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Infinitesimal stability of stable maps on manifolds with boundary

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Abstract

Let X be a compact orientable smooth m-manifold with boundary, $V_1, V_2, \ldots, V_r$ disjoint compact connected submanifolds in X where each V_i is also disjoint from ∂X if it is not a component of ∂X, and Y a smooth n-manifold. Then we show that a smooth map $f: X \to Y$ is strongly stable with respect to $V_1, V_2, \ldots, V_r$ if and only if f is strongly infinitesimally stable with respect to $V_1, V_2, \ldots, V_r$.

1. Introduction

When X is a closed smooth manifold and Y is a smooth manifold without boundary, Mather [3] showed that a smooth map $f: X \to Y$ is stable if and only if f is infinitesimally stable. In addition, von Essen [1] proved that for a compact smooth manifold with boundary X and a smooth manifold with boundary Y, if a smooth map $f: X \to Y$ is infinitesimally stable then it is stable. However von Essen [1] did not explain that the stability of $f: X \to Y$ induces the infinitesimal stability or not. Since it is a natural question, we want to prove that if $f: X \to Y$ is stable then it is infinitesimally stable.

Our main result in this paper is a generalized theorem which solves the above question positively. (See Theorem 1.) In the previous paper [5], we had stated this theorem in the restricted form with an outline of proof. However, the precise verification of the proof needs a minute modification of the past theory not in brief, and in this paper, the theorem will be proved by a more extended and wide-applicable form. Hence we believe that the publishing of this detailed proof for the generalized theorem is worth apart from the contents of our previous paper [5].

Let X be a compact smooth manifold with boundary and Y a smooth manifold with boundary. We define a notion of strong stability as follows: Let $V_1, V_2, \ldots, V_r$ be disjoint compact connected submanifolds in X where each V_i is also disjoint from ∂X if it is not a component of ∂X. Then $f: X \to Y$ is *strongly stable with respect to* $V_1, V_2, \ldots, V_r$ if f is stable and the diffeomorphism $X \to X$ of the right-left equivalence keeps $V_1, V_2, \ldots, V_r$ (see the detailed description in §2).

Next, we define a notion of strong infinitesimal stability which is defined as follows: Let α: $X \to Y$ be a smooth map and π_Y: $TY \to Y$ the canonical projection, and let $V_1, V_2, \ldots, V_r$ be disjoint compact connected submanifolds in X where each V_i is also disjoint from ∂X if it is not a component of ∂X. A smooth map $w: X \to TY$ is called a *vector field along* α if ω satisfies $\alpha = \pi_Y \circ w$. Then we say that α is *strongly infinitesimally stable with respect to* $V_1, V_2, \ldots, V_r$ if for every w, a vector field along α, there always exist a vector field s on X whose restriction to ∂X is a vector field on ∂X (i.e., each vector of s on ∂X is tangent to ∂X) and whose restrictions to $V_1, V_2, \ldots, V_r$; $\partial V_1, \partial V_2, \ldots, \partial V_k$ are vector fields on $V_1, V_2, \ldots, V_r$; $\partial V_1, \partial V_2, \ldots, \partial V_k$ respectively, and a vector field t on Y whose restriction to ∂Y is a vector field on ∂Y such that

$$w = (d\alpha) \circ s + t \circ \alpha,$$

where $d\alpha$: $TX \to TY$ is the differential of α.

Then as Theorem 1, we prove that if f is strongly stable with respect to $V_1, V_2, \ldots, V_r$ then f is strongly infinitesimally stable with respect to $V_1, V_2, \ldots, V_r$. Since we may assume that boundary is disjoint submanifolds of the manifold, our paper with the paper of von Essen [1] completes a proof of the equivalence between the stability and the infinitesimal stability of smooth maps on manifolds with boundary.

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2. Preliminaries

Let X be an m-dimensional smooth manifold with or without boundary and Y a smooth n-manifold, and let $V_1, V_2, \ldots, V_r$ be disjoint compact connected submanifolds with or without boundary in X where each V_i is also disjoint from ∂X if it is not a component of ∂X. We denote by $C^\infty(X, Y)$ the set of the smooth maps of X into Y with the Whitney C^∞ topology. A smooth map $f: X \to Y$ is *stable* if there exists an open neighborhood $N(f)$ of f in $C^\infty(X, Y)$ such that every g in $N(f)$ is *right-left equivalent* to f; i.e., there exist diffeomorphisms ϕ: $X \to X$ and φ: $Y \to Y$ satisfying $g = \varphi \circ f \circ \phi^{-1}$, and $f: X \to Y$ is *strongly stable with respect to* $V_1, V_2, \ldots, V_r$ if there exists an open neighborhood $N(f)$ of f in $C^\infty(X, Y)$ such that for every g in $N(f)$, there exist diffeomorphisms ϕ: $X \to X$ such as $\phi(V_i) = V_i$ $(1 \leq i \leq r)$ and φ: $Y \to Y$ satisfying $g = \varphi \circ f \circ \phi^{-1}$.

Trivially, if a smooth map f: $X \to Y$ is strongly stable then f is stable.

When X is a closed manifold, Y is a smooth manifold without boundary and f: $X \to Y$ is a smooth map, it is known that the following (i), (ii) and (iii) are equivalent.

(i) f is stable.

(ii) f is transverse stable.

(iii) f is infinitesimally stable.

The definition of the above each stability for manifolds without boundary is given in Golubitsky-Guillemin [2].

3. Strong stability is equivalent to strong infinitesimal stability

Throughout this section, X is a compact smooth manifold with boundary, Y is a smooth manifold with boundary and $V_1, V_2,\dots, V_r$ are disjoint connected closed submanifolds in X where each V_i is also disjoint from ∂X if it is not a component of ∂X. However, $V_1, V_2,\dots, V_r$ will be disjoint compact connected submanifolds with boundary in X in the end of this section.

Proposition 1. *$f: X\to Y$ is strongly stable with respect to $V_1, V_2,\dots, V_r$ if and only if f is strongly infinitesimally stable with respect* to $V_1, V_2,\dots, V_r$.

Corollary 1. *A smooth map $f: X\to Y$ is stable if and only if f is strongly infinitesimally stable (due to the definition* in [5]).

Proof. We can regard that ∂X is closed submanifolds in X. Hence the stability of f is equivalent to the strong stability of f with respect to ∂X. Similarly, the strong infinitesimal stability of f due to the definition in [5] of the case with boundary is equivalent to the strong infinitesimal stability with respect to ∂X. Therefore the proof is a direct consequence of Proposition 1. □

Before proving Proposition 1, we prepare a notion of strong transverse stability.

Definition 1. We define an s-fold k-multijet bundle $J^k_s(X, Y)$. Let $X^s=\underbrace{X\times\cdots\times X}_{s}$ and $X^{(s)}=\{(x_1,\dots,x_s)\in X^s | x_i\neq x_j \text{ for } 1\leq i<j\leq s\}$. For $\alpha: J^k(X, Y)\to X$ the source map, we define $\alpha^s: J^k(X, Y)^s=\underbrace{J^k(X, Y)\times\cdots\times J^k(X, Y)}_{s}\to X^s$ as the induced map by α. Then

$$J^k_s(X, Y)=(\alpha^s)^{-1}(X^{(s)})$$

is called an *s-fold k-multijet bundle.*

$X^{(s)}$ is a manifold since it is an open subset of X^s. Thus $J^k_s(X, Y)$ is an open subset of $J^k(X, Y)^s$ and is also a smooth manifold.

Let $f: X\to Y$ be smooth. Then we can also define $j^k_s f: X^{(s)}\to J^k_s(X, Y)$ by $j^k_s f(x_1,\dots,x_s)=(j^k f(x_1),\dots,j^k f(x_s))$.

Definition 2. We define $\overline{\mathrm{Diff}(X)}=\{\phi \in \mathrm{Diff}(X) | \phi(V_i)=V_i\ (^\forall i)\}$. Then let σ be an s-fold multijet in $J^k_s(X, Y)$ and $\overline{D^s_\sigma}$ be the orbit through σ under the action of $\overline{\mathrm{Diff}(X)}\times \mathrm{Diff}(Y)$ on $J^k_s(X, Y)$. We can prove that for each σ, $\overline{D^s_\sigma}$ is an immersed submanifold without boundary of $J^k_s(X, Y)$, furthermore, $\overline{D^s_\sigma}$ is a submanifold without boundary in $J^k_s(X, Y)$. Then f is said to be *strongly transverse stable with respect to* $V_1, V_2, \ldots, V_r$ if for every s with $1\leq s\leq n+1$ and any diagonal element σ in $J^n_s(X, Y)$, it is shown that $j^n_s f$ is transverse to $\overline{D^s_\sigma}$.

For a proof of Proposition 1, we prepare several lemmas. The proofs of these lemmas are simply modification of the case without boundary, thus if the reader needs their proofs then we recommend to see corresponding lemmas in Golubitsky-Guillemin [2].

Lemma 1. *If $f: X\to Y$ is strongly stable with respect to $V_1, V_2, \ldots, V_r$ then f is strongly transverse stable with respect to $V_1, V_2, \ldots, V_r$.*

Notation 1. $\overline{C^\infty(X, X)} \overset{\mathrm{def}}{=} \{u \in C^\infty(X, X) | u|V_i \in C^\infty(V_i, V_i)\ (^\forall i),\ u|\partial X \in C^\infty(\partial X, \partial X)\}$
$\overline{C^\infty(Y, Y)} \overset{\mathrm{def}}{=} \{u \in C^\infty(Y, Y) | u|\partial Y \in C^\infty(\partial Y, \partial Y)\}$
$\overline{J^k(X, X)} \overset{\mathrm{def}}{=} \{u \in J^k(X, X) |$ representative of u is in $\overline{C^\infty(X, X)}\}$
$\overline{J^k(Y, Y)} \overset{\mathrm{def}}{=} \{u \in J^k(Y, Y) |$ representative of u is in $\overline{C^\infty(Y, Y)}\}$
$\overline{C^\infty(TX)} \overset{\mathrm{def}}{=} \{v \in C^\infty(TX) | v|V_i \in C^\infty(TV_i)\ (^\forall i),\ v|\partial X \in C^\infty(T\partial X)\}$
$\overline{C^\infty(TY)} \overset{\mathrm{def}}{=} \{v \in C^\infty(TY) | v|\partial Y \in C^\infty(T\partial Y)\}$
$\overline{J^k(TX)} \overset{\mathrm{def}}{=} \{v \in J^k(TX) |$representative of v is in $\overline{C^\infty(TX)}\}$
$\overline{J^k(TY)} \overset{\mathrm{def}}{=} \{v \in J^k(TY) |$representative of v is in $\overline{C^\infty(TY)}\}$

As the second preparation, we prepare the following key lemma.

Lemma 2. *Let $X\to Y$ be a smooth map. Then, f is strongly infinitesimally stable with respect to $V_1, V_2, \ldots, V_r$ if and only if for any q* in Y *and any finite set $S\subset f^{-1}(q)$* with $\#S\leq n+1$,

$$J^n(f^*TY)_S=(df)(\overline{J^n(TX)}_S)+f^*(\overline{J^n(TY)}_q)$$

holds and $f(X)\subset \mathrm{Int}\, Y$ is satisfied.

Proof. Since we may assume $q\notin\partial Y$, the proof is reduced to simple modification of the case without boundary. □

Definition 3. Let $f: X\to Y$ be a smooth map with $q\in Y$ and put $S=\{p_1, p_2, \ldots, p_t\}\subset f^{-1}(q)$. Then f is *strongly simultaneously locally infinitesimally stable at $p_1, p_2, \ldots, p_t$ with respect to* V_1,

$V_2,\ldots, V_r$ if for germs of vector field along f, $[\tau_1]_{p_1}$, $[\tau_2]_{p_2},\ldots,[\tau_t]_{p_t}$, there exist $\zeta_i \in \overline{C^\infty(TX)}$ and $\eta \in \overline{C^\infty(TY)}$ such that $[\tau_i]_{p_i}=[(df)(\zeta_i)]_{p_i}+[\eta\circ f]_{p_i}$ $({}^\forall i)$ holds.

Lemma 3. *Let $f: X\rightarrow Y$ be a smooth map with $q\in Y$ and put $S=\{p_1, p_2,\ldots,p_t\}\subset f^{-1}(q)$. Then f is strongly simultaneously locally infinitesimally stable at $p_1, p_2,\ldots,p_t$ with respect to V_1, $V_2,\ldots, V_r$ if and only if*

$$J^n(f^*TY)_S=(df)(\overline{J^n(TX)_S})+f^*(\overline{J^n(TY)_q})$$

holds.

Proof of Proposition 1.

Sufficiency: It is obtained by Theorem 2.3 of [1, p.199].

Necessity: To begin with, we know that if $f: X\rightarrow Y$ is strongly stable with respect to V_1, $V_2,\ldots$, V_r then f is strongly transverse stable with respect to V_1, $V_2,\ldots$, V_r, by Lemma 1 and know that the strong stability implies $f(X)\subset \operatorname{Int} Y$.

Next, we show that if f is strongly transverse stable with respect to V_1, $V_2,\ldots$, V_r and $f(X)\subset \operatorname{Int} Y$ holds then f is strongly infinitesimally stable with respect to V_1, $V_2,\ldots$, V_r, thus we will show the equivalence condition of strong infinitesimal stability in Lemma 2. For the later arguments, we use the same notations in [2] in the sense of our case with submanifolds. When we review the proof of the case without boundary which is noted in [2, p. 140], we must check that the mapping

$$(d\gamma_2^S)\oplus(d\gamma_1^S): \overline{C^\infty(TX)}_S\oplus\overline{C^\infty(TY)}_q\rightarrow T_\sigma\overline{D_\sigma^s}$$

is onto or not, where $q\in \operatorname{Int} Y$, $S=\{p_1, p_2,\ldots,p_s\}\subset f^{-1}(q)$, and $\sigma=j_S^kf(S)$.

Let v be in $T_\sigma\overline{D_\sigma^s}$ and $c(t)$ a curve representing v in $\overline{D_\sigma^s}$. Then it is verified that if there exists a curve of diffeomorphisms $t\mapsto(g_t, h_t)$ in $\overline{\mathrm{Diff}(X)}\times\mathrm{Diff}(Y)$ such as $c(t)=j_S^kh_t(q)\circ\sigma\circ j_S^kg_t^{-1}(g_t(S))$, then v is an image of $(d\gamma_2^S)\oplus(d\gamma_1^S)$ by the similar arguments of [2], because if we define vector fields $\zeta\in C^\infty(TX)$ and $\eta\in C^\infty(TY)$ such as $\zeta_S=\frac{dg_t}{dt}(S)|_{t=0}$ and $\eta_q=\frac{dh_t}{dt}(q)|_{t=0}$ (for ${}^\forall S$ and ${}^\forall q$), then $\zeta\in\overline{C^\infty(TX)}$ and $\eta\in\overline{C^\infty(TY)}$ hold automatically. So, if we explain that the existence of a curve of diffeomorphisms $t\mapsto(g_t, h_t)$ in $\overline{\mathrm{Diff}(X)}\times\mathrm{Diff}(Y)$ such as $c(t)=j_S^kh_t(q)\circ\sigma\circ j_S^kg_t^{-1}(g_t(S))$, then to prove that $(d\gamma_2^S)\oplus(d\gamma_1^S)$ is onto is completed. For $p\in X$, let ω be a k-jet $(k>0)$ in $\overline{J^k(X, X)_{p,p}}$. Then we say that ω is invertible if any representative of ω is a diffeomorphism on a neighborhood of p. The invertible k-jets form a group under composition and a manifold since they are an open subset of $\overline{J^k(X, X)_{p,p}}$. Then we denote the set of invertible k-jets at p in $\overline{J^k(X, X)_{p,p}}$ by $\overline{G^k(X)_p}$. Also $\overline{G^k(Y)_q}$ is defined based on $\overline{J^k(Y, Y)_{q,q}}$.

Since $\overline{\overset{\circ}{G}}=\overline{G^k(\overset{\circ}{X})_S}\times\overline{G^k(\overset{\circ}{Y})_q}$ has a structure of Lie group and $\overline{D^s_\sigma}$ is the $\overline{\mathrm{Diff}(X)}\times\mathrm{Diff}(Y)$-orbit of σ, the arguments of the case without boundary is also effective to our case with submanifolds for showing the existence of (g_t, h_t).

Based on the above, we will show that f satisfies the condition of

$$J^n(f^*TY)_S=(df)(\overline{J^n(TX)_S})+f^*(\overline{J^n(TY)_q})$$

with $1\leq \#S\leq n+1$ due to Lemma 2, and then, Lemma 3 implies that it is sufficient to show that for $S=\{p_1, p_2,\dots,p_s\}\subset f^{-1}(q)$ $(1\leq s\leq n+1)$, f is strongly simultaneously locally infinitesimally stable at $p_1, p_2,\dots,p_s$.

We put $\sigma=j^k_Sf(S)$ again. With respect to our situation, an injection map λ^S: $J^n_S(f^*TY)_S\to T_\sigma J^n_S(X, Y)$ is defined as follows. For $\sigma_i=j^nf(p_i)$, let $\omega\in J^n(f^*TY)_{p_i}$ and a vector field along f τ: $X\to TY$ its representative. Since we may assume $q\in \mathrm{Int}\,Y$, there exists a deformation of f, F_t, such as $\frac{dF_t}{dt}|_{t=0}=\tau$. We take a path $t\to j^nF_t(p_i)$ in $J^n(X, Y)$ and define $\lambda_i(\omega)$ using F_t as a tangent vector at $t=0$, and then we define λ^S as $\lambda^S=\oplus^s_{i=1}\lambda_i$. Since the condition of strong transverse stability of f guarantees the existence of $w\in T_\sigma\overline{D^s_\sigma}$ and $v\in T_SX^{(s)}$ so that $\lambda^S(\tau)=w+(dj^n_Sf)(v)$ holds, the remainder of the proof is similar to that of the case without boundary. □

Remark 1. In the above proof of Proposition 1, we have shown that the notion of strong transverse stability with $f(X)\subset\mathrm{Int}\,Y$ is equivalent to the notion of strong infinitesimal stability. When we include the information of ∂Y into the definition of the map j^k_Sf beforehand, we will not need the additional condition $f(X)\subset\mathrm{Int}\,Y$ for our equivalence relation.

Theorem 1. *A smooth map $f: X\to Y$ is strongly stable with respect to disjoint compact connected submanifolds with boundary $V_1, V_2,\dots, V_r$ in X if and only if f is strongly infinitesimally stable with respect to $V_1, V_2,\dots, V_r$, where each V_i is also disjoint from ∂X if it is not a component of ∂X.*

Proof. It is obtained by repeating the similar modification with the proof of Proposition 1. □

Corollary 2. *If a smooth map $f: X\to Y$ is strongly stable with respect to $V_1, V_2,\dots, V_r$ (of Theorem 1) then $f|\cup_iV_i$ is also stable.*

Proof. Theorem 1 implies that if f is strongly stable with respect to $V_1, V_2,\dots, V_r$ then $f|\cup_iV_i$

is strongly infinitesimally stable due to the definition of [5], and hence, Corollary 1 implies that $f|\bigcup_i V_i$ is stable. □

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