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Parabolic type Monge-Ampère equation with zero initial boundary value*†

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Abstract

The initial boundary value problem of parabolic Monge-Ampère equation we are dealing with is $-u_t \det D_x^2 u=f$ in $Q:=\Omega\times(0, T]$, $u=0$ on $\partial_p Q$ where $\Omega\subset\mathbf{R}^n$ is a bounded convex set, f is a nonnegative bounded measurable function defined on Q. The existence and uniqueness of generalized solution of the above problem is proved through geometric method.

1. Introduction

In 1958, A. D. Aleksandrov [1] gave the definition of the generalized solution of Monge-Ampère equation $\det D_x^2 u=f$ in Ω, $u=\varphi$ on $\partial\Omega$, where $D_x^2 u$ is the Hessian matrix of a function u, by means of the gradient mapping of a convex function, and obtained its existence and uniqueness:

Definition A. Let Ω be a bounded strictly convex domain in $\mathbf{R}^n$, u be a convex function defined on Ω, ∇u denote the gradient mapping of u: $\nabla u(y)=\{p\in\mathbf{R}^n|u(x)\geq u(y)+p(x-y),\ \forall x\in\Omega\}$. u is said to be a generalized solution of $\det D_x^2 u=f$ in Ω, if $|\nabla u(U)|=\int_U f(x)dx$, $\forall$ Borel set $U\subset\Omega$, where $|\cdot|$ is the Lebesgue measure.

Theorem A. *If f is a nonnegative bounded measurable function and $\varphi\in C(\partial\Omega)$, then there exists a unique generalized solution of $\det D_x^2 u=f$ in Ω which satisfies $u|_{\partial\Omega}=\varphi$.*

Later, in 1976, N. V. Krylov [5] introduced three kinds of parabolic analogues of Monge-Ampère equation:

$$\det(D_{x_i x_j}u - \delta_{ij}u_t) = f^n(x,t),$$
$$\det(D_x^2 u)=[(f(x, t) + u_t)_+]^n,$$
$$-u_t\det(D_x^2 u)=f^{n+1}(x, t).$$

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Monge-Ampère equation $\det D_x^2 u = f$ played an important role in the proof of Aleksandrov maximum principle for second-order elliptic equations. In 1985, [4], Kaising Tso's proof of Aleksandrov-Bakel'man type maximum principle for second-order parabolic equations indicates that $-u_t \det D_x^2 u = f$ is the appropriate analogue of Monge-Ampère equation in a sense. It also relates to a kind of curvature flow problem $-u_t \frac{\det D^2 u}{(1+|Du|^2)^{\frac{n+2}{2}}} = f$ whose generalized solutions would be discussed in our following papers.

In this paper, we consider the following initial boundary value problem

$$-u_t \det D_x^2 u = f(x,t) \text{ in } Q \tag{1.1}$$

$$u = 0 \text{ on } \partial_p Q \tag{1.2}$$

Our definition of the generalized solution is from paper [8] which belongs to Rouhuai Wang and Guanglie Wang:

The Legendre transformation generated by $u(x, t)$ is: L_u: $(x, t) \in Q \to (p, h) \in \mathbf{R}^n \times \mathbf{R}$, $p \in \nabla u(x, t)$, $h = p \cdot x - u(x, t)$, where $u(x, t) \in C(Q)$ is a convex-monotone function, i.e. u is convex in x and non-increasing in t.

Definition 1.1. A convex-monotone function $u \in C(Q)$ is said to be a generalized solution of (1.1), if the Radon measure in Q defined by

$$\omega_u(E) = |L_u(E)| \text{ for any Borel set } E \text{ of } Q$$

is absolutely continuous and its Radon-Nikedym derivative is equal to f in Q, where $|\cdot|$ denotes the $n+1$-dimensional Lebesgue measure. $u \in C(\overline{Q})$ is said to be a generalized solution of (1.1), (1.2) if it is a generalized solution of (1.1) and $u=0$ on $\partial_p Q$.

The result we obtained in this paper is:

Theorem 1.1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded convex set and f be a nonnegative bounded measurable function defined on Q. Then there exists a unique generalized solution of problem* (1.1), (1.2).

There are also other definitions of generalized solution of problem (1.1), (1.2). N. V. Krylov in his paper [5] gave a kind of generalized solution obtained from a sequence of convex functions defined on Ω. In a sense, it is an approximate generalized solution which is not completely compared to the elliptic case. In 1992, J. L. Spiliotis [9] proved the existence of another kind of generalized solution of (1.1), (1.2) with probability method on condition that $\Omega \subset \mathbf{R}^n$ is a uniformly convex open set, $\partial\Omega \in C^2$, $f \in C(\overline{\Omega} \times [0, \infty))$, $f \geq 0$ and $f = 0$ on $(\partial\Omega \times [0,\infty)) \cup (\overline{\Omega} \times [T, \infty))$, $D_t f, D_{tt} f \in C(\overline{\Omega} \times [0, \infty))$. In [9], the author specifically pointed out that he hadn't have the uniqueness and regularity of it. Compare our result, Theorem 1.1, with J. L. Spiliotis'. The former is better, not only because of its weaker hypothesis, but also better results, existence and uniqueness. The main reason is in the definition of the generalized solution. In a sense, in J. L. Spiliotis' definition, variables x and t were separated, and in the definition of [8], Legendre transformation treats x and t as a unity. So in our opinion, [8]'s definition is better. Under the assumption of regular data, paper [7] had already get the regularity of [8]'s generalized solution. We will get the $W^{2,1}_{p,\infty}$ regularity of it under weaker conditions in our succeeding paper.

The paper is arranged in the following way: In § 2, we prove the comparison principle and uniqueness of a generalized solution by geometric method involving ideas by K. S. Tso [4]. § 3 is devoted to the Hölder continuity of generalized solution in t, which plays a main role in § 4, in the proof of the existence result. Our proof is parallel to the elliptic case which was given by I. Ya. Bakel'man [2]. The idea is an approximation by convex-monotone polyhedron. It is worth pointing out that the construction of convex-monotone polyhedron and its property after Legendre transformation are carefully scrutinized.

2. Comparison principle and uniqueness

The Geometric Property of Legendre transformation:

For convex-monotone function $u \in C(Q)$, let $L_u: Q \ni (x_1, t_1) \to (p_1, h_1)$, $p_1 \in \nabla u(x_1, t_1)$, $h_1 = p_1 \cdot x_1 - u(x_1, t_1)$, be the Legendre transformation generated by u. If $t \leq t_1$, it holds that, for $\forall x \in \Omega$,

$$\begin{aligned} u(x, t) &\geq u(x, t_1) \geq p_1 \cdot (x - x_1) + u(x_1, t_1) \\ &= p_1 \cdot x - (p_1 \cdot x_1 - u(x_1, t_1)) \\ &= p_1 \cdot x - h_1 \end{aligned}$$

and so $-h_1$ is the intercept of the supporting hyperplane of $u(x, t_1)$ at x_1.

The following proposition is obvious, so we omit the proof of it.

Lemma 2.1. *u is a convex-monotone function, $x_0 \in \Omega$, $p_0 \in \mathbf{R}^n$, $\bar{u}(x, t) = u(x, t) + p_0 \cdot (x - x_0)$, $\bar{u}(x - x_0, t) = u(x, t)$, then $\forall$ Borel set $E \subset Q$, we have*

$$L_{\bar{u}}(E) = L_u(E) + (p_0, p_0 \cdot x_0), \quad \omega_{\bar{u}}(E) = \omega_u(E),$$
$$\omega_u(E) = \omega_{\bar{u}}(E - (x_0, 0)),$$

where $E - (x_0, 0)$ is the parallel transform.

We give an important lemma which will be the comparison principle and uniqueness.

Lemma 2.2. *u_1, u_2 are convex-monotone function and $u_1, u_2 \in C(\overline{Q})$, if there is an open set $G \subset Q$ such that $u_1 > u_2$ in G, $u_1 = u_2$ on $\partial G \setminus \{t = T\}$, denote $T_1 = \inf\{t | (x, t) \in G\}$, $T_2 = \sup\{t | (x, t) \in G\}$. Then $\forall t_0 \in (T_1, T_2)$, we have*

$$L_{u_1}(G \cap \{t \leq t_0\}) \subset L_{u_2}(G \cap \{t \leq t_0\}),$$

and

$$\omega_{u_1}(G \cap \{t \le t_0\}) < \omega_{u_2}(G \cap \{t \le t_0\}).$$

Proof. For all $(p_1, h_1) \in L_{u_1}(G \cap \{t \le t_0\})$, there exists $(x_1, t_1) \in G \cap \{t \le t_0\}$ such that $p_1 \in \nabla u_1(x_1, t_1)$, $h_1 = p_1 \cdot x_1 - u_1(x_1, t_1)$. From $u_1 = u_2$ on $\partial G \cap \{t \le t_1\}$ and $u_2(x_1, t_1) < u_1(x_1, t_1)$, hyperplane $H = p_1 \cdot x - h_1$ must contact u_2 in $G \cap \{t \le t_1\} \subset G \cap \{t \le t_0\}$. Let (x_2, t_2) be the point that makes t_2 the minimum in the contact set. Then $H(x, t_2)$ in the supporting plane of convex function $u_2(x, t_2)$, namely, $p_1 \in \nabla u_2(x_2, t_2)$. By the geometric property of Legendre transformation, we have $L_{u_2}(x_2, t_2) = (p_1, h_1)$, i.e. $(p_1, h_1) \in L_{u_2}(G \cap \{t \le t_0\})$.

Next we prove: $\omega_{u_1}(G \cap \{t \le t_0\}) < \omega_{u_2}(G \cap \{t \le t_0\})$.

$\forall (x_0, t_0) \in G \cap \{t = t_0\}$, $\forall p_0 \in \nabla u_2(x_0, t_0)$, let $\bar{u}_2(x, t) = u_2(x, t) - p_0 \cdot (x - x_0)$, $\bar{u}_1(x, t) = u_1(x, t) - p_0 \cdot (x - x_0)$. By lemma 2.1, it suffices to prove that

$$\omega_{\bar{u}_1}(G \cap \{t \le t_0\}) < \omega_{\bar{u}_2}(G \cap \{t \le t_0\}).$$

Let $\tilde{u}_2(x - x_0, t) = \bar{u}_2(x, t)$, $\tilde{u}_1(x - x_0, t) = \bar{u}_1(x, t)$ and, also by lemma 2.1, it suffices to prove that

$$\omega_{\tilde{u}_1}(G \cap \{t \le t_0\} - (x_0, 0)) < \omega_{\tilde{u}_2}(G \cap \{t \le t_0\} - (x_0, 0)).$$

From the above transformations, $\tilde{u}_1$, $\tilde{u}_2$ are convex-monotone functions defined on $\overline{Q} - (x_0, 0)$, and $\tilde{u}_2(0, t_0) = \inf_{G \cap \{t \le t_0\} - (x_0, 0)} \tilde{u}_2$, $\tilde{u}_2 < \tilde{u}_1$ in $G - (x_0, 0)$, then $\exists \eta > 0$ s. t.

$$\tilde{u}_1(x, t) \ge \tilde{u}_2(0, t_0) + 2\eta, \quad (x, t) \in G \cap \{t \le t_0\} - (x_0, 0)$$

Let $\bar{u}_2(x, t) = \tilde{u}_2(x, t) - \tilde{u}_2(0, t_0) - \eta$, $\bar{u}_1 = \tilde{u}_1(x, t) - \tilde{u}_2(0, t_0) - \eta$. Then we have $\bar{u}_2(0, t_0) = -\eta$, and $\bar{u}_1(x, t) \ge \bar{u}_2(0, t_0) + 2\eta = \eta$ in $G \cap \{t \le t_0\} - (x_0, 0)$.

Let $O = \{(y, s) | \bar{u}_2(y, s) < 0, s \le t_0\} \subset G \cap \{t \le t_0\} - (x_0, 0)$, and then we have

$$\bar{u}_1(x, t) > \bar{u}_2(y, s) + \eta, \ (y, s) \in O \ (x, t) \in G \cap \{t \le t_0\} - (x_0, 0).$$

Let $A = \{(p, h) : |p| \le \frac{\eta}{d_G}, L_{\bar{u}_2}^{-1}(p, h) \in O\}$, where $d_G = \sup\{|x - y| : (x, t), (y, t) \in G\}$.

Now we claim that $A \subset L_{\bar{u}_2}(G \cap \{t \le t_0\} - (x_0, 0)) \setminus L_{\bar{u}_1}(G \cap \{t \le t_0\} - (x_0, 0))$ and $|A| > 0$.

In fact, $\forall (p, h) \in A$, $\exists (y, s) \in O$ *s. t.* $L_{\bar{u}_2}(y, s) = (p, h)$, and $(p, h) \in L_{\bar{u}_2}(G \cap \{t \le t_0\} - (x_0, 0))$. Note that, for a hyperplane $H = p \cdot x - h$,

$$\begin{aligned} H = p \cdot x - h = p \cdot (x - y) + \bar{u}_2(y, s) &\le \frac{\eta}{d_G}|x - y| + \bar{u}_2(y, s) \\ &\le \eta + \bar{u}_2(y, s) < \bar{u}_1(x, t), \qquad (x, t) \in G \cap \{t \le t_0\} - (x_0, 0), \end{aligned}$$

which indicates that H locates strictly lower than $\bar{u}_1$ in $G \cap \{t \le t_0\} - (x_0, 0)$. By the geometric property of Legendre transformation, we have $(p, h) \notin L_{\bar{u}_1}(G \cap \{t \le t_0\} - (x_0, 0))$.

Next, we show that $|A|>0$. Consider a set $D=\{(p, h): |p|\leq\frac{\eta}{d_G}, d_G|p|<h<\eta\}$ and, for $\forall(p, h)\in D$, let a hyperplane $H=p\cdot x-h$. Then we have

$$\bar{u}_2(0, t_0)=-\eta<-h=H(0, t),\ \bar{u}_2(x, t)=0>H \text{ on } \partial O\backslash\{t=t_0\},$$

so H must contact with $\bar{u}_2$ in O. Let (x_1, s_1) be the point that makes s_1 the minimum in the contact set. Then $H(x, s_1)$ is the supporting plane of convex function $\bar{u}_2(x, s_1)$, namely, $p\in\nabla\bar{u}_2(x_1, s_1)$. Thus $L_{\bar{u}_2}(x_1, s_1)=(p, h)$, i.e. $(p, h)\in L_{\bar{u}_2}(O)$ and so, $D\subset A$. Noting that

$$|D|=\sigma_n\int_0^{\frac{\eta}{d_G}}(\eta-rd_G)r^{n-1}dr=\sigma_n[n(n+1)]^{-1}\eta^{n+1}d_G^{-n}>0,$$

we get $|A|>0$.

Theorem 2.1. (Comparison principle) *If $u, v\in C(\overline{Q})$ are generalized solutions of*

$$\begin{cases}-u_t\det D^2u=f & \text{in } Q\\ u=0 & \text{on } \partial_pQ\end{cases}$$

and

$$\begin{cases}-v_t\det D^2v=g & \text{in } Q\\ v=0 & \text{on } \partial_pQ\end{cases}$$

separately, f, g, are nonnegative measurable functions and $f\geq g$ in Q, then $u\leq v$ in Q.

Proof. If there exists a point $(x_0, t_0)\in Q$ such that $u(x_0, t_0)>v(x_0, t_0)$, since $u=v=0$ on ∂_pQ and $u, v\in C(\overline{Q})$, there must exist an open set $G\subset Q$ such that $(x_0, t_0)\in G$, $u(x, t)>v(x, t)$ in G and $u=v$ on $\partial G\backslash\{t=T\}$. Then, by lemma 2.2,

$$\int_{G\cap\{t\leq t_0\}}fdxdt=\omega_u(G\cap\{t\leq t_0\})<\omega_v(G\cap\{t\leq t_0\})=\int_{G\cap\{t\leq t_0\}}gdxdt,$$

which is impossible by $f\geq g$ in Q.

The uniqueness of generalized solution can be easily obtained from the comparison principle.

Theorem 2.2. *If $u_1, u_2\in C(\overline{Q})$ are generalized solution of problem (1.1) (1.2), then $u_1=u_2$ in $\overline{Q}$.*

3. Hölder continuity of generalized solutions

In this section we will prove the Hölder continuity in t of the generalized solution to problem (1.1), (1.2), which is the substantial ingredient in the proof of existence of the generalized solution. Here in the next theorem we only require that $f\in L^p$ with $p>1$.

Theorem 3.1. *If u is the generalized solution of (1.1), (1.2) and $f(x, t) \in L^p(Q)$ for some $p>1$, then*

$$|u(x, t_1)-u(x, t_2)|\leq C|t_1-t_2|^{\frac{1}{(n+1)q}}, \quad x \in \Omega,\ t_1,\ t_2 \in [0, T].$$

where $C = C(n, d, \|f\|_{L^p})$, q is the dual exponent of p and d is the diameter of Ω.

Proof. $\forall x_0 \in \Omega$, $\forall t_1, t_2 \in [0, T]$, $t_1 < t_2$, let $\Pi_1(x)$ be the supporting hyperplane of $u(x, t_1)$ at x_0. Then $\forall x \in \overline{\Omega}$, $\Pi_2(x) \leq u(x, t_1)$. Specially, we have $u(x_0, t_2) < \Pi_2(x_0)$. Let $\bar{u}(x, t)=u(x, t)-\Pi_1(x)$, $P_{\Pi_1}=\{(x, t)\in \Omega\times(t_1,t_2]|u\leq\Pi_1\}$ and then $\bar{u}\leq 0$ in P_{Π_1}. Let $M=\sup_{P\Pi_1}|\bar{u}|=\bar{u}(x^*, t^*)$. Make another transformation, $\tilde{u}(x-x^*, t) = \bar{u}(x, t)$. It can be proved that

$$\begin{aligned} D&=\{(p, h): |p|\leq\frac{M}{d},\ d|p|<h<M\} \\ &\subset\ A=\{(p, h): |p|\leq\frac{M}{d},\ L\tilde{u}^{1}(p, h)\in P_{\Pi_1}-(x^*, 0)\}\subset L\tilde{u}(P_{\Pi_1}-(x^*, 0)). \end{aligned}$$

The latter inclusion relation is obvious. It remains to prove $D\subset A$. $\forall(p, h)\in D$, consider a hyperplane $H=p\cdot x-h$,

$$\tilde{u}(0, t^*)=-M<-h=H(0, t^*)$$

$$\tilde{u}(x, t)=0>H(x, t), \text{ on } \partial_p(P_{\Pi_1}-(x^*, 0))$$

where $\partial_p(P_{\Pi_1}-(x^*, 0))$ is the parabolic boundary of $P_{\Pi_1}-(x^*, 0)$. So H must contact with $\tilde{u}$ in $P_{\Pi_1}-(x^*, 0)$. Let (y, τ) be the point that makes τ the minimun in the contact set, then $H(x, \tau)$ is the supporting hyperplane of convex function $\tilde{u}(x, \tau)$. Thus

$$L\tilde{u}(y, \tau)=(p, h)\in A.$$

By lemma 2.1, we have

$$\int_{Lu(\Omega\times(t_1,t_2])}dpdh\geq\int_{Lu(P_{\Pi_1})}dpdh\geq\int_{L\tilde{u}(P_{\Pi_1}-(x^*,0))}\geq|D|=\frac{\sigma_n}{n(n+1)}M^{n+1}d^{-n},$$

that is,

$$\sup(\Pi_1(x)-u(x.\ t_2))=M\leq\left(\frac{|D_{12}|}{\sigma_n}n(n+1)d^n\right)^{\frac{1}{n+1}},$$

where

$$|D_{12}|\leq\int_{Lu(\Omega\times(t_1,t_2])}dpdh=\int_{\Omega\times(t_1,t_2]}fdxdt.$$

By the Hölder inequality, we have

$$\int_{\Omega\times(t_1,t_2]}fdxdt \leq \|f\|_{L^p}(|\Omega|\cdot|t_1 - t_2|)^{\frac{1}{q}},\ \frac{1}{p} + \frac{1}{q} = 1.$$

Then

$$\sup(\Pi_1(x) - u(x, t_2)) \leq C(n)\|f\|_{L^p}^{\frac{1}{n+1}} d^{\frac{n}{n+1}+\frac{n}{(n+1)p}}|t_2-t_1|^{\frac{1}{(n+1)p}}.$$

Thus we complete the proof.

4. Existence

In this section, we turn to prove the existence of generalized solution by approximation. To get the uniform convergence, we have to suppose f has compact support in x at first. Then construct another approximation problem to remove our requirement on f.

Theorem 4.1. *Let Ω be a bounded convex set in $\mathbf{R}^n$ and f be a nonnegative bounded measurable function with support $Q=\Omega'\times(0, T]$ for a compact subset Ω' of Ω with $d(\Omega', \Omega) = \delta$. Then* (1.1), (1.2) *admit a unique generalized solution and*

$$|u(x_1, t)-u(x_2, t)|\leq L|x_1-x_2|, \quad x_1, x_2\in\overline{\Omega}$$

$$|u(x, t_1)-u(x, t_2)|\leq C|t_1-t_2|^{\frac{1}{n+1}}, \quad t_1, t_2\in[0, T],$$

where $L=L(n, \delta, \operatorname{diam}\Omega, \sup f)$, $C=C(n, \operatorname{diam}\Omega, \sup f)$ are positive constants.

We need some preparations to prove Theorem 4.1.

Construction of the convex-monotone polyhedron defined on $\overline{Q}$:

Step 1. Let $X_k\in\Omega$, $1\leq k\leq l$, $Y_i\in\partial\Omega$, $1\leq i\leq m$, be points in Ω which satisfy that $\{X_k\}_1^l\subset\Omega'$ are interior points of the convex polyhedron $P\subset\Omega$ whose vertices are $\{Y_i\}_1^m$. Thus $P\times(0, T]$ is a convex cylinder in $\mathbf{R}^{n+1}$, which is denoted by P_T. Let $\{t_j\}_0^h$ satisfy $0=t_0<t_1<\cdots<t_h=T$. We give the following notations, $B_{ij}=(Y_i, t_j)$, $B_k^{(0)}=(X_k, t_0=0)\in\partial_p P_T$, $A_{kj}=(X_k, t_j)\in P_T$, $j>0$.

Step 2. We dissect P_T into simplex in the following way.

First, dissect each section of P_T, $P\times\{t_j\}$, $j=0, 1,\cdots, h$, in the same manner so that the vertices of each n-simplex$\{a_0^\tau, a_1^\tau,\cdots, a_n^\tau\}$, $\tau=1,\cdots, M$, are subset of $\{(Y_i, t_j)\}_{i=1}^m \cup \{(X_k, t_j)\}_{k=1}^l$. Second, between each pair of hyperplane $\{t=t_{j-1}\}$, $\{t=t_j\}$, connect points (X_k, t_{j-1}), (X_k, t_j), and (Y_i, t_{j-1}), (Y_i, t_j) with segments. Now we obtain that $P\times[0, T]$ has been divided into some columns $\{a_0^\tau, a_1^\tau,\cdots, a_n^\tau\}\times[t_{j-1}, t_j]$, $\tau=1,\cdots, M$. Third, dissect each column (without appending new vertices), $\{a_0^\tau, a_1^\tau,\cdots, a_n^\tau\}\times\{t_{j-1}, t_j\}$, on the manner that all the segments in $\{t_{j-1}, t_j\}$ with vertices Bij are on the boundary of P_T. Up to now, we have obtained a finite $(n+1)$-simplex $\{b_0^\gamma, b_1^\gamma,\cdots, b_{n+1}^\gamma\}$, $\gamma=1,\cdots, N$. Through the above process, we obtain a dissection of P_T with vertices $\{(Y_i, t_j)\}$, $\{(X_k, t_j)\}$.

Step 3. The convex-monotone polyhedron Zon P_T is that $Z\in C(\overline{P}_T)$, convex in x, $Z(x_0, t)$ strictly decreasing in t for $x_0\in P$, affine on each $\{b_0^\gamma, b_1^\gamma,\cdots, b_{n+1}^\gamma\}\subset P_T$. Namely, the projections of the vertices of Z on hyperplane $\{Z=0\}$ are $\{(Y_i, t_j), (X_k, t_j)\}$.(Here we use the same notation Z to represent the convex-monotone polyhedron and the function determined it.)

Consequently, after Legendre transformation generated by Z, according to the dissection, P_T has the following properties:

1. $|L_z(A_{kj})|=0$, $k=1,\cdots, l$, $j=1,\cdots, h$.

2. If $\{a_0, a_1\}\subset\{t=t_j\}$ for some $j=1,\cdots, h$, where $\{a_0, a_1\}$ is a 1-dimensional face of $\{b_0^\gamma, b_1^\gamma,\cdots, b_{n+1}^\gamma\}$, $\gamma=1,\cdots, N$, then $|L_Z(\{a_0, a_1\})|=0$.

If $\{c_0, c_1\}\subset P\times(t_{j-1}, t_j)$ for some $j=1,\cdots, h$, where $\{c_0, c_1\}$ is a 1-dimensional face of $\{b_0^\gamma, b_1^\gamma,\cdots, b_{n+1}^\gamma\}$, $\gamma=1,\cdots, N$ and $c_0\in\{t=t_{j-1}\}$, $c_1\in\{t=t_j\}$, then $|L_Z(\{c_0, c_1\})|$ needn't to be 0 and $\forall(A_1, A_2)\subset\{c_0, c_1\}$ with $A_1\in\{t=t^1\}$, $A_2\in\{t=t^2\}$, $t^1<t^2$, we have

$$|L_Z((A_1, A_2))| = \frac{t^2-t^1}{t_j-t_{j-1}}|L_Z(\{c_0, c_1\})|.$$

3. If $\{d_0, d_1,\cdots, d_i\}$, $i=2,\cdots, n+1$, is an i-dimensional face of $\{b_0^\gamma, b_1^\gamma,\cdots, b_{n+1}^\gamma\}$, $\gamma=1,\cdots, N$, then $|L_Z(\{d_0, d_1,\cdots, d_i\})|=0$. Notice that $\{d_0, d_1,\cdots, d_{(n+1)}\}$ is $\{b_0^\gamma, b_1^\gamma,\cdots, b_{n+1}^\gamma\}$.

In fact we can prove them in the following.

1. Let $L_Z(A_{kj})=E$. Since $p\in\nabla Z(A_{kj})$, $h=p\cdot X_k-Z(A_{kj})$ holds for any $(p, h)\in L_Z(A_{kj})$, we have that E is a bounded subset of some n-dimensional hyprplane. So $|L_Z(A_{kj})|=0$.

2. By [8] Prop.1.1, note that $\forall(p, h)\in L_Z(\{a_0, a_1\})$, $\exists(x_1, t_j), (x_2, t_j)\in\{a_0, a_1\}$ such that $\nabla Z(x_1, t_j)=\nabla Z(x_2, t_j)=p$ and $p\cdot x_1-Z(x_1, t_j)=p\cdot x_2-Z(x_2, t_j)$. Thus $|L_Z(\{a_0, a_1\})|=0$.

$\forall(p^1, h^1)\in L_Z(A_1)$, $H^1(x)=p^1\cdot x-h^1$ is the supporting plane of Z at A_1. By the fact that Z is affine on each simplex, we can suppose $Z(A_2)-Z(A_1)=q(t^2-t^1)$, where q is a constant depending on $\{c_0, c_1\}$. Then

$$H^2(x)=p^1\cdot x-h^1+Z(A^2)-Z(A^1)=p^1\cdot x-h^1+q(t_2-t_1)$$

is the supporting plane of Z at A_2, that is to say $\nabla Z(A_1)=\nabla Z(A_2)$, $(p^1, h^1-q(t^2-t^1))\in L_Z(A_2)$ and h is affine in $t\in(t_{j-1}, t_j)$. So we have

$$\begin{aligned}|L_Z((A_1, A_2))|&=\int_{\{p^1|(p^1,h)\in L_Z((A_1,A_2))\}}dp\int_{L_Z((A_1,A_2))\cap\{p^1\}}dh\\&=\int_{\{p^1|(p^1,h)\in L_Z(\{c_0,c_1\})\}}dp\int_{L_Z(\{c_0,c_1\})\cap\{p^1\}}\frac{t^2-t^1}{t_j-t_{j-1}}dh\\&=\frac{t^2-t^1}{t_j-t_{j-1}}|L_Z(\{c_0, c_1\})|.\end{aligned}$$

3. Since $\{X_k\}_1^l\subset\Omega'$ and Z is convex in x, we have $\forall p\in\nabla Z(P_T)$, $|p|\leq C(\sup|Z|, \delta)$. When $i\geq2$, since Z is affine on$\{d_0, d_1,\cdots, d_i\}$, we have $H^n(\nabla Z(\{d_0, d_1,\cdots, d_i\}))=0$. Noting that $\forall(p, h)\in L_Z(\{d_0, d_1,\cdots, d_i\})$, $|h|=|p\cdot x-Z(x, t)|\leq C(p, \sup|Z|)=C(\sup|Z|, \delta)$, we have $|L_Z(\{d_0, d_1,\cdots, d_i\})|=0$.

Construction of the approximation problem:

By the above process, we get I (a finite integer) "good" sets l_β, $\beta=1,\cdots,I$, whose image by Legendre transformation might have positive measure, with $H^{n+1}(l_\beta)=0$, $\forall\beta\in[1, I]$, and $l_\beta\cap l_{\beta'}=\emptyset$. So $\exists\{U_\beta\}_{\beta=1}^{I}\subset R^{n+1}$ satisfies: $l_\beta\subset U_\beta$ and $U_\beta\subset P\times[t_{j-1}, t_j]$ for some j, $|U_\beta|\neq 0$, $\bigcup_{\beta=1}^{I}U_\beta=P\times[0, T]$, and $\forall\beta, \beta'\in[1, I]$, U_β and $U_{\beta'}$ have no common interior.

Here we point out that from the construction of $\{U_\beta\}$, $\{l_\beta\}$, any subset of Q can be approximated by $\{U_\beta\}$ if the density of $\{X_k\}$, $\{Y_i\}$, $\{t_j\}$ is sufficiently large.

Let $\sigma_\beta=\int_{U_\beta}fdxdt>0$, $\beta=1,\cdots,I$. Since f has support $\Omega'\times(0, T)$, we have

$$\sum_{\beta=1}^{I}\sigma_\beta=\sum_{\beta=1}^{I}\int_{U_\beta}fdxdt=\int_Q fdxdt:=F.$$

To prove Theorem 4.1, we first get the existence of convex-monotone polyhedron.

Theorem 4.2. *There exists a convex-monotone polyhedron $Z(x, t)$ with vertices $Z(A_{kj})$, $j\neq 0$, $Z(B_{ij})$ and $Z(B_k^{(0)})$ such that*

$$Z(B_{ij})=Z(B_k^{(0)})=0,$$

$$|L_Z(\bigcup_{\beta\in a_{kj}}U_\beta)|=|L_Z(\bigcup_{\beta\in a_{kj}}l_\beta)|=\sum_{\beta\in a_{kj}}\sigma_\beta,\quad k=1,\cdots,l,\ j=1,\cdots,h,$$

where $a_{kj}\subset[1, I]$ contains those β such that $l_\beta\subset P\times(t_{j-1}, t_j)$ with A_{kj} as its vertices, and the following estimates hold

$$|Z(x_1, t)-Z(x_2, t)|\leq L|x_1-x_2|,$$

$$|Z(x, t_1)-Z(x, t_2)|\leq C|t_1-t_2|^{\frac{1}{n+1}},$$

where L depends on n,δ, $diam\Omega$, $sup\ f$, and C on n, $diam\Omega$, $sup\ f$.

Proof. Let S be a set of convex-monotone polyhedrons such that each $Z\in S$ satisfies the followings:

1) The vertices of Z are $Z(A_{kj})_{j\neq 0}$, $Z(B_{ij})$, $Z(B_k^{(0)})$.

2) $|Lz(\bigcup_{\beta\in a_{kj}}U_\beta)|=|Lz(\bigcup_{\beta\in a_{kj}}l_\beta)|\leq\sum_{\beta\in a_{kj}}\sigma_\beta$, $k=1,\cdots,l$, $j=1,\cdots,h$.

3) $Z(B_{ij})=Z(B_k^{(0)})=0$.

$\forall X_{k_1}\in\{X_k\}_1^l$, $\forall t_0\in[0, T]$, let $Z_0(x, t_0)$ be a cone whose interior vertices is a single point $Z_0(X_{k_1})=-\varepsilon_1 t_0$ and $Z_0(x, t_0)=0$ on ∂P. Then, if ε_1 is sufficiently small, we have $Z\in S$, namely, $S\neq\emptyset$.

$\forall Z_0\in S$, convex in x and decreasing in t, we can prove

$$-\left(\frac{F}{\sigma_n}n(n+1)d^n\right)^{\frac{1}{n+1}}\leq Z\leq 0, \tag{4.1}$$

where σ_n is the volume of unit sphere and d is the diameter of P. In fact, the convex -monotonicity of Z contributes the latter inequality of (4.1). To prove the former inequality, suppose $M=\sup_{P_t}|Z|=-Z(x_0, t_0)$. Let $\tilde{Z}(x-x_0, t)=Z(x, t)$. It can be proved that

$$D=\{(p, h): |p|\leq\frac{M}{d}, d|p|<h<M\}$$
$$\subset A=\{(p, h): |p|\leq\frac{M}{d}, L_{\tilde{z}}^{-1}(p, h)\in P_T-(x_0, 0)\}\subset L_{\tilde{z}}(P_T-(x_0, 0)).$$

Since the latter inclusion relation is obvious, we only need to prove $D\subset A$. $\forall(p, h)\in D$, consider a hyperplane $H=p\cdot x-h$,

$$\tilde{Z}(0, t_0)=-M<-h=H(0, t),$$
$$\tilde{Z}(x, t)=0>H(x, t), \text{ on } \partial_p(P_T-(x_0, 0)).$$

So H must contact with $\tilde{Z}$ in $P_T-(x_0, 0)$. Let (y, τ) be the point that makes τ the minimum in the contact set, then $H(x, \tau)$ is the supporting hyperplane of convex function $\tilde{Z}(x, \tau)$. Thus

$$L_{\tilde{z}}(y, \tau)=(p, h)\in A.$$

By Lemma 2.1, we have

$$F=\int_{L_z(P_T)}dpdh=\int_{L_{\tilde{z}}(P_T-(x_0,0))}dpdh\geq|D|=\frac{\sigma_n}{n(n+1)}M^{n+1}d^{-n},$$

which means we have completed the proof of (4.1). Noting that $\{X_k\}\subset\Omega'$, we have

$$|Z(x_1, t) - Z(x_2, t)| \leq L|x_1-x_2|,$$

where $L=\delta^{-1}[(\frac{F}{\sigma_n}n(n+1)d^n)^{\frac{1}{n+1}}]$.

Another a prior estimate for Z is the Hölder continuity in t which can be proved similarly to the proof of Theorem 3.1. The estimate is that $\exists\, C(n,d,sup\ f)>0$ such that

$$|Z(x, t_1)-Z(x, t_2)|\leq C|t_1-t_2|^{\frac{1}{n+1}} \quad \forall t_1, t_2\in[0, T]. \tag{4.2}$$

However, note that the function we considered here is the convex monotone polyhedron. The Legendre transformation determined by which is something special (see Properties 1-3 in the beginning of the Construction of the convex monotone polyhedron defined on $\overline{Q}$). For the completeness, we prove (4.2) below. $\forall t_1, t_2\in[0, T]$, $t_1<t_2$, $\forall x_0\in P$, by the convexity of Z in x, $\exists\, p\in\nabla Z(x_0, t_1)$ s.t.

$$H=Z(x_0, t_1)+p\cdot(x-x_0)\leq 0 \text{ on } \partial P.$$

Let $\overline{Z}(x, t)=Z(x, t)-Z(x_0, t_1)-p\cdot(x-x_0)$ and again use the method of [4] Prop.2.1. Now

$P_{(x_0,t_1)}=\{(x, t)\in P\times(t_1, t_2]|\bar{Z}\leq 0\}$ is the domain. Let (x_1, t_2) be the point that $\bar{Z}(x, t_2)$ achieve its minimum in $\{t=t_2\}$. We have

$$\bar{Z}(x_0, t_1)-\bar{Z}(x_1, t_2)\leq\left(\frac{|D_{12}|}{\sigma_n}n(n+1)d^n\right)^{\frac{1}{n+1}},$$

where

$$|D_{12}|\leq\int_{L_{\bar{z}}(P\times(t_1,t_2])}dpdh=\int_{L_{\bar{z}}(P\times(t_1,t_2])}dpdh.$$

Suppose $(t_1, t_2)\subset(t_{j_1}, t_{j_2})$. By properties 1-3, we have

$$\begin{aligned}
&\int_{L_{\bar{z}}(P\times(t_1,\ t_2])}dpdh\\
&=\int_{L_{\bar{z}}(P\times(t_1,\ t_{j_1}+1])}dpdh+\int_{L_{\bar{z}}(P\times(t_{j_1}+1,t_{j_1}+2])}dpdh\\
&\quad+\cdots+\int_{L_{\bar{z}}(P\times(t_{j_2}-2,t_{j_2}-1])}dpdh+\int_{L_{\bar{z}}(P\times(t_{j_2}-1,t_2])}dpdh\\
&=\frac{t_{j_1+1}-t_1}{t_{j_1+1}-t_{j_1}}\int_{L_{\bar{z}}(P\times(t_{j_1},t_{j_1}+1])}dpdh+\int_{L_{\bar{z}}(P\times(t_{j_1}+1,t_{j_1}+2])}dpdh\\
&\quad+\cdots+\int_{L_{\bar{z}}(P\times(t_{j_2}-2,t_{j_2}-1])}dpdh+\frac{t_2-t_{j_2-1}}{t_{j_2}-t_{j_2-1}}\int_{L_{\bar{z}}(P\times(t_{j_2}-1,t_{j_2}])}dpdh\\
&\leq \sup f\cdot d^n\cdot(t_{j_1+1}-t_1+t_{j_1+2}-t_{j_1+1}+\cdots+t_2-t_{j_2-1})\\
&=\sup f\cdot d^n\cdot|t_2-t_1|.
\end{aligned}$$

Then

$$\bar{Z}(x_0, t_1)-\bar{Z}(x_1, t_2)\leq C(n)(\sup f)^{\frac{1}{n+1}}d^{\frac{2n}{n+1}}|t_2-t_1|^{\frac{1}{n+1}}.$$

From the fact that $\bar{Z}$ is decreasing in t

$$\begin{aligned}
0<\bar{Z}(x_0, t_1)-\bar{Z}(x_0, t_2)&=\bar{Z}(x_0, t_1)-\bar{Z}(x_1, t_2)+\bar{Z}(x_1, t_2)-\bar{Z}(x_0, t_2)\\
&\leq\bar{Z}(x_0, t_1)-\bar{Z}(x_1, t_2)\leq C\cdot|t_2-t_1|^{\frac{1}{n+1}}.
\end{aligned}$$

Consequently, (4.2) has been proved by the fact $\bar{Z}(x_1, t_2)=\min_{\{t=t_2\}}\bar{Z}(x, t_2)$.

$\forall\{Z_j\}_1^\infty\in S$, by Ascoli-Arzela lemma, there exists a subsequence $\{Z_{jk}\}$ such that Z_{jk} uniformly converge to $Z\in S$ in $C(\bar{P}_T)$. Thus, S is a compact set of $C(\bar{P}_T)$. Let $V(Z)$ be the volume of Z, (its lower base is Z, its upper base is $Z=0$). $V(Z)$ is a continuous function on $Z\in S$, so there exists a $\tilde{Z}\in S$ such that

$$V(\tilde{Z})=V_0=\sup_{Z\in S} V(Z)<+\infty.$$

Now we claim that $\tilde{Z}$ is the convex monotone polyhedron which solved theorem 4.2. Otherwise, we have $|L_{\tilde{z}}(\bigcup_{\beta\in\alpha_{kj}} l_\beta)|\leq\sum_{\beta\in\alpha_{kj}}\sigma_\beta$, $k=1,\cdots, l$, $j=1,\cdots, h$, and there exist an $A_{k_1j_1}$ s.t.

$$|L_{\tilde{z}}(\bigcup_{\beta\in a_{k_1j_1}} l_\beta)| < \sum_{\beta\in a_{k_1j_1}} \sigma_\beta.$$

Note that $\tilde{Z}$ is strictly decreasing in t and displace $\tilde{Z}$ (X_{k_1}, t_{j_1}) in a distance of suffciently small ε_0 by the new convex-monotone polyhedron $\bar{Z}\varepsilon_0$. By the special construction of $\bar{Z}_\varepsilon$, $\bar{Z}_\varepsilon$ is different from $\tilde{Z}$ only at $a_{k_1j_1}$. Obviously, $\bar{Z}\varepsilon_0$ converges uniformly to $\tilde{Z}$ as $\varepsilon_0 \to 0$. By [8] Prop. 1.2, we have $\omega_{\bar{z}_{\varepsilon_0}} \to w_{\tilde{z}}$, so that

$$|L_{\bar{z}_{\varepsilon_0}}(\bigcup_{\beta\in a_{kj}} l\beta| \leq \sum_{\beta\in a_{kj}} \sigma\beta,\ k=1,\cdots, l,\ j=1,\ \cdots,\ h.$$

In particular, the above inequality holds with $a_{k_1j_1}$.

On the other hand, $V(\bar{Z}\varepsilon_0) > V(\tilde{Z})$ and $\bar{Z}\varepsilon_0 \in S$, which is impossible. Thus the assertion on $\tilde{Z}$ is true.

Proof of Theorem 4.1. Increase the density of the net of points $\{Y_i\}$, $\{X_k\}$, $\{t_j\}$ such that $\{X_k\} \subset \Omega'$. By theorem 4.2, we have a sequence of convex-monotone polyhedrons $\{u_m\} \subset C(\bar{Q})$ (here the sequence needn't to be unique because we haven't obtained the uniqueness of convex-monotone polyhedron in therem 4.2) which is uniformly Lipschitz continuous in x,

$$\frac{|u_m(x_1, t) - u_m(x_2, t)|}{|x_1 - x_2|} \leq C(\delta,\ n,\ d,\ \sup f),$$

and is uniformly Hölder continuous in t,

$$\frac{|u_m(x, t_1) - u_m(x, t_2)|}{|t_1 - t_2|^{\frac{1}{n+1}}} \leq C(n,\ d,\ \sup f).$$

Thus, by Ascoli-Arzela lemma, there exists a subsequence $\{u_m\}$ (still denoted by $\{u_m\}$) uniformly converging to some $u \in C(\bar{Q})$. By [8] Prop. 1.2, we have $w_{u_m} \to w_u$ in Q.

On the other hand, $w_{u_m}(\bigcup_{\beta\in a^m_{kj}} l^m_\beta) = \sum_{\beta\in a^m_{kj}} \sigma\beta = \int_{\bigcup_{\beta\in a^m_{kj}} u^m_\beta} f dx dt :\ = \mu m(\bigcup_{\beta\in a^m_{kj}} l^m_\beta)$, with $\bigcup_\beta U^m_\beta = Q$. It remains to prove there exists a subsequence $\{\mu_m\}$ (still denoted by $\{\mu_m\}$) which converges to a Radon measure μ defined by $\mu(E) = \int_E f dx dt$, $\forall$ Borel $E \subset Q$. For each compact $K \subset Q$, $\forall m \in N$, $\exists \{A^m_{k_1j_1}\}$, $\{A^m_{k_2j_2}\} \subset \{A^m_{kj}\}$, $k=1,\cdots,l, j=1,\cdots,h$, such that $\bigcup_{\beta_1\in\cup a^m_{k_1j_1}} U^m_{\beta_1} \subset K \subset \bigcup_{\beta_2\in a^m_{k_2j_2}} U^m_{\beta_2}$ and $|\bigcup_{\beta_2\in\cup a^m_{k_2j_2}} U^m_{\beta 2} \setminus \bigcup_{\beta_1\in\cup a^m_{k_1j_1}} U^m_{\beta 1}| \to 0,\ (m \to \infty)$.

Therefore $\forall \varepsilon > 0$, we have

$$\begin{aligned}\limsup_{m\to\infty} \mu_m(K) &\leq \limsup_{m\to\infty} \mu_m(\bigcup_{\beta_2\in\cup a^m_{k_2j_2}} U^m_{\beta_2}) \\ &= \limsup_{m\to\infty} \sum_{\beta_2\in\cup a^m_{k_2j_2}} \int_{U^m_{\beta_2}} f dx dt \leq \mu(K) + \varepsilon.\end{aligned}$$

Similarly, for each relatively compact open set $G \subset Q$, $\forall m$, $\exists \{A^m_{k_3j_3}\}$, $\{A^m_{k_4j_4}\} \subset \{A^m_{kj}\}$, $k=1,\cdots, l$, $j=1,\cdots,\ h$, such that $\bigcup_{\beta_3\in\cup a^m_{k_3j_3}} U^m_{\beta_3} \subset G \subset \bigcup_{\beta_4\in\cup a^m_{k_4j_4}} U^m_{\beta_4}$ and $|\bigcup_{\beta_4\in\cup a^m_{k_4j_4}} U^m_{\beta_4} \setminus \bigcup_{\beta_3\in\cup a^m_{k_3j_3}} U^m_{\beta_3}| \to 0,\ (m \to \infty)$.

Thus,

$$\liminf_{m\to\infty}\mu_m(G) \geq \liminf_{m\to\infty}\mu_m\Big(\bigcup_{\beta_3\in\cup a^m_{k_3j_3}} U^m_{\beta_3}\Big)$$

$$= \liminf_{m\to\infty}\sum_{\beta_3\in\cup a^m_{k_3j_3}}\int_{U^m_{k_3j_3}} f dx dt \geq \mu(G)-\varepsilon.$$

By the arbitrariness of ε, we have $\mu_m\to\mu$. Owing to $\omega_{\mu_m}=\mu_m$, we get $\omega_u=\mu$.

The initial boundary value condition $u=0$ on $\partial_p Q$ can be obtained directly from the construction of μ_m and the fact that μ_m uniformly converges to u in Q. Hence, u is the generalized solution of (1.1), (1.2).

The uniqueness can be proved by Theorem 2.2.

Proof of Theorem 1.1

Theorem 2.2 ensures the uniqueness. Here we only need to prove the existence. The idea is an approximation by a generalized solution obtained from Theorem 4.1. Let

$$f_m=\begin{cases} f+\dfrac{1}{m} & \text{in } Q_m=\Omega_m\times(0, T], \\ 0 & \text{in } Q\backslash Q_m, \end{cases}$$

where $\Omega_m=\{x\in\Omega \mid d(x, \partial\Omega)\geq\frac{1}{m}\}$. The f_m is a nonnegative bounded measurable function of Q with support $\Omega_m\times\{0, T]$.

Consider the approximation problem

$$\begin{cases} -(u_m)_t\det(D_x^2 u_m)=f_m & \text{in } Q, \\ u_m=0 & \text{on } \partial_p Q. \end{cases} \tag{4.3}$$

By Theorem 4.1, there exists a sequence of unique generalized solutions u_m which has a uniform bound $|u_m|\leq C(n, d, \sup f)$, and

$$|u_m(x, t_1)-u_m(x, t_2)|\leq C|t_1-t_2|^{\frac{1}{n+1}},\ x\in\Omega,\ t_1, t_2\in[0, T], \tag{4.4}$$

where C depends on n, d and $\sup f$.

On the other hand, $\forall$ compact set $K\subset\Omega$, u_m has a uniform Lipschitz constant in x (depending on K), a uniformly Hölder module in t. By Ascoli-Arzela lemma, there exists a subsequence of $\{u_m\}$, still denoted by $\{u_m\}$, which uniformly converges to u in $K\times(0, T]$. Then by [8] Prop. 1.2,

$$\int_Q g d\omega_{u_m}\to\int_Q g d\omega_u,\ \forall g\in C_0(Q),\ \text{supp} g\subset K.$$

We also have

$$\int_Q g d\omega_{u_m}=\int_Q g f_m dx dt\to\int_Q g f dx dt,\ \forall g\in C_0(Q).$$

So, we have proved that u is the generalized solution of equation (1.1).

Now we turn to prove the boundary condition, i.e. $\forall(x_0, t_0)\in\partial_p Q$, we have $\liminf_{(x,t)\to(x_0,t_0)} u(x, t)=0$. By (4.4), we have

$$|u(x, t_1)-u(x, t_2)|\le C|t_1-t_2|^{\frac{1}{n+1}},\ \forall x\in\Omega,\ t_1,\ t_2\in[0, T]. \tag{4.5}$$

Then $\liminf_{(x,t)\to(x_0,0)} u(x, t)\equiv\lim_{t\to 0} u(x_0, 0),\ \forall x_0\in\Omega$.

So we only need to prove $\liminf_{(x,t)\to(x_0,t_0)} u(x, t)=0,\ x_0\in\partial\Omega$.

$\forall x_0\in\partial\Omega$, let $u(x_0, t_0)=\liminf_{(x,t)\to(x_0,t_0)} u(x, t)$. By (4.5), we have $\liminf_{(x,t)\to(x_0,t_0)} u(x, t)=\liminf_{x\to x_0} u(x, t_0)$.

We first prove that when $x_0\in\partial\Omega$, $u(x_0, t)\in C^{\frac{1}{n+1}}([0, T])$. In fact, $\forall t_1, t_2\in[0, T]$, $(t_2<t_1)$,

$$u(x_0, t_1)=\liminf_{x\to x_0} u(x, t_1)=\liminf_{x\to x_0}\lim_{m\to\infty} u_m(x, t_1),$$

so $\exists x_m\to x_0$ s.t. $u(x_0, t_1)=\lim_{m\to\infty} u_m(x_m, t_1)$. Because $u(x_0, t_2)=\liminf_{x\to x_0}\lim_{m\to\infty} u_m(x, t_2)$, $\exists\{x_{m_j}\}\subset\{x_m\}$ s.t.

$$\lim_{m_j\to\infty} u_{m_j}(x_{m_j}, t_2) \text{ exists and} \ge u(x_0, t_2).$$

Here we denote $\{x_{m_j}\}$ by $\{x_m\}$. Then $\forall\varepsilon>0$, $\exists M>0$ such that, $\forall m>M$, we have

$$\begin{aligned}0 &\le u(x_0, t_2)-u(x_0, t_1)\le u_m(x_m, t_2)-u(x_0, t_1)+\varepsilon\\ &\le u_m(x_m, t_2)-u_m(x_m, t_1)+|u_m(x_m, t_1)-u(x_0, t_1)|+\varepsilon\\ &\le C|t_1-t_2|^{\frac{1}{n+1}}+|u_m(x_m, t_1)-u(x_0, t_1)|+\varepsilon\end{aligned}$$

Let $m\to\infty$ and note that $\varepsilon>0$ in arbitrary, so that

$$0\le u(x_0, t_2)-u(x_0, t_1)\le C|t_1-t_2|^{\frac{1}{n+1}}$$

If $u\not\equiv 0$ on $\partial\Omega\times[0, T]$, then $\exists(x_0, t_0)\in\partial\Omega\times[0, T]$ s.t. $u(x_0, t_0)<0$. If $t_0=0$ then

$$0>u(x_0, t_0)=\lim_{x\to x_0} u(x, t_0)=0,$$

which is impossible. Therefore $\exists\tau_0>0$ s.t. $u(x_0, t)<0$, $t\in[t_0-\tau_0, t_0]\subset[0, T]$. We can suppose $u(x_0, t_0-\tau_0)>u(x_0, t_0)$. Otherwise, $u(x_0, t_0-\tau_0)=u(x_0, t_0)$. Then $\exists\tau_1>0$ s.t. $u(x_0, t)<0$, $t\in[t_0-\tau_0-\tau_1, t_0]$, and thus we can suppose $u(x_0, t_0-\tau_0-\tau_1)>u(x_0, t_0)$. Then after finite times there are two cases happen, one is $\exists\tau>0$ s.t. $u(x_0, t)<0$, $t\in[t_0-\tau, t_0]$ and $u(x_0, t_0-\tau)>u(x_0, t_0)$, the other is $u(x_0, 0)<0$ which will not happen as we have proved.

By similar method used in [6] by Pogorelov, $\forall t^*\in[t_0-\tau, t_0]$, for sufficiently large $M>0$, $\exists\hat{\Omega}\subset\Omega$ s.t.

$$|\nabla u(x, t^*)|_{(n)}>M,\ x\in\hat{\Omega},$$

where $|\cdot|_{(n)}$ denotes the n-dimensional Lebesgue measure. Fix $\bar{t}\in[t-\tau_0, t_0]$, let $V(x, \bar{t})$ be a cone whose vertices are $u(x_0, \bar{t})$, and $V(x, \bar{t})=0$ on $\partial\Omega$. So $V(x, t)$ is continuous by the continuity of $u(x_0, t)$ in t, and $V(x, t)>u(x, t)$, so that $L_V(\Omega\times[t_0-\tau, t_0]\subset Lu(\Omega\times[t_0-\tau, t_0])$. We have $\exists\tilde{\Omega}\subset\Omega$ s.t.

$$\int_Q f dxdt \geq |L_u(\tilde{\Omega}\times[t_0-\tau, t_0])|\geq|L_V(\tilde{\Omega}\times[t_0-\tau, t_0])|$$
$$\geq C(u(x_0, t_0-\tau)-u(x_0, t_0))\cdot M,$$

which is impossible since M can be arbitrary large and f is bounded. Thus we have completed the proof.

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