 0 & d & \beta_3\end{pmatrix}.$$

Hence we may suppose $b=0$.

(i) : If $c=0$, $d\neq 0$, then by $[A_i, B_i]=0$ $(i=2, 3)$ we have

$$B_2=\begin{pmatrix}\beta_2 & 0 & 0\\ 0 & \beta_2 & 0\\ 0 & d & \beta_2\end{pmatrix},\quad B_3=\begin{pmatrix}\beta_3 & 1 & 0\\ 0 & \beta_3 & 0\\ 0 & -d & \beta_3\end{pmatrix},$$

$$A_2=\begin{pmatrix}\alpha_2 & x & 0\\ 0 & \alpha_2' & 0\\ y & z & \alpha_2'\end{pmatrix},\quad A_3=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ 0 & \alpha_{22} & 0\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{pmatrix}.$$

These reduce to

$$B_2=\begin{pmatrix}0&0&0\\0&0&0\\0&d&0\end{pmatrix},\quad B_3=\begin{pmatrix}0&1&0\\0&0&0\\0&-d&0\end{pmatrix},$$

$$A_2=\begin{pmatrix}a_2&x&0\\0&0&0\\y&z&0\end{pmatrix},\quad A_3=\begin{pmatrix}da_{13}&a_{12}&a_{13}\\0&0&0\\a_{31}&a_{32}&\frac{a_{31}}{d}\end{pmatrix}.$$

By $[A_2, A_3]=0$ we get

$$\begin{cases} ya_{13}=0, & (a_2+\frac{y}{d})a_{31}=yda_{13},\\ a_2a_{13}=0, & (x+\frac{z}{d})a_{31}=ya_{12},\\ a_2a_{12}=(xd+z)a_{13}. & \end{cases}$$

(i_1) If $a_{13}\neq 0$, then $y=a_2=0$ and $xd+z=0$. Then the nilpotent matrices

$$N_2=\begin{pmatrix}0&0&0\\0&0&0\\0&d&0\end{pmatrix},\quad N_3=\begin{pmatrix}0&0&0\\0&0&0\\0&-d&0\end{pmatrix}$$

are commtative and commute with every element of H and $N_2+N_3=0$. Thus we see B_2 is redundant.

(i_2) If $a_{13}=0$, then $a_2a_{12}=0$, $(a_2+\frac{y}{d})a^{31}=0$, $(x+\frac{z}{d})a_{31}=ya_{12}$. When $ya_{12}\neq 0$, we get $a_2=a_{31}=0$. These lead to a contradiction. Thus $ya_{12}=0$.

If $y=0$, then $a_2a_{31}=0$, that is $a_2=0$ or $a_{31}=0$. In both of these cases the singular set is redundant.

If $y\neq 0$, then $a_{12}=0$. When $a_2=0$, we see $a_{31}=0$. In the same way as in the case (i_1) we have A_3 is redundant. When $a_2\neq 0$, we may assume that $a_{31}\neq 0$ because B_2 is redundant if $a_{31}=0$. Then A_i, B_i $(i=2, 3)$ are simultaneously similar to

$$B_2=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix},\quad B_3=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},$$

$$A_2=\begin{pmatrix}0&0&0\\0&a_2&0\\0&0&0\end{pmatrix},\quad A_3=\begin{pmatrix}0&0&0\\0&0&0\\0&0&\frac{a_{31}}{d}\end{pmatrix}.$$

Since we have $\tau_{23}\sigma H=(A_1, B_3, B_2, B_1, A_3, A_2)$, these are nothing but the (G_2) case with $1-\gamma=1$

$-\gamma'=0$.

(ii): If $d=0$, $c\neq0$, then we can classify in the same way as above, in which (G_1) case appears.

Case II-2: We suppose that B_2 satisfies (R) and B_3 does not satisfy (R). Then by $[B_2, B_3]=0$ we have

$$B_2=\begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & d & \beta_2 \end{pmatrix}, \quad B_3=\begin{pmatrix} 0 & 1-b & -c \\ 0 & 0 & 0 \\ 0 & -d & -\beta_2 \end{pmatrix}.$$

(i) $c=d=0$: By the assumption we have $\beta_2 b=0, \beta_2(1-b)\neq0$. Hence we see $\beta_2\neq0, b=0$. Then H is decomposable by $[A_i, B_i]=0$ $(i=2, 3)$.

(ii) $c=0, d\neq0$: By the assumption we have $\beta_2 b=0, \beta_2(1-b)\neq0$. Hence $\beta_2\neq0, b=0$. Then A_i, B_i $(i=2, 3)$ are simultaneously similar to

$$B_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta_2 \end{pmatrix}, \quad B_3=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\beta_2 \end{pmatrix},$$

$$A_2=\begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \end{pmatrix}, \quad A_3=\begin{pmatrix} 0 & \alpha'_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\beta_2}{d}\alpha'_{32} \end{pmatrix}.$$

by $[A_i, B_i]=0$ $(i=2, 3)$. From $[A_2, A_3]=0$, we see H is elementary or decomposable.

(iii) $c\neq0, d=0$: In the same way as above, we see H is decomposable.

(iv) $cd\neq0$: By the assumption we have $\beta_2 b-cd=0, -\beta_2(1-b)-cd\neq0$. Hence $\beta_2=\frac{cd}{b}\neq0$, $b\neq0$. Then B_2, B_3 are simultaneously similar to

$$B_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{cd}{b} \end{pmatrix}, \quad B_3=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{cd}{b} \end{pmatrix}.$$

By $[A_i, B_i]=0$ $(i=2, 3)$ and $[A_2, A_3]=0$, we have H is elementary or decomposable.

Case II-3: We suppose that both B_2 and B_3 don't satisfy (R). Then by $[B_2, B_3]=0$ we have

$$B_2=\begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & d & \beta_2 \end{pmatrix}, \quad B_3=\begin{pmatrix} 0 & 1-b & -c \\ 0 & 0 & 0 \\ 0 & -d & -\beta_2 \end{pmatrix}.$$

(i) $c=d=0$: We have $\beta_2 b\neq0, \beta_2(1-b)\neq0$ by the assumption. Then H is decomposable by

$[A_i, B_i]=0\ (i=2, 3)$.

(ii) $c=0, d\neq 0$ (*resp.* $c\neq 0, d=0$): By the assumption we have $\beta_2 b\neq 0, \beta_2(1-b)\neq 0$. Then in the same way as in (ii) of Case II-2 we see H is decomposable.

(iii) $cd\neq 0$: By the assumption we have $\beta_2 b-cd\neq 0, -\beta_2(1-b)-cd\neq 0$. We may assume $\beta_2=0, b=0$. By $[A_i, B_i]=0\ (i=2, 3)$ we have

$$B_2=\begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & d & 0 \end{pmatrix}, \qquad B_3=\begin{pmatrix} 0 & 1 & -c \\ 0 & 0 & 0 \\ 0 & -d & 0 \end{pmatrix},$$

$$A_2=\begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} \\ 0 & 0 & 0 \\ 0 & \frac{d}{c}\alpha_{13} & 0 \end{pmatrix}, \quad A_3=\begin{pmatrix} 0 & \alpha'_{12} & \alpha'_{13} \\ 0 & 0 & 0 \\ 0 & \frac{d}{c}\alpha'_{13} & 0 \end{pmatrix}.$$

Then H is elementary.

If $\beta_2\neq 0, b(1-b)\neq 0$, then by $[A_i, B_i]=0\ (i=2, 3)$, we see that H reduces to

$$B_2=\begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & d & \beta_2 \end{pmatrix}, \qquad B_3=\begin{pmatrix} 0 & 1-b & -c \\ 0 & 0 & 0 \\ 0 & -d & -\beta_2 \end{pmatrix},$$

$$A_2=\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 0 & \alpha_{22} & 0 \\ 0 & \alpha_{32} & \alpha_{33} \end{pmatrix}, \quad A_3=\begin{pmatrix} \alpha'_{11} & \alpha'_{12} & \alpha'_{13} \\ 0 & \alpha'_{22} & 0 \\ 0 & \alpha'_{32} & \alpha'_{33} \end{pmatrix},$$

where

$$\left[\begin{array}{ll} \alpha_{33}-\alpha_{11}=\frac{\beta_2}{c}\alpha_{13}, & \alpha_{33}-\alpha_{22}=\frac{\beta_2}{d}\alpha_{32}, \\ \alpha_{22}-\alpha_{11}=\frac{d}{b}\alpha_{13}-\frac{c}{b}\alpha_{32}, & \alpha'_{33}-\alpha'_{11}=\frac{\beta_2}{c}\alpha'_{13}, \\ \alpha'_{33}-\alpha'_{22}=\frac{\beta_2}{d}\alpha'_{32}, & \alpha'_{22}-\alpha'_{11}=\frac{d}{b-1}\alpha'_{13}-\frac{c}{b-1}\alpha'_{32}. \end{array}\right. \tag{3.6}$$

A_2 and A_3 reduce to

$$A_2=\begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} \\ 0 & \alpha_{22}-\alpha_{11} & 0 \\ 0 & \alpha_{32} & \alpha_{33}-\alpha_{11} \end{pmatrix}, \quad A_3=\begin{pmatrix} 0 & \alpha'_{12} & \alpha'_{13} \\ 0 & \alpha'_{22}-\alpha'_{11} & 0 \\ 0 & \alpha'_{32} & \alpha'_{33}-\alpha'_{11} \end{pmatrix}.$$

By (3.6) and (I), these are simultaneously similar to

$$B_2=\begin{pmatrix} 0 & b-\frac{c}{\beta_2}d & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta_2 \end{pmatrix}, \qquad B_3=\begin{pmatrix} 0 & 1-b+\frac{c}{\beta_2}d & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\beta_2 \end{pmatrix},$$

$$A_2=\begin{pmatrix} 0 & \alpha_{12}-\frac{c}{\beta_2}\alpha_{32} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_{33}-\alpha_{11} \end{pmatrix}, \quad A_3=\begin{pmatrix} 0 & \alpha'_{12}-\frac{c}{\beta_2}\alpha'_{32} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha'_{33}-\alpha'_{11} \end{pmatrix}.$$

Thus H is decomposable.

In the same way as above, if $\beta_2\neq 0$, $b=0$, then H is decomposable by $[A_i, B_i]=0\ (i=2, 3)$.

III: The case of $B=\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ $a\neq b,\ b\neq c,\ c\neq a$

If both B_2 and B_3 don't satisfy (R) or B_2 (*resp.* B_3) doesn't satisfy (R), then H is elementary or decomposable by the relation (I). We suppose that both B_2 and B_3 satisfy (R). By $[B, B_i]=0\ (i=2, 3)$ and (3.1), we have

$$B_2=\begin{pmatrix} \beta'_2 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_2 \end{pmatrix}, \quad B_3=\begin{pmatrix} \beta_3 & 0 & 0 \\ 0 & \beta'_3 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}.$$

By $[A_i, B_i]=0\ (i=2, 3)$, we have

$$A_2=\begin{pmatrix} \alpha_2 & 0 & 0 \\ 0 & w_2 & x_2 \\ 0 & y_2 & z_2 \end{pmatrix}, \quad A_3=\begin{pmatrix} w_3 & 0 & x_3 \\ 0 & \alpha_3 & 0 \\ y_3 & 0 & z_3 \end{pmatrix}.$$

By (3.2), we get

$$\begin{aligned} &x_2y_3=0, && x_3y_2=0, \\ &\alpha_3x_2=z_3x_2, && \alpha_2x_3=z_2x_3, \\ &\alpha_3y_2=z_3y_2, && \alpha_2y_3=z_2y_3. \end{aligned}$$

If $x_2=0$, $y_2=0$ or $x_3=0$, $y_3=0$, then H is decomposable.

If $x_2=0$, $x_3=0$, $y_2y_3\neq 0$, then A_i, B_i $(i=2, 3)$ reduce to

$$B_2=\begin{pmatrix}\beta_2'-\beta_2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\qquad B_3=\begin{pmatrix}0 & 0 & 0\\ 0 & \beta_3'-\beta_3 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$A_2=\begin{pmatrix}0 & 0 & 0\\ 0 & w_2-\alpha_2 & 0\\ 0 & y_2 & 0\end{pmatrix},\qquad A_3=\begin{pmatrix}w_3-\alpha_3 & 0 & 0\\ 0 & 0 & 0\\ y_3 & 0 & 0\end{pmatrix}.$$

These are simultaneously similar to

$$B_2=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \beta_2'-\beta_2\end{pmatrix},\qquad B_3=\begin{pmatrix}0 & 0 & 0\\ 0 & \beta_3'-\beta_3 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$A_2=\begin{pmatrix}0 & 1 & 0\\ 0 & w_2-\alpha_2 & 0\\ 0 & 0 & 0\end{pmatrix},\qquad A_3=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & w_3-\alpha_3\end{pmatrix}.$$

Thus these reduce to the (G_1) case.

If $y_2=0$, $y_3=0$, $x_2x_3\neq 0$, then we see that these reduce to the (G_2) case.

IV: The case of $B=\begin{pmatrix}a & 1 & 0\\ 0 & a & 0\\ 0 & 0 & b\end{pmatrix}$ $a\neq b$

By $[B, B_i]=0$ $(i=2, 3)$ we have

$$B_2=\begin{pmatrix}\beta_2 & m_2 & 0\\ 0 & \beta_2 & 0\\ 0 & 0 & \beta_2'\end{pmatrix},\quad B_3=\begin{pmatrix}\beta_3 & m_3 & 0\\ 0 & \beta_3 & 0\\ 0 & 0 & \beta_3'\end{pmatrix}.$$

Case IV-1. $m_2m_3\neq 0$: If $\beta_2\neq\beta_2'$, $\beta_3\neq\beta_3'$, then H is elementary by $[A_i, B_i]=0$ $(i=2, 3)$ and (3.2).

If $\beta_2=\beta_2'$, $\beta_3\neq\beta_3'$, then by $[A_i, B_i]=0$ $(i=2, 3)$, we have

$$A_2=\begin{pmatrix}\alpha_2 & n_2 & n_2'\\ 0 & \alpha_2 & 0\\ 0 & n_2'' & \alpha_2'\end{pmatrix},\quad A_3=\begin{pmatrix}\alpha_3 & n_3 & 0\\ 0 & \alpha_3 & 0\\ 0 & 0 & \alpha_3'\end{pmatrix}.$$

Put

$$N_i=\begin{pmatrix}0 & n_i & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad M_j=\begin{pmatrix}0 & m_j & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\quad (i, j=2, 3).$$

These are nilpotent, commutative and commute with every element of H. Thus we get B_2 is redundant.

Case IV-2. $m_2=1$, $m_3=0$: If $\beta_2\neq\beta_2'$, $\beta_3\neq\beta_3'$, then H is decomposable by $[A_i, B_i]=0\ (i=2, 3)$.

If $\beta_2=\beta_2'$, $\beta_3\neq\beta_3'$, then by $[A_i, B_i]=0\ (i=2, 3)$, we have

$$A_2=\begin{pmatrix}\alpha_2 & n_2 & n_2'\\ 0 & \alpha_2 & 0\\ 0 & n_2'' & \alpha_2'\end{pmatrix},\quad A_3=\begin{pmatrix}x & y & 0\\ z & w & 0\\ 0 & 0 & v\end{pmatrix}.$$

By $[A_2, A_3]=0$, we also have

$$\begin{aligned} &zn_2=0, && xn_2=wn_2,\\ &zn_2'=0, && xn_2'=vn_2',\\ &zn_2''=0, && wn_2''=vn_2''. \end{aligned}$$

If $z\neq 0$, we get $n_2=n_2'=n_2''$ and H is decomposable.

We suppose that $z=0$. If $n_2n_2'n_2''\neq 0$, then $x=w=v$. In the same way as above, the singular set is redundant.

If only one of n_2, n_2' and n_2'' are zero or $n_2\neq 0$, $n_2'=n_2''=0$, then the singular set is redundant in the same way.

If $n_2=n_2'=0$, $n_2''\neq 0$, we get $w=v$. Hence A_i, B_i $(i=2, 3)$ reduce to

$$B_2=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\qquad B_3=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \beta_3'-\beta_3\end{pmatrix},$$

$$A_2=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & n_2'' & \alpha_2'-\alpha_2\end{pmatrix},\quad A_3=\begin{pmatrix}x-w & y & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

These are simultaneously similar to

$$B_2=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},\qquad B_3=\begin{pmatrix}0&0&0\\0&0&0\\0&0&\beta_3'-\beta_3\end{pmatrix},$$

$$A_2=\begin{pmatrix}0&0&0\\0&0&0\\1&0&\alpha_2'-\alpha_2\end{pmatrix},\quad A_3=\begin{pmatrix}0&0&0\\0&x-w&0\\0&0&0\end{pmatrix}.$$

If we transform y to $y-1$ in the system (3.5), then A_3 becomes B_2. Since $(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{B}_1, \tilde{B}_2, \tilde{B}_3) := \sigma H=(A_1, A_3, A_2, B_1, B_3, B_2)$, these reduce to the (G_2) case with $1-\gamma=0$.

Similarly, if $n_2=n_2''=0$, $n_2'\neq 0$, we observe that these reduce to the (G_1) case with $1-\gamma=0$.

V: The case of $B=\begin{pmatrix}a&1&0\\0&a&1\\0&0&a\end{pmatrix}$

By $[B,B_i]=0$ $(i=2, 3)$, we have

$$B_2=\begin{pmatrix}\beta_2&m_2&m_2'\\0&\beta_2&m_2\\0&0&\beta_2\end{pmatrix},\quad B_3=\begin{pmatrix}\beta_3&m_3&m_3'\\0&\beta_3&m_3\\0&0&\beta_3\end{pmatrix}.$$

Case V-1. $m_2m_3\neq 0$: We see H is elementary by $[A_i, B_i]=0$ $(i=2, 3)$.

Case V-2. $m_2=0$ (*reap.* $m_3=0$): We have

$$B_2=\begin{pmatrix}\beta_2&0&m_2'\\0&\beta_2&0\\0&0&\beta_2\end{pmatrix},\quad B_3=\begin{pmatrix}\beta_3&1&m_3'\\0&\beta_3&1\\0&0&\beta_3\end{pmatrix}.$$

By $[A_i, B_i]=0$ $(i=2, 3)$, we see

$$A_2=\begin{pmatrix}\alpha_2&n_2&n_2'\\0&\alpha_2'&n_2''\\0&0&\alpha_2\end{pmatrix},\quad A_3=\begin{pmatrix}\alpha_3&n_3&n_3'\\0&\alpha_3&n_3\\0&0&\alpha_3\end{pmatrix}.$$

By (3.2), it holds that $n_3\alpha_2'=n_3\alpha_2$ and $n_2n_3=n_2''n_3$.

If $n_3\neq 0$, then $\alpha_2'=\alpha_2$, $n_2=n_2''$. Thus this case reduces to Case V-1.

If $n_3=0$, then it is easy to see the singular set is redundant.

VI: The case of $B=\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$

Then by transformation of solution we may suppose $B=0$. Hence $B=B_2+B_3=0$ or $B_2=-B_3$. From the condition (I) we get $[A_2, B_3]=0, [A_3, B_2]=0$. Therefore every element of H is commutative with each other. Thus H is elementary. □

3.3. The case that the singular set is redundant in two elements of 6-tuple.

We consider the case that the singular set is redundant in the 5-tuple of **section 3.2.** It is sufficient to classify 4-tuples.

Theorem 3.3. *Assume that the* 6*-tuple H satisfies (I) and (S). If the singular set is redundant in two elements of H, then H is decomposable or elementary, or the singular set is redundant in three elements of H.*

Proof. It suffices to consider the cases of $A_1=B_1=0$ or $A_2=B_1=0$. If $A_1=B_1=0$, then every element of H is commutative with each other. Thus H is elementary. We suppose that $A_2=B_1=0$. Then by (I) and (S) we have

$$[A_3, B_3]=0, [B_2, B_3]=0, [A_1, B_3]=0, A_1+A_3+B_2+B_3=0. \tag{3.7}$$

Hence B_3 is commutative with A_1, A_3, B_2 and it suffices to consider the following 5 cases I~V of B_3.

$$\mathrm{I}: \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}, \quad \mathrm{II}: \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix},$$

$$\mathrm{III}: \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad \mathrm{IV}: \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}, \quad \mathrm{V}: \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}.$$

We see that B_3 is redundant in the case II. The other cases are easily seen to be elementary or decomposable by virtue of (3.7). □

Remark 3.4. In the case II above in which the singular set is redundant, the resulting Pfaffian system can be integrated using the Gaussian hypergeometric function.

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