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(p,q,r)-Generations and nX-complementary generations of the sporadic group Ly

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Abstract

Let G be a finite group and nX be a conjugacy class of G. G is called (p,q,r)-generated, p,q and r are distinct primes, if it is a quotient group of the triangle group $T(p,q,r)=\langle x,y,z \mid x^p=y^q=z^r=xyz=1\rangle$. Also, G is said to be nX-complementary generated if, for any arbitrary $x\in G-\{1\}$, there is a $y\in nX$ such that $G=\langle x,y\rangle$. In some research papers the problem is posed to find all possible (p,q,r)-generations and nX-complementary generations for the non-abelian finite simple groups.

In this paper we partially answer this question for the sporadic group Ly. We find all (p,q,r)-generations and nX-complementary generations, for the Lyons's group Ly.

1. Introduction

Let G be a group and nX a conjugacy class of elements of order n in G. Following Woldar [21], the group G is said to be nX-complementary generated if, for any arbitrary non-identity element $x\in G$, there exists a $y\in nX$ such that $G=\langle x,y\rangle$. The element $y=y(x)$ for which $G=\langle x,y\rangle$ is called complementary. Furthermore, a group G is said to be (lX,mY,nZ)-generated (or (l,m,n)-generated for short) if there exist $x\in lX$, $y\in mY$ and $z\in nZ$ such that $xy=z$ and $G=\langle x,y\rangle$. If G is (l,m,n)-generated, then we can see that for any permutation π of S_3, the group G is also $((l)\pi,(m)\pi,(n)\pi)$-generated. Therefore we may assume that $l\le m\le n$. By [2], if the non-abelian simple group G is (l,m,n)-generated, then either $G\cong A_5$ or $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}<1$. Hence for a non-abelian finite simple group G and divisors l,m,n of the order of G such that $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}<1$, it is natural to ask if G is a (l,m,n)-generated group. The motivation for this question came from the calculation of the genus of finite simple groups [22]. It can be shown that the problem of finding the genus of a finite simple group can be

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reduced to one of generations.

In a series of papers, [10], [11], [12], [13], [14], [15] and [16], Moori and Ganief established all possible (p,q,r)-generations, where p,q,r are distinct primes, of the sporadic groups J_1, J_2, J_3, J_4, HS, McL, Co_3, Co_2, and F_{22}. In [5], [6], [7], [8] and [9], Darafsheh, Ashrafi and Moghani did the same work for the sporadic groups Co_1 and ON. The motivation for this study is outlined in these papers and the reader is encouraged to consult these papers for background material as well as basic computational techniques.

Throughout this paper we use the same notation as in the mentioned papers. In particular, $\Delta(G)=\Delta(lX,mY,nZ)$ denotes the structure constant of G for the conjugacy classes lX, mY,nZ, whose value is the cardinality of the set $\Lambda=\{(x,y) \mid xy=z\}$, where $x \in lX$, $y \in mY$ and z is a fixed element of the conjugacy class nZ. In Table V, we list the values $\Delta(pX,qY,rZ)$, where p, q and r are distinct prime divisors of $|Ly|$, using the character table of Ly [4]. Also, $\Delta^*(G)=\Delta^*_G(lX,mY,nZ)$ and $\Sigma(H_1 \cup H_2 \cup \cdots \cup H_r)$ denote the number of pairs $(x,y) \in \Lambda$ such that $G=\langle x,y \rangle$ and $\langle x,y \rangle \subseteq H_i$ (for some $1 \leq i \leq r$), respectively. The number of pairs $(x,y) \in \Lambda$ generating a subgroup H of G will be given by $\Sigma^*(H)$ and the centralizer of a representative of lX will be denoted by $C_G(lX)$. A general conjugacy class of a subgroup H of G with elements of order n will be denoted by nx. Clearly, if $\Delta^*(G)>0$, then G is (lX,mY,nZ)-generated and (lX,mY,nZ) is called a generating triple for G. The number of conjugates of a given subgroup H of G containing a fix element z is given by $\chi_{N_G(H)}(z)$, where $\chi_{N_G(H)}$ is the permutation character of G with action on the conjugates of H (cf. [20]). In most cases we will calculate this value from the fusion map from $N_G(H)$ into G stored in GAP, [17].

Now we discuss techniques that are useful in resolving generation type questions for finite groups. We begin with a theorem that, in certain situations, is very effective at establishing non-generations.

Theorem 1.1. ([3]) *Let G be a finite centerless group and suppose lX,mY and nZ are G-conjugacy classes for which $\Delta^*(G)=\Delta^*_G(lX,mY,nZ)<|C_G(z)|$, $z \in nZ$. Then $\Delta^*(G)=0$ and therefore G is not (lX,mY,nZ)-generated.*

Further useful results that we shall use are:

Lemma 1.2. ([12]). *Let G be a $(2X,sY,tZ)$-generated simple group then G is $(sY,sY,(tZ)^2)$-generated.*

Lemma 1.3. ([13]). *If G is nX-complementary generated and $(sY)^k=nX$, for some integer k, then G is sY-complementary generated.*

Lemma 1.4. *Let G be a finite simple group and H a maximal subgroup of G containing a fixed element x. Then the number h of conjugates of H containing x is $\chi_H(x)$, where χ_H is the permutation character of G with action on the conjugates of H. In particular,*

$$h=\sum_{i=1}^{m}\frac{|C_G(x)|}{|C_H(x_i)|}$$

where $x_1, x_2, \cdots, x_m$ are representatives of the H-conjugacy classes that fuse to the G-conjugacy class of x.

In the present paper we investigate the (p,q,r)-generations and nX- complementary generations, where p, q and r are distinct primes and n is an element order, for the Lyons's group Ly. We prove the following results:

Theorem A. *The Lyons's group Ly is (p,q,r)-generated if and only if $(p,q,r)\neq(2,3,5)$.*

Theorem B. *The Lyons's group Ly is nX-complementary generated if and only if $n\geq 3$ and $nX\neq 3A$.*

2. (p,q,t)-Generations for Ly

In this section we obtain all the (pX,qY,rZ)-generations of the Lyons's group Ly, which is a sporadic group of order 51765179004000000. Since $11A^{-1}=11B$, the group Ly is $(pX,qY,11A)$-generated if and only if it is $(pX,qY,11B)$-generated. Therefore, it is enough to investigate the $(pX,qY,11A)$-generation of Ly.

We will use the maximal subgroups of Ly listed in the ATLAS extensively, especially those with order divisible by 31. We listed in Table I, all the maximal subgroups of Ly and in Table IV, the fusion maps of these maximal subgroups into Ly (obtained from GAP) that will enable us to evaluate $\Delta^*_{Ly}(pX,qY,rZ)$, for prime classes pX, qY and rZ. In this table h denotes the number of conjugates of the maximal subgroup H containing a fixed element z (see

Table I
The maximal subgroups of Ly

Group	*Order*	*Group*	*Order*
$G_2(5)$	$2^6.3^3.5^6.7.31$	$3.McL:2$	$2^8.3^7.5^3.7.11$
$5^3.PSL(3,5)$	$2^5.3.5^6.31$	$2.A_{11}$	$2^8.3^4.5^2.7.11$
$5^{1+4}_+:4S_6$	$2^6.3^2.5^6$	$3^5:(2\times M_{11})$	$2^5.3^7.5.11$
$3^{2+4}:2A_5.D_8$	$2^6.3^7.5$	$67:22$	$2.11.67$
$37:18$	$2.3^2.37$		

Lemma 1.4). For basic properties of the group Ly and information on its maximal subgroups the reader is referred to [4]. It is a well known fact that Ly has exactly 9 conjugacy classes of maximal subgroups, as listed in Table I.

If the group Ly is (2,3,p)-generated, then by Conder's result [2], $\frac{1}{2}+\frac{1}{3}+\frac{1}{p}<1$. Thus we only need to consider the cases $p=7,11,31,37,67$.

Woldar, in [22] determined which sporadic groups other than Fi_{22}, F_{23}, F'_{24}, Th, J_4, B and M are Hurwitz groups, i.e. generated by elements x and y with order $o(x)=2$, $o(y)=3$ and $o(xy)=7$. In fact, G is a Hurwitz group if and only if G is (2,3,7)-generated. By hisresult, Ly is a Hurwitz group and so it is (2,3,7)-generated. For the sake of completeness, we reprove this result by using the character table of Ly [4].

Lemma 2.1. *The Lyons group Ly is not $(2A,3A,7A)$-generated, but it is $(2A,3B,7A)$-generated.*

Proof. It is easy to see that $\Delta_G(2A,3A,7A)<|C_G(7A)|$. Therefore, by Theorem 1.1, $\Delta^*(G)=0$ and therefore Ly is not $(2A,3A,7A)$-generated. Next we consider the tripe $(2A,3B,7A)$. The maximal subgroups of Ly, up to isomorphisms, that contain $(2A,3B,7A)$-generated subgroups are $G_2(5)$, $3.McL.2$ and $2.A_{11}$. Using the structure constants, Table V, we have,

$$\Delta(Ly)=8680,\ \Sigma(G_2(5))=546,\ \Sigma(3.McL.2)=49 \text{ and } \Sigma(2.A_{11})=112.$$

Therefore $\Delta^*(Ly)\geq 8680-8(546)-4(49)-112>0$, and so Ly is$(2A,3B,7A)$-generated. □

In the following results we not only prove for certain triples (p,q,r) that Ly is (p,q,r)-generated, but we also find all generating triples (pX,qY,rZ). We will use some of these generating triples later to find conjugacy classes nX for which Ly is nX-complementary generated.

Lemma 2.2. *Let $p\geq 11$ be a prime divisor of $|Ly|$. The Lyons group Ly is $(2A,3X,pY)$-generated if and only if $X\neq A$.*

Proof. It is easy to see that $\Delta_{Ly}(2A,3A,pX)<|C_{Ly}(pX)|$, for all prime class pX, $p\geq 5$. Therefore, by Theorem 1.1, $\Delta^*(Ly)=0$ and therefore Ly is not $(2A,3A,pX)$-generated. Suppose $X=B$ and pX is a prime class with $p\geq 11$. We will treat each triple separately.

Case (2A,3B,11A). From the list of maximal subgroups of Ly, Table I, we observe that, up to isomorphisms, $3.McL.2$, $2.A_{11}$ and $3^5:(2\times M_{11})$ are the only maximal subgroups of

Ly that admit $(2A,3B,11A)$-generated subgroups. From the structure constants, we calculate $\Delta(Ly)=6974$, $\Sigma(3.McL.2)=11$, $\Sigma(2.A_{11})=110$ and $\Sigma(3^5:(2\times M_{11}))=33$. Thus,

$$\Delta^*(Ly)\geq\Delta(Ly)-11-3(110)-3(33)>0.$$

This shows that the Lyons group Ly is $(2A,3B,11A)$-generated.

Case $(2A,3B,31X)$, $X\in\{A,B,C,D,E\}$. For every X, $X\in\{A,B,C,D,E\}$, the maximal subgroups of Ly that have non-empty intersection with the classes $2A$, $3B$ and $31X$ are, up to isomorphism, $G_2(5)$ and $5^3.PSL(3,5)$. We calculate that $\Delta(Ly)=7254$, $\Sigma(G_2(5))=496$ and $\Sigma(5^3.PSL(3,5))=310$. From Table V, it follows that

$$\Delta^*(Ly)\geq 7254-496-2(310)>0.$$

Hence Ly is $(2A,3B,31X)$-generated.

Case $(2A,3B,37X)$, $X\in\{A,B\}$. For every $X\in\{A,B\}$, the only maximal subgroups that may contain $(2A,3B,37X)$-generated subgroups is isomorphic to $37:18$. We calculate that $\Sigma(37:18)=0$. Since $\Delta(Ly)=7252$, we have $\Delta^*(Ly)>0$. This proves generation by these triples.

Case $(2A,3B,67X)$, $X\in\{A,B,C\}$. From the structure constants, Table V, we can see that $\Delta(Ly)=7705$. But, for $X\in\{A,B,C\}$, there is no maximal subgroups containing the conjugacy classes $2A$, $3B$ and $67X$. This shows that $\Delta^*(Ly)=\Delta(Ly)>0$. Thus, $(2A,3B,67X)$, $X\in\{A,B,C\}$ is a generating triple for the group Ly, which completes the proof. □

Lemma 2.3. *For every prime divisor p and q of $|Ly|$ with $5\leq p\leq q$, the Lyons group Ly is $(2A,pX,qY)$-generated.*

Proof. Suppose (pX,qY) is one of the pairs in the Table II. Then we can see that there is no maximal subgroups with non-empty intersection with $(2A,pX,qY)$. From the structure constants, Table V, we calculate $\Delta(Ly)>0$. Therefore, $\Delta^*(Ly)=\Delta(Ly)>0$ and Ly is $(2A,pX,qY)$-generated. We investigate other triples case by case.

Case (2A,5A,7A). In this case $\Delta(Ly)=532$ and the maximal subgroups of Ly that may contain $(2A,5A,7A)$-generated proper subgroups are isomorphic to $G_2(5)$, $3.McL.2$ and $2.A_{11}$. We calculate that $\Sigma(G_2(5))=21$, $\Sigma(3.McL.2)=7$ and $\Sigma(2.A_{11})=28$. Our calculations give:

$$\Delta^*\geq 532-8(21)-4(7)-28>0$$

This implies the generation of Ly by this triple.

Case (2A,5B,7A). From the list of maximal subgroups of Ly, Table I, we observe that, up to isomorphisms, $G_2(5)$,$3.McL.2$ and $2.A_{11}$ are the only maximal subgroups of Ly that admit

Table II

(5A,37A)	(5A,37B)	(5B,37A)	(5B,37B)	(5A,67A)	(5A,67B)
(5A,67C)	(5B,67A)	(5B,67B)	(5B,67C)	(7A,37A)	(7A,37B)
(7A,67A)	(7A,67B)	(7A,67C)	(11A,31A)	(11A,31B)	(11A,31C)
(11A,31D)	(11A,31E)	(11A,37A)	(11A,37B)	(31A,67A)	(31A,67B)
(31A,67C)	(31B,67A)	(31B,67B)	(31B,67C)	(31C,67A)	(31C,67B)
(31C,67C)	(31D,67A)	(31D,67B)	(31D,67C)	(31E,67A)	(31E,67B)
(31E,67C)					

$(2A,5B,7A)-$generated proper subgroups. From the structure constants, we calculate $\Delta(Ly)=346425$, $\Sigma(G_2(5))=630$, $\Sigma(3.McL.2)=476$ and $\Sigma(2.A_{11})=504$. Thus,

$$\Delta^*(Ly)\geq\Delta(Ly)-8(630)-4(476)-504>0.$$

This shows that the Lyons group Ly is $(2A,5B,7A)$-generated.

Using similar argument as in above, we can prove the $(2A,pX,qY)-$generation of other triples.□

Lemma 2.4. *Suppose p,q and r are prime divisors of $|Ly|$ with $7\leq p\leq q\leq r$. Then the Lyons group Ly is $(pX,qY,rZ)-$generated.*

Proof. Using the fusion maps of maximal subgroups into Ly, Table IV, we have $\Delta^*(Ly)=\Delta(Ly)>0$, proving the lemma.□

By the previous lemma, it is enough to investigate the $(pX,qY,rZ)-$generation of Ly, only for $p=3,5$.

Lemma 2.5. *The Lyons group Ly is $(3A,5B,pY)-$and $(3B,5X,pY)-$generated, in which $p\geq7$ is a prime divisor of $|Ly|$.*

Proof. Form the fusion maps of maximal subgroups into Ly (Table IV), we can see that for each triple $(3A,5B,37X)$, $(3A,5B,67X)$, $(3B,5X,37Y)$ and $(3B,5X,67Y)$, there is no maximal subgroup of Ly that contains elements from all conjugacy classes listed in that triples. Thus, the Lyons group Ly is $(3A,5B,37X)$-, $(3A,5B,67X)$-, $(3B,5X,37Y)$-and $(3B,5X,67Y)$-generated. We investigate other triples case by case.

Case (3A,5B,7A). Amongst the maximal subgroups of Ly with order divisible by $3\times5\times7$, the only maximal subgroups with non-empty intersection with any conjugacy class in this triple are isomorphic to $G_2(5)$, $3.McL.2$ and $2.A_{11}$. Using Table V, we can see that $\Delta(Ly)=$ 10528, $\Sigma(G_2(5))=21$, $\Sigma(3.McL.2)=280$ and $\Sigma(2.A_{11})=0$. Our calculations give,

$$\Delta^*(Ly)\geq\Delta(Ly)-8(21)-4(280)>0,$$

proving the generation of Ly by this triple.

Case (3A,5B,11A). In this case, $\Delta(Ly)=5071$ and the only maximal subgroup with non-empty intersection with any conjugacy class in this triple are isomorphic to $3.McL.2$, $2.A_{11}$ and $3^5:(2\times M_{11})$. We calculate, $\Sigma(3.McL.2)=418$, $\Sigma(2.A_{11})=11$ and $\Sigma(3^5:(2\times M_{11}))=0$. Our calculations give, $\Delta^*(Ly)\geq\Delta(Ly)-418-3(11)>0$. Therefore, Ly is $(3A,5B,11A)$-generated.

Case $(3A,5B,31X)$, $X\in\{A,B,C,D,E\}$. We have $\Delta(Ly)=4030$. The $(3A,5B,31X)$-generated proper subgroups of Ly are contained in the maximal subgroups isomorphic to $G_2(5)$. We calculate further that $\Sigma(G_2(5))=31$. From Table V, we conclude that $\Delta^*(Ly)\geq 4030-31>0$ and the generation of Ly by this triple follows.

Using similar argument as in above, we can prove the $(2A,pX,qY)$-generation of Ly for other triples.□

Lemma 2.6. *The Lyons group Ly is $(3X,7A,pY)$-generated, for all prime class pY with $p\geq 11$.*

Proof. Form the fusion maps of maximal subgroups into Ly, Table IV, we can see that for every triples $(3X,7A,37Y)$ and $(3X,7A,67Y)$, there is no maximal subgroup of Ly that contains conjugacy classes of one of these triples. Thus, the Lyons group Ly is $(3X,7A,37Y)$- and $(3X,7A,67Y)$-generated. We investigate other triples case by case.

Case (3A,7A,11A). From the list of maximal subgroups of Ly (Table I), we observe that, up to isomorphisms, $3.McL.2$ and $2.A_{11}$ are the only maximal subgroups of Ly that admit $(3A,7A,11A)$-generated proper subgroups. From the structure constants, we calculate $\Delta(Ly)=$ 104940, $\Sigma(3.McL.2)=1452$ and $\Sigma(2.A_{11})=0$. Thus,

$$\Delta^*(Ly)\geq\Delta(Ly)-1452>0.$$

This shows that the Lyons group Ly is $(3A,7A,11A)$-generated.

Case $(3A,7A,31X)$, $X\in\{A,B,C,D,E,\}$. In this case, $\Delta(Ly)=114390$ and the only maximal subgroup with non-empty intersection with any conjugacy class in this triple is isomorphic to $G_2(5)$. We calculate that $\Sigma(G_2(5))=806$. Our calculations give, $\Delta^*(Ly)\geq\Delta(Ly)-806>0$. Therefore, Ly is $(3A,7A,31X)$-generated.

Using similar argument as in above, we can prove the generation of other triples.□

Lemma 2.7. *Suppose p and q are prime divisors of $|Ly|$ with $11 \leq p \leq q$. Then the Lyons group Ly is $(3X, pY, qZ)$-generated.*

Proof. Using the fusion maps of maximal subgroups into Ly, Table IV, we can see that there is no maximal subgroups which intersects the conjugacy classes $3X$, pY and qZ with $11 \leq p \leq q$. Now from the structure constants, Table V, we have $\Delta^*(Ly) = \Delta(Ly) > 0$, proving the lemma. □

Lemma 2.8. *Suppose p, q and r are prime divisors of $|Ly|$ with $5 \leq p \leq q$. Then the Lyons group Ly is (pX, qY, rZ)-generated.*

Proof. The proof is similar to the above lemmas and so omitted. □

We are now ready to state one of main results of this paper.

Theorem A. *The Lyons group Ly is (p,q,r)-generated for all $p, q \in \{2,3,5,7,11,31,37,67\}$ with $p < q < r$, except when $(p,q,r) = (2,3,5)$.*

Proof. The proof follows from the Lemmas 2.1 to 2.8. □

3. nX-Complementary Generations of Ly

As we mentioned in the introdoction of this paper, a group G is said to be nX-complementary generated if, for any arbitrary non-identity element $x \in G$, there exists a $y \in nX$ such that $G = \langle x, y \rangle$. In this section we investigate the nX-complementary generations of the sporadic-group Ly. As a consequence of a result in [20], Woldar proved that a group G is nX-complementary generated if and only if G is (pY, nX, t_pZ)-generated, for all conjugacy classes pY with representatives of prime order and some conjugacy class t_pZ (depending on pY). Using this result, we obtain all of the conjugacy class nX for which Ly is nX-complementary generated.

Lemma 3.1. *The sporadic group Ly is not $3A$-complementary generated.*

Proof. Consider the conjugacy class $2A$. From the structure constants, Table V, we can see that $\Delta(2A, 3A, t_pZ) < |C_{Ly}(t_pZ)|$, for each conjugacy class t_pZ. Therefore, Ly is not $3A$-generated. □

Table III
The Power Maps of Ly

$(6B)^2=3B$	$(6C)^2=3B$	$(9A)^3=3B$	$(12B)^4=3B$	$(15B)^5=3B$
$(18A)^6=3B$	$(24B)^8=3B$	$(24C)^8=3B$	$(30B)^{10}=3B$	$(8A)^2=4A$
$(8B)^2=4A$	$(12A)^3=4A$	$(20A)^5=4A$	$(24A)^6=4A$	$(28A)^7=4A$
$(40A)^{10}=4A$	$(40B)^{10}=4A$	$(10A)^2=5A$	$(15A)^3=5A$	$(25A)^5=5A$
$(10B)^2=5B$	$(15C)^3=5B$	$(30A)^5=6A$	$(14A)^2=7A$	$(21A)^3=7A$
$(21B)^3=7A$	$(22A)^2=11A$	$(22B)^2=11A$	$(33A)^3=11A$	$(33B)^3=11A$

Lemma 3.2. *The sporadic group Ly is $nX-$complementary generated, for $n\geq 3$ and $nX\neq 3A$.*

Proof. First of all, we show that Ly is not $2X$-complementary generated. To see this, we notice that for any positive integer n, $T(2,2,n)\cong D_{2n}$, the dihedral group of order $2n$. Thus if G is a finite group which is not isomorphic to some dihedral group, then G is not $(2X,2X,nY)$-generated, for all classes of involutions and any G-class nY. Thus, Ly is not $2X$-complementary generated.

By Woldar's result [21], every sporadic simple group is $pX-$complementary generated, for the greatest prime divisor p of the order of the group. So, Ly is $67A-$, $67B-$ and $67C-$complementary generated. We now assume that nX is a conjugacy class different from $2X$, $3A$ and $67X$.

Set $T=\{3B,4A,5A,5B,6A,7A,11A,11B,31A,31B,31C,31D,31E,37A,37B\}$. Consider the conjugacy class $67A$. Then with a tedious calculations, we can see that $\Delta^*_{Ly}(pY,nX,67A)=\Delta_{Ly}(pY,nX,67A)>0$, for each prime class pY. So for every $nX\in T$, the group Ly is nX $-$complementary generated.

Finally, we assume that $nX\notin T$. From the character table of Ly [4], we consider the Table III.
Then by this table and Lemma 1.3, the proof is complete.□

We are now ready to state the second main result of this paper.

Theorem B. *The Lyons group Ly is nX-complementary generated if and only if $n\geq 3$ and $nX\neq 3A$.*

Proof. The proof follows from the Lemmas 3.1 and 3.2.□

Table IV
Partial Fusion Maps of Maximal Subgroups into *Ly*

$G_2(5)$-classes	2a	3a	3b	5a	5b	5c	5d	5e	7a	31a
→*Ly*	2A	3A	3B	5A	5A	5B	5B	5B	7A	31B
h									8	1
$G_2(5)$-classes	31b	31c	31d	31e						
→*Ly*	31C	31D	31E	31A						
h	1	1	1	1						
3.McL.2-classes	2a	2b	3a	3b	3c	3d	5a	5b	7a	11a
→*Ly*	2A	2A	3A	3A	3B	3B	5A	5B	7A	11A
h									4	1
3.McL.2-classes	11b									
→*Ly*	11B									
h	1									
$5^3.PSL(3,5)$-classes	2a	3a	5a	5b	5c	5d	31a	31b	31c	31d
→*Ly*	2A	3B	5A	5A	5B	5B	31B	31A	31E	31D
$5^3.PSL(3,5)$-classes	31e	31f	31g	31h	31i	31j				
→*Ly*	31C	31B	31A	31E	31D	31C				
h	2	2	2	2	2					
$2.A_{11}$-classes	2a	2b	3a	3b	3c	5a	5b	7a	11a	11b
→*Ly*	2A	2A	3A	3B	3B	5A	5B	7A	11A	11B
h								1	3	3
$5_+^{1+4}:4S_6$-classes	2a	2b	3a	3b	5a	5b	5c	5d	5e	5f
→*Ly*	2A	2A	3A	3B	5A	5A	5B	5A	5B	5B
$3^5:(2\times M_{11})$-classes	2a	2b	2c	3a	3b	3c	3d	5a	11a	11b
→*Ly*	2A	2A	2A	3A	3B	3B	3B	5B	11A	11B
h									3	3
$3^{2+4}:2A_5.D_8$-classes	2a	2b	2c	3a	3b	3c	3d	3e	3f	5a
→*Ly*	2A	2A	2A	3A	3B	3B	3A	3B	3B	5A
67 : 22-classes	2a	11a	11b	11c	11d	11e	11f	11g	11h	11i
→*Ly*	2A	11B	11A	11B	11B	11B	11A	11A	11A	11B
67 : 22-classes	11j	67a	67b	67c						
→*Ly*	11A	67A	67B	67C						
37 : 18-classes	2a	3a	3b	37a	37b					
→*Ly*	2A	3B	3B	37A	37B					

Table V
The Structure Constants of Ly

pY	$\Delta(2A,3B,pY)$	$\Delta(2A,5A,pY)$	$\Delta(2A,5B,pY)$	$\Delta(2A,7A,pY)$
7A	8680	532	312200	-
11A	6974	396	355960	7804071
31X	7254	589	346425	7719372
37X	7252	629	330595	7560025
67X	7705	871	320930	7511102

pY	$\Delta(2A,11A,pY)$	$\Delta(2A,31X,pY)$	$\Delta(2A,37X,pY)$	$\Delta(3A,5B,pY)$
7A	-	-	-	10528
11A	-	-	-	5071
31X	19645413	-	-	4030
37X	19767583	41833125	-	8658
67X	19724130	41833125	34610391	7169

pY	$\Delta(3B,5A,pY)$	$\Delta(3B,5B,pY)$	$\Delta(3A,7A,pY)$	$\Delta(3B,7A,pY)$
7A	148197	76620040	-	-
11A	131824	80152600	104940	1767039153
31X	124651	78192850	114390	1761121098
37X	141192	78319010	135050	1750265353
67X	140231	77608445	133330	1749468088

pY	$\Delta(3A,11A,pY)$	$\Delta(3B,11A,pY)$	$\Delta(3X,31Y,pY)$	$\Delta(3X,37Y,pY)$
31X	294810	4485289467	-	-
37X	274096	4487698476	629000	-
67X	276777	4489529166	619750	563537

pY	$\Delta(5A,7A,pY)$	$\Delta(5B,7A,pY)$	$\Delta(5A,11A,pY)$	$\Delta(5B,11A,pY)$
11A	139215681	81836325450	-	-
31X	136944918	82166950800	348203997	209102876325
37X	132934747	82809311500	351751378	208778152675
67X	132495314	82829855500	351043686	208734902100

pY	$\Delta(5A,31X,pY)$	$\Delta(5B,31X,pY)$	$\Delta(5A,37X,pY)$	$\Delta(5B,37X,pY)$
37X	741364375	445187954375	-	-
67X	742150625	445291924375	61815123	374586995225

pY	$\Delta(7A,11A,pY)$	$\Delta(7A,31X,pY)$	$\Delta(7A,37X,pY)$	$\Delta(11A,31X,pY)$
31X	4668576991056	-	-	-
37X	4664806712182	9939550500000	-	25301095453125
67X	4664812116078	9939550500000	8341173788256	25300674000000

pY	$\Delta(11A,37X,pY)$	$\Delta(31X,37U,pY)$
67X	21189314475000	45130930953125

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