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A characterization of ${}^2E_6(q)$

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Abstract

The order of every finite group G can be expressed as a product of coprime positive integers $m_1,\dots,m_t$ such that $\pi(m_i)$ is a connected component of the prime graph of G. The integers $m_1,\dots,m_t$ are called the order components of G. It is known that some non-abelian simple groups are uniquely determined by their order components. As the main result of this paper, we show that groups ${}^2E_6(q)$ are also uniquely determined by their order components. As corollaries of this result, the validity of a conjecture of J. G. Thompson and a conjecture of W. Shi and J. Bi both on ${}^2E_6(q)$ is obtained.

1. Introduction

If n is an integer, then $\pi(n)$ is the set of prime divisors of n and if G is a finite group then $\pi(G)$ is defined to be $\pi(|G|)$. *The prime graph* $\Gamma(G)$ of a group G is a graph whose vertex set is $\pi(G)$, and two distinct primes p and q are linked by an edge if and only if G contains an element of order pq. Let π_i, $i=1,2,\dots, t(\Gamma(G))$ be the connected components of $\Gamma(G)$. For $|G|$ even, π_1 will be the connected component containing 2. Then $|G|$ can be expressed as a product of some positive integers m_i, $i=1,2,\dots, t(\Gamma(G))$ with $\pi(m_i)=\pi_i$. The integers m_i's are called the order components of G. The set of order components of G will be denoted by $OC(G)$. If the order of G is even, we will assume that m_1 is the even order component and $m_2,\dots, m_{t(\Gamma(G))}$ will be the odd order components of G. The order components of non-abelian simple groups having at least three prime graph components are obtained by G. Y. Chen [8, Tables 1, 2, 3]. Similarly the order components of non-abelian simple groups with two order components can be obtained by using the tables in [10, 14, 15, 22]. The following groups are uniquely determined by their order components: $G_2(q)$ where $q\equiv 0 \pmod 3$ [2], Sporadic simple groups [3], Suzuki-Ree groups [6], $E_8(q)$ [7], $PSL_2(q)$ [8], $PSL_3(q)$ where q is an odd prime power [12], $PSL_3(q)$ where $q=2^n$ [13] *and* $F_4(q)$ where q is even [11]. In this paper, we prove that ${}^2E_6(q)$

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are also uniquely determined by their order components, that is we have:

The Main Theorem. *Let G be a finite group and* $M={}^2E_6(q)$. *Then* $OC(G)=OC(M)$ *if and only if* $G\cong M$.

2. Preliminary results

In order to prove the main theorem, first we bring some lemmas.

Definition 2.1. ([9]) A finite group G is called a 2-Frobenius group if it has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K and G/H are Frobenius groups with kernels H and K/H, respectively.

Lemma 2.2. ([22, Theorem A]) *If G is a finite group with its prime graph having more than one component, then G is one of the following groups:*

(a) *a Frobenius or 2-Frobenius group;*

(b) *a simple group;*

(c) *an extension of a* π_1*-group by a simple group;*

(d) *an extension of a simple group by a* π_1*-solvable group;*

(e) *an extension of a* π_1*-group by a simple group by a* π_1*-group.*

Lemma 2.3. ([22, Lemma 3]) *Suppose that G is a non-solvable group and not a Frobenius group. If G is a finite group with more than one prime graph component and has a normal series* $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ *such that H and G/K are* π_1*-groups and K/H is simple, then H is a nilpotent group.*

The next lemma follows from Theorem 2 in [1]:

Lemma 2.4. ([1]) *Let G be a Frobenius group of even order and H, K be Frobenius complement and Frobenius kernel of G, respectively. Then* $t(\Gamma(G))=2$, *and the prime graph components of G are* $\pi(H)$, $\pi(K)$ *and G has one of the following structures:*

(a) $2\in\pi(K)$ and all Sylow subgroups of H are cyclic.

(b) $2\in\pi(H)$, K is an abelian group, H is a solvable group, the Sylow subgroups of odd order of H are cyclic groups and the 2-Sylow subgroups of H are cyclic or generalized quaternion groups.

(c) $2\in\pi(H)$, K is an abelian group and there exists $H_0\leq H$ such that $|H:H_0|\leq 2$, $H_0=Z\times$

$SL(2, 5)$, $(|Z|, 2. 3. 5)=1$ and the Sylow subgroups of Z are cyclic.

The next lemma follows from Theorem 2 in [1] and Lemma 2. 3:

Lemma 2.5. ([1]) *Let G be a 2-Frobenius group of even order. Then $t(\Gamma(G))=2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that*

(a) $\pi_1=\pi(G/K) \cup \pi(H)$ and $\pi(K/H)=\pi_2$;

(b) *G/K and K/H are cyclic, $|G/K|$ divides $|Aut\ (K/H)|$, $(|G/K|,|K/H|)=1$ and $|G/K|<|K/H|$;*

(c) *H is nilpotent and G is a solvable group.*

Lemma 2.6. ([5, Lemma 8]) *Let G be a finite group with $t(\Gamma(G))\geq 2$ and let N be a normal subgroup of G. If N is a π_i-group for some prime graph component of G and $m_1,m_2,\ldots, m_r$ are some order components of G but not π_i-numbers, then $m_1\ m_2 \ldots m_r$ is a divisor of $|N|-1$.*

Lemma 2.7. ([4, Lemma 1.4]) Suppose G and M are two finite groups satisfying $t(\Gamma(M))\geq 2$, $N(G)=N(M)$, where $N(G)=\{n|G$ has a conjugacy class of size n$\}$, and $Z(G)=1$. Then $|G|=|M|$.

The next lemma follows from Lemma 1.5 in [4].

Lemma 2.8. *Let G_1 and G_2 be finite groups satisfying $|G_1|=|G_2|$ and $N(G_1)=N(G_2)$. Then $t(\Gamma(G_1))=t(\Gamma(G_2))$ and $OC(G_1)=OC(G_2)$.*

Lemma 2.9. *Let G be a finite group and M is a non-abelian simple group with $t(\Gamma(M))=2$ satisfying $OC(G)=OC(M)$,* then:

(1) *Let $|M|=m_1\ m_2$, $OC(M)=\{m_1,m_2\}$,* and *$\pi(m_i)=\pi_i$ for $i=1$ or 2. Then $|G|=m_1\ m_2$ and one of the following holds:*

(a) *G is a Frobenius or 2-Frobenius group;*

(b) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/K is a π_1-group, H is a nilpotent π_1-group, and K/H is a non-abelian simple group. Moreover $OC(K/H)=\{m'_1,m'_2,\ldots,m'_s,m_2\}$, $|K/H|=m'_1m'_2\ldots m'_sm_2$* and *$m'_1m'_2\ldots m'_s|m_1$* where *$\pi(m_j')=\pi_j'$, $1\leq j\leq s$.* Note that *$\pi_j',1\leq j\leq s$ are the connected components* of *$\Gamma(K/H)$.*

(2) $|G/K|\,|\,|$ Out $(K/H)|$.

Proof. (1) follows from the above lemmas. Since $t(\Gamma(G))\geq 2$, we have $t(\Gamma(G/H))\geq 2$. Otherwise $t(\Gamma(G/H))=1$ and hence $t(\Gamma(G))=1$, since H is a π_1-group, a contradiction.

Moreover we have $Z(G/H)=1$. For any $xH \in G/H$ and $xH \notin K/H$, xH induces an automorphism of K/H and this automorphism is trivial if and only if $xH \in Z(G/H)$. Therefore $G/K \leq Out\ (K/H)$ for $Z(G/H)=1$ and (2) follows. □

Lemma 2.10. Let $M={}^2E_6(q)$. *Suppose* $D(q)=(q^6-q^3+1)/(3,q+1)$. *Then*:

(a) *If* $p \in \pi(M)$, *then* $|S_p| \leq q^{36}$ *where* $S_p \in Syl_p(M)$;

(b) *If* $q>2$, $p \in \pi_1(M)$, $p^\alpha \| |M|$ *and* $p^\alpha - 1 \equiv 0 (\text{mod } D(q))$, *then* $p^\alpha = q^{18}$ or q^{36};

(c) *If* $q>2$, $p \in \pi_1(M)$, $p^\alpha \| |M|$ and $p^\alpha + 1 \equiv 0 (\text{mod } D(q))$ *then* $p^\alpha = q^9$ or q^{27}.

Proof. (a) Observe that $|M| = q^{36}(q-1)^4(q+1)^6(q^2+q+1)^2(q^2-q+1)^3(q^4-q^2+1)(q^4+1)(q^2+1)^2(q^4-q^3+q^2-q+1)\frac{q^6-q^3+1}{(3,q+1)}$. Now let p be a prime number, such that $p^\alpha \| |M|$. Since q is coprime with respect to other factors of $|M|$, one of the possibilities of p^α is $p^\alpha | q^{36}$. Also we have $(q+1, q-1)|2$, $(q+1, q^2+q+1)=1$, $(q+1, q^2-q+1)|3$, $(q+1, q^2+1)|2$, $(q+1, q^4-q^2+1)=1$, $(q+1, q^4+1)|2$ and $(q+1, q^4-q^3+q^2-q+1)|5$, therefore another possibility of p^α is $p^\alpha | 5\times 3^3 \times 2^7(q+1)^6$. By using this method we can see that p^α divides q^{36}, $2^9\times 3^2(q-1)^4$, $5\times 3^3\times 2^7(q+1)^6$, $3\times 2^{11}(q^2+1)^2$, $3^6(q^2-q+1)^3$, $3^4(q^2+q+1)^2$, $2^{12}(q^4+1)$, $3^2(q^4-q^2+1)$ or $5^6(q^4-q^3+q^2-q+1)$. Hence (a) follows.

(b) Let $p \in \pi_1(M)$, $p^\alpha \| |M|$ and $p^\alpha - 1 \equiv 0(\text{mod } D(q))$. Obviously $p^\alpha > D(q)$. Now we consider two cases:

Case 1. $3 \nmid q+1$. Hence $D(q)=q^6-q^3+1$. If p^α does not divide q^{36}, then we must consider every possibility of p^α which presented in the proof of part (a). Since they are similar, for convenience we consider only a few of them.

First note that if $q \leq 19$ then numerical calculations show that (b) holds. So let $q>19$.

If $p^\alpha | 3^4(q^2+q+1)^2$, then for $q<13$ numerical calculations show that there exist no p^α such that $p^\alpha | 3^4(q^2+q+1)^2$, and $p^\alpha - 1 \equiv 0(\text{mod } D(q))$. If $q \geq 13$ then $3^4(q^2+q+1)^2 < D(q)$, but $p^\alpha - 1 \equiv 0(\text{mod } D(q))$ and so $D(q) < p^\alpha$ which is impossible.

If $p^\alpha | 5\times 3^3 \times 2^7(q+1)^6$ then p^α divides $5(q+1)^6$, $3^3(q+1)^6$ or $2^7(q+1)^6$. If $p^\alpha | 5(q+1)^6$ then $p^\alpha = \frac{5(q+1)^6}{s}$, for some $s>0$. Also $\frac{5(q+1)^6}{s} - 1 = t.D(q)$, for some $t>0$. So $t.D(q) < \frac{5(q+1)^6}{s}$, which implies that $D(q) < \frac{5(q+1)^6}{st}$. But for $q>9$ we have $\frac{5(q+1)^6}{10} < D(q)$, and hence $st<10$. Since p^α is a power of a prime number, s can be equal to 1 or 5. But then $\frac{5(q+1)^6}{s} - 1 = t.D(q)$ is not satisfied for $st<10$, which is a contradiction. If $p^\alpha | 3^3(q+1)^6$ we can proceed similarly and get a contradiction. We must note that by using a Mathematical software, for example Maple, easily we can get a contradiction. In fact if $p^\alpha | 3^3(q+1)^6$ then $p^\alpha = 3^3(q+1)^6/s$ for some $s>0$. Also $p^\alpha - 1 = t.D(q)$, which implies that $t.D(q) < p^\alpha$ and hence $D(q) < \frac{3^3(q+1)^6}{st}$. But for $q \geq 19$ we have $\frac{3^3(q+1)^6}{60} < D(q)$ which implies that $st<60$. Hence $p^\alpha = \frac{3^3(q+1)^6}{s}$ and $\frac{3^3(q+1)^6}{s} - 1 = t(q^6-q^3+1)$ where $st<60$. Now by a simple program in Maple we can see that there is no p^α

which satisfies these equations. If $p^a|2^7(q+1)^6$ then we proceed similarly.

This method can be used for another cases and hence we have $p^a|q^{36}$.

If p^a divides q^{36}, then $q=p^n$ for some $n>0$. Also since $p^a>D(q)$, we have $q^3|p^a$. Let $p^r|q^9$ and $p^r+1\equiv 0(\text{mod } D(q))$. In this case $p^r+1=s.D(q)=s(q^6-q^3+1)$ where $1\leq s\leq q^3+1$, and hence $q^3|s-1$. Therefore $s=1$ or $s=q^3+1$. Obviously $s\neq 1$ and hence $p^r=q^9$.

Now if $p^a-1=s.D(q)$ then $q^3|s+1$, similarly. If $p^a\leq q^9$ then $1\leq s<q^3+1$ and $q^3|s+1$, which implies that $s=q^3-1$, but then $p^a-1=q^9-2q^6+2q^3-1$ which is impossible. Therefore $p^a>q^9$ and hence $p^a=q^9.p^m$ where $m>0$. Now we have

$$s.D(q)=p^a-1=p^mD(q)(q^3+1)-p^m-1,$$

which implies that $p^m+1\equiv 0(\text{mod } D(q))$. If we suppose $p^a\leq q^{18}$ then $p^m=q^9$ and hence $p^a=q^{18}$.

Now let $q^9<p^\gamma$ and $p^\gamma+1\equiv 0(\text{mod } D(q))$. Then similarly we have $p^\gamma=q^9p^k$ where $p^k\leq q^{18}$. Hence

$$s'D(q)=p^\gamma+1=p^kD(q)(q^3+1)-p^k+1,$$

which implies that $p^k=q^{18}$ and hence $p^\gamma=q^{27}$. By using this fact,if $p^a>q^{18}$ then similarly we can see that $p^a=q^{36}$.

Case 2. If $3|q+1$ then similarly we prove that (b) holds.

(c) Similar arguments show that (c) holds.

Lemma 2.11. *Let G be a finite group and $M={}^2E_6(q)$. If $OC(G)=OC(M)$ then G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. G is not a Frobenius group otherwise by Lemma 2.4 $OC(G)=\{|H|,|K|\}$ where K and H are Frobenius kernel and Frobenius complement of G, respectively. Since $|H|\,|(|K|-1)$, we have $|H|<|K|$. Therefore $2\nmid |H|$, and hence $2||K|$. Therefore $|H|=\frac{q^6-q^3+1}{(3,q+1)}, |K|=\frac{|G|}{|H|}$. Since $2^7(q+1)^6>1$, there exists a prime p such that $p^a|2^7(q+1)^6$. If P is a p-Sylow subgroup of K then since K is nilpotent, $P\lhd G$ and hence $\frac{q^6-q^3+1}{(3,q+1)}|(|P|-1)$, by Lemma 2.6, which implies that $p^a=q^{18}$ or q^{36}, by Lemma 2.10(b). But $q^{18}\nmid 2^7(q+1)^6$ and it is a contradiction. Therefore G is not a Frobenius group.

Let$G$ be a 2-Frobenius group. By Lemma 2.5 there is a normal series $1\lhd H\unlhd K\unlhd G$ such that $|K/H|=\frac{q^6-q^3+1}{(3,\ q+1)}<3^6(q^2-q+1)^3$ and $|G/K|<|K/H|$. Thus there exists a prime p such that $p|3^6(q^2-q+1)^3$ and $p||H|$. If P is a $p-$Sylow subgroup of H, since H is nilpotent, P must be a normal subgroup of K with $P\subseteq H$ and $|K|=\frac{q^6-q^3+1}{(3,q+1)}|H|$. Therefore, $\frac{q^6-q^3+1}{(3,\ q+1)}|(|P|-1)$, by Lemma 2.6,and hence $p^a-1\equiv 0(\text{mod } D(q))$, so $q^{18}||P|$, by Lemma 2.10(b), which is impossible since $q^{18}\nmid 3^6(q-q+1)^3$. □

Lemma 2.12. *Let G be a finite group and $M={}^2E_6(q)$. If* OC(G)=OC(M), *then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is a simple group. Moreover, the odd order component of M is equal to some of those of K/H, in particular, $t(\Gamma(K/H)) \geq 2$.*

Proof. The first part of the lemma follows from the above lemmas since the prime graph of M has two prime graph components. For primes p and q, if K/H has an element of order pq, then G has one. Hence, by the definition of prime graph component, the odd order component of G must be an odd order component of K/H. □

3. Some related results

As an application of the main theorem we have:

Remark 3. 1. It is a well known conjecture of J. G. Thompson that if G is a finite group with $Z(G)=1$ and M is a non-abelian simple group satisfying $N(G)=N(M)$, then $G \cong M$.

We can give a positive answer to this conjecture by our characterization of the groups under discussion.

Corollary 3.2. *Let G be a finite group with $Z(G)=1$ and $M={}^2E_6(q)$. If $N(G)=N(M)$, then $G \cong M$.*

Proof. By Lemmas 2.7 and 2.8, if G and M are two finite groups satisfying the conditions of Corollary 3.2, then OC(G)=OC(M). So the main theorem implies this corollary. □

Remark 3.3. Wujie Shi and Bi Jianxing in [19] put forward the following conjecture:

Conjecture. Let G be a group and M a finite simple group. Then $G \cong M$ if and only if

(i) $|G|=|M|$, and

(ii) $\pi_e(G)=\pi_e(M)$, where $\pi_e(G)$ denotes the set of orders of elements in G.

This conjecture is valid for sporadic simple groups [16] , alternating groups [20] , and some simple groups of Lie type [17, 18, 19] . As a consequence of the main theorem, we prove the validity of this conjecture for the groups under discussion.

Corollary 3.4. *Let G be a finite group and $M={}^2E_6(q)$. If $|G|=|M|$ and $\pi_e(G)=\pi_e(M)$, then $G \cong M$.*

Proof. By assumption we must have $OC(G)=OC(M)$, then the corollary follows by the main theorem. □

4. Proof of The Main Theorem

By Lemma 2. 12, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups,

K/H is a non-abelian simple group, $t(\Gamma(K/H))\geq 2$ and the odd order component of M is an odd order component of K/H.

For the proof of the main theorem first suppose $q=2$. In this case $t(\Gamma({}^2E_6(q)))=4$. So $t(\Gamma(K/H))\geq 4$, by Lemma 2. 12, and so K/H must be one of the following groups:

M_{22} J_1, J_4, ON, Ly, F'_{24}, F_1, $E_8(q)$, $A_2(4)$, ${}^2B_2(q)$ where $q=2^{2n+1}>2$, ${}^2E_6(2)$.

The odd order components of ${}^2E_6(2)$ are 13, 17, 19 and so K/H can be equal to ${}^2B_2(q)$, ${}^2E_6(2)$ or $E_8(q)$.

If $K/H\cong {}^2B_2(q)$ where $q=2^{2n+1}>2$, then q-1 must be equal to 13, 17 or 19 which is impossible. If $K/H\cong E_8(q)$ then since the odd order components of $E_8(q)$ are greater than 13 it is a contradiction. So $K/H\cong {}^2E_6(2)$. Now we proceed similar to step 13 of the proof.

Now we suppose $q>2$ and hence $t(\Gamma({}^2E_6(q)))=2$. In this case we proceed the proof in the following steps:

Step 1. Let $K/H\cong A_n$ where $n=p$, $p+1$, $p+2$ and $p\geq 5$ is a prime number. Then if $gcd(3,q+1)=1$ and $D(q)=q^6-q^3+1=p$ then $p-2=q^6-q^3-1$. But by a simple calculation we can see that $(q^6-q^3-1,\ q^2+1)|5$, $(q^6-q^3-1,\ q^4-q^2+1)|5$, $(q^6-q^3-1,\ q^4-q^3+q^2-q+1)|11$, $(q^6-q^3-1,\ q^4+1)|3$, and q^6-q^3-1 is coprime with respect to other factors of $|G|$. Also $(3,\ q^4+1)=1$ which implies that $(q^6-q^3-1,\ q^4+1)=1$. Therefore $(p-2,\ |G|)|5^3\times 11$, and so p-2 must be equal to 1, 5, 25, 125, 11, 55, 275, 1375. But p is a prime number and $p=D(q)$. Also $D(3)=703$, $D(7)=117307$ and therefore it is impossible. Similarly if $D(q)=p-2$ we get a contradiction. If $gcd(3,\ q+1)=3$ then we consider $(q^6-q^3-5)/3$ and proceed similarly to get a contradiction.

For simplicity suppose $X=\{q^{18},\ q^{36}\}$ and $Y=\{q^9,\ q^{27}\}$.

Step 2. If $K/H\cong A_r(q')$ then we distinguish the following 6 cases:

2.1. $K/H\cong A_{p'-1}(q')$ where $(p',q')\neq(3,\ 2),(3,\ 4)$. Then $q'^{p'}-1\equiv 0(\text{mod } D(q))$ which implies that $q'^{p'}\in X$. Now since these cases are similar, through the proof of the main theorem we consider the hardest case i.e. q^{18} and the other case is easier than it. If $q'^{p'}=q^{18}$ then if $p'>5$ then $q'^{p'(p'-1)/2}>q^{36}$ so K/H has a Sylow subgroup of size greater than q^{36}, and it is a contradiction by Lemma 2.10(a). Since p' is an odd prime number, we must check cases $p'=3$, 5.

If $p'=3$ then $q'=q^6$ and $\frac{q'^3-1}{(q'-1)(3,\ q'-1)}=\frac{q^9+1}{(q^3+1)(3,\ q+1)}$. Hence $(q^6-1)(3,\ q^6-1)=(q^9-1)(q^3+1)(3,\ q+1)$ which is impossible. Similarly for $p'=5$ we get a contradiction.

2.2. $K/H\cong A_{p'}(q')$ where $(q'-1)|(p'+1)$. Then if $p'>5$,K/H has a Sylow subgroup of size greater than q^{36}, and it is a contradiction by Lemma 2. 10(a). Otherwise $p'=3$, 5 *and* $q'-1|p'+1$ *and* $q'^{p'}=q^{18}$ which is impossible.

2.3. $K/H\cong A_1(q')$, where $4|(q'+1)$. If $D(q)=\frac{q'-1}{2}$ then $q'\in X$.

If $q'=q^{18}$ then $\frac{q^6-q^3+1}{(3,\ q+1)}=\frac{q'-1}{2}$, and $q'=q^{18}$ and hence $(3,\ q+1)(q^3+1)(q^9-1)=2$, which is impossible.

If $D(q)=q'$ then $q'+1=\frac{q^6-q^3+1+(3,q+1)}{(3,q+1)}$ and $(q'+1)||G|$. Now we get a contradiction similarly to step 1.

2.4. $K/H \cong A_1(q')$ where $4|(q'-1)$. The possibility $D(q)=q'$ was discussed in 2.3. If $D(q)=\frac{q'+1}{2}$ then $q' \in Y$. If $q'=q^9$ then $\frac{q^9+1}{2}=D(q)$ which is impossible.

2.5. $K/H \cong A_1(q')$ where $4|q'$. If $D(q)$ equals to $q'-1$, then $q' \in X$. If $q'=q^{18}$ or q^{36} then $D(q)=q^{18}-1$or $q^{36}-1$ which is a contradiction.

If $D(q)=q'+1$ then $q' \in Y$. If $q'=q^9$ or q^{27} then $D(q)=q^9+1$ or $q^{27}+1$ which is impossible.

2.6. $K/H \cong A_2(2)$ or $A_2(4)$ then $D(q)$ must be equal to 3, 5, 7, 9 which is impossible.

Step 3. If $K/H \cong {}^2A_r(q')$ then we consider 3 cases:

3.1. $K/H \cong {}^2A_{p'-1}(q')$

Then $q'^{p'}+1 \equiv 0 (\bmod D(q))$, and so $q'^{p'} \in Y$, by Lemma 2.10(c). Since the proofs are similar, we do only one of them. If $q'^{p'}=q^9$ then for $p' \geq 11$ we have $q'^{p'(p'-1)/2} > q^{36}$ and so K/H has a Sylow subgroup of size greater than q^{36}, which is a contradiction, by Lemma 2.10(a). Since p' is an odd prime number, we must check cases $p'=$ 3, 5, 7.

If $p'=3$ then $q'=q^3$. Let $q=p^n_o$. Then

$$|K/H|=|{}^2A_2(p^3)|=|{}^2A_2(p^{3n}_o)|=|q^9(q^3+1)(q^6-1)\frac{(q^6-q^3+1)}{(3,q+1)}$$

and $|G|=|H|.|K/H|.|G/K|$, therefore $|H|.|G/K|=q^{27}(q^{12}-1)(q^8-1)(q^5+1)(q^2-1)$. But $|G/K|||Out(K/H)|=|Out({}^2A_2(p^{3n}_o))| \leq 12n$ [21], which implies that $|H| \neq 1$ and so we can consider a p-Sylow subgroup of H, say P. Since H is nilpotent, $P \lhd G$ and hence $D(q)|(|P|-1)$. So we can choose P such that $D(q) \nmid (|P|-1)$, and so it is a contradiction.

If $p'=5$, then $q'^5=q^9$ and $(q'+1)(5,q'+1)=(q^3+1)(3,q+1)$, which is impossible. Similarly for $p'=7$ we get a contradiction.

3.2. $K/H \cong {}^2A_{p'}(q')$ where $(q'+1)|(p'+1)$ and $(p',q') \neq (3,3),(5,2)$. Then if $p'>7$, K/H has a Sylow subgroup of size greater than q^{36} which is a contradiction, by Lemma 2.10(a). Otherwise $p'=3$, 5, 7 and $q'+1|p'+1$ which is impossible.

3.3. $K/H \cong {}^2A_3(2)$, ${}^2A_3(3)$ or ${}^2A_5(2)$. Then $D(q)$ must be equal to 5, 7, 11 which is impossible.

Step 4. If $K/H \cong D_r(q')$ where $(r,q')=(p',q')(p' \geq 5, q'=2,3,5)$ or $(r,q')=(p'+1,q')(q'=2,3)$ then $q'^{p'} \in X$ and since $p' \geq 5$, we can get a contradiction.

Step 5. $K/H \ncong Br(q')$ and $C_r(q')$. For example if $K/H \cong B_r(q')$ then we consider 2 cases:

5.1. $K/H \cong B_r(q')$ where $r=2^t \geq 4$ and q' is odd. Then $q'^r+1 \equiv 0$ (mod D(q)). By Lemma 2.10(c), $q'^r \in Y$, which is a contradiction since $\frac{q'^r+1}{2} \neq D(q)$.

5.2. $K/H \cong B_p(3)$. Then $3^p \in X$, which is impossible since p is an odd prime number.

Step 6. If $K/H \cong {}^2D_r(q')$ then we consider 3 cases:

6.1. $K/H \cong {}^2D_r(q')$ where $r=2^t>2$. Then $q'^r \in Y$. For example if $q'^r=q^9$ then $\frac{q^9+1}{(2,q)}=\frac{q^6-q^3+1}{(3,q+1)}$, which is impossible.

6.2. If $K/H \cong {}^2D_r(2)$ where $r=2^t+1 \geq 5$ or ${}^2D_p(3)$ where $p=2^t+1$, $t \geq 2$ or ${}^2D_{p+1}(2)$ or ${}^2D_r(3)$ where $r=2^t+1 \neq p, t \geq 2$ then we proceed similar to 6.1.

6.3. If $K/H \cong {}^2D_p(3)$ where $5 \leq p \neq 2^r+1$ then $3^p \in Y$ which is impossible since p is an odd prime number.

Step 7. If $K/H \cong {}^2B_2(q')$ where $q'=2^{2t+1}>2$, then

If $D(q)=q'-1$ then $q' \in X$ which is impossible since $q^{18}-1 \neq D(q)$.

If $D(q)=q' \pm \sqrt{2q'}+1$. Then $q'^2+1 \equiv 0(\mathrm{mod}\ D(q))$. Therefor $q'^2 \in Y$ which is a contradiction.

Step 8. $K/H \ncong G_2(q')$ and ${}^3D_4(q')$.

For example if $K/H \cong G_2(q')$ then we consider 3 cases:

8.1. $K/H \cong G_2(q')$ where $2<q' \equiv 1(\mathrm{mod}\ 3)$. Then $D(q)=q'^2-q'+1$ and hence $q'^3+1 \equiv 0(\mathrm{mod}\ D(q))$ so $q'^3 \in Y$. Hence $q'=q^3$ or q^9. So $q'^3-1=q^9-1$ or $q^{27}-1$, but $q^9-1 \nmid |G|$.

8.2. $K/H \cong G_2(q')$ where $2<q' \equiv -1(\mathrm{mod}\ 3)$. Then $q'^3 \in X$. If $q'^3=q^{18}$ then $q'=q^6$. But then $q^{12}+q^6+1=\frac{q^6-q^3+1}{(3,\ q+1)}$, which is impossible.

8.3. $K/H \cong G_2(q')$ where $3|q'$. We proceed similar to 8.1 and 8.2.

Step 9. If $K/H \cong E_7(2)$ or $E_7(3)$ or ${}^2E_6(2)$ or ${}^2F_4(2)'$ then $D(q)$ must be equal to 13, 17, 19, 73, 127, 757, 1093 which have no solution in Z.

Step 10. If $K/H \cong F_4(q')$ then we consider 2 cases.

10.1. If $D(q)=q'^4-q'^2+1$ then we proceed similar to step 8.

10.2. If $D(q)=q'^4+1$, then $q'^4 \in Y$ which is impossible, by Lemma 2.10(a).

Step 11. $K/H \ncong {}^2F_4(q')$ where $q'=2^{2r+1}>2$, ${}^2G_2(q')$ where $q'=3^{2r+1}$ and $E_6(q')$.

For example if $K/H \cong {}^2G_2(q')$ where $q'=3^{2r+1}$ then $D(q)=q' \pm \sqrt{3q'}+1$. So $q'^3 \in Y$ but $D(q) \neq q^3 \pm \sqrt{3q^3}+1$.

Step 12. If K/H is a sporadic simple group then $D(q)$ must be equal to 5, 7,11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 59, 67, 71 which have no solution in **Z**.

Step 13. If $K/H \cong {}^2E_6(q')$, then easily we conclude that $q=q'$, so $K/H={}^2E_6(q)$. Then $|G|=|{}^2E_6(q)|=|K/H|=|K|/|H|$ which implies that $|H|=1$ and $|K|=|G|=|{}^2E_6(q)|$. Therefore, $K={}^2E_6(q)$ and hence $G={}^2E_6(q)$.

Step 14. If $K/H \cong E_8(q')$, then since all odd order components are less than or equal to q'^9, we have $q<q'^9$ or $q^{10}<q'^{90}$, which is a contradicton by Lemma 2.10(a).

The proof of the main theorem is now completed.

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