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On confluences of hypergeometric integrals

Yoshishige Haraoka

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Abstract

In [KHT1] we introduced the generalized confluent hypergeometric functions (GCHGFs, for short) as Radon transforms of characters of linear abelian groups (see also [GRS]). The GCHGFs contain the Gauss hypergeometric function, the Kummer conflunet hypergeometric function, the Bessel function, the Hermite-Weber function, the Airy integral and their generalizations in several variables. It is known that the Kummer confluent hypergeometric differential equation is obtained from the Gauss hypergeometric differential equation by the confluence of two of the singular points, and the Hermite-Weber differential equation is obtained from the Kummer's differential equation also by the confluence of the singular points. In [KHT2] we gereralized these process for the systems of differential equations satisfied by the GCHGFs, and called the generalized process the confluence. We can also consider the confluence of the GCHGFs by using the integral representations coming from the Radon transforms. In this sense the confluence is a limit process which gives an integral representation of a GCHGF of some confluence type as a limit of an integral representation of another GCHGF of different confluence type.

In this note we prove convergences of the limit process of the confluence of GCHGFs which are represented by one dimensional integrals. Convergences of similar limits are discussed in [H1] and [H2], and the present result is a generalization of them.

Define polynomials $\theta_j(x_1,\dots,x_j)$ $(j=1, 2,\dots)$ by the generating function

$$\log(1+x_1T+x_2T^2+\cdots)=\sum_{j=1}^{\infty}\theta_j(x_1,\dots,x_j)T^j. \tag{1}$$

For $j=0, 1, 2,\dots$, take $z_{0j}, z_{1j}\in C$ and $\alpha_j\in C$, and assume that

$$z_{10}\neq 0,\ \begin{vmatrix} z_{00} & z_{01} \\ z_{10} & z_{11}\end{vmatrix}\neq 0.$$

Set

$$\ell_j = \ell_j(t) := z_{0j} + t z_{1j} \quad (j=0, 1, 2, \ldots),$$

and denote by x_0 the zero of $\ell_0(t)$; i.e. $x_0 = -z_{00}/z_{10}$. For $m \geq 1$ we consider the integral

$$\int_\Delta \ell_0^{a_0} e^{\sum_{j=1}^{m} a_j \theta_j(\ell_1/\ell_0, \ldots, \ell_j/\ell_0)} f(t)\,dt, \tag{2}$$

where $f(t)$ is a function holomorphic at $t = x_0$. The GCHGFs on $Z_{2,n+1}$ (see [H1], [H2] for the definition) can be represeted by the integral of the form (2). We call the integral (2) a hypergeometric integral of confluence type m.

Following [KHT2] we explain the formal confluence of hypgergeometric integrals of confluence type m. We assume that $f(t)$ can be written as

$$f(t) = \ell_{m+1}(t)^{a_{m+1}} \hat{f}(t),$$

where $\hat{f}(t)$ is holomorphic at $t = x_0$. Let ε be a parameter, and set

$$\begin{aligned} V_\varepsilon(t) = {} & \ell_0^{a_0 - \varepsilon^{-m-1} a_{m+1}} e^{\sum_{j=1}^{m} (a_j - \varepsilon^{-m-1+j} a_{m+1}) \theta_j(\ell_1/\ell_0, \ldots, \ell_j/\ell_0)} \\ & \times (\ell_0 + \varepsilon \ell_1 + \cdots + \varepsilon^{m+1} \ell_{m+1})^{\varepsilon^{-m-1} a_{m+1}}. \end{aligned} \tag{3}$$

Then we have

$$\lim_{\varepsilon \to 0} V_\varepsilon(t) = \ell_0^{a_0} e^{\sum_{j=1}^{m+1} a_j \theta_j(\ell_1/\ell_0, \ldots, \ell_j/\ell_0)} =: \hat{V}(t). \tag{4}$$

Hence a hypergeometric integral $\int_\Delta V_\varepsilon(t) \hat{f}(t)\,dt$ dt of confluence type m depending on a parameter ε converges formally to a hypergeometric integral $\int_\Delta \hat{V}(t) \hat{f}(t)\,dt$ of confluence type $m+1$. Here formally means that we discuss only the convergence of the integrands.

Now we consider integrals over the chain

$$\Delta_\vartheta := \{t = x_0 + \delta e^{\sqrt{-1}\vartheta};\ 0 < \delta < R\}, \tag{5}$$

where R and ϑ are fixed real numbers. In [H1] and [H2] we showed the convergence

$$\int_{\Delta_\vartheta} V_\varepsilon(t) f(t)\,dt \to \int_{\Delta_\vartheta} \hat{V}(t) f(t)\,dt \quad (\varepsilon \to 0) \tag{6}$$

under the conditions $-\pi/2 < \arg(\varepsilon^{-m-1} a_{m+1}) < \pi/2$ and $\arg(\varepsilon^{-m-1} a_{m+1}) = \pi/2$, respectively. In this note we shall show the convergence of (6) when $\arg \varepsilon$ takes any value but fixed.

Theorem. *Let v be any real number. Set*

$$\sigma := \frac{1}{m+1} \arg a_{m+1} + \arg\left(\frac{z_{00} z_{11} - z_{10} z_{01}}{z_{10}^2}\right), \tag{7}$$

and assume that ϑ satisfies the inequalities

$$\frac{\pi}{2}+2k_1\pi<-\nu+\frac{1}{m+1}\arg a_{m+1}+m(\sigma-\vartheta)<\frac{3}{2}\pi+2k_1\pi, \tag{8}$$

$$-\frac{\pi}{2}+2k_2\pi<(m+1)(\sigma-\vartheta)<\frac{\pi}{2}+2k_2\pi, \tag{9}$$

where k_1 and k_2 are integers taken so that the intervals in ϑ defined by (8) and (9) intersect. Moreover, if we assume that $\hat{f}(t)$ is bounded on the chain Δ_ϑ, then we have

$$\int_{\Delta_\vartheta} V_\varepsilon(t)\hat{f}(t)dt \rightarrow \int_{\Delta_\vartheta} \hat{V}(t)\hat{f}(t)dt$$

as ε tends to 0 satisfying $\arg\varepsilon=\nu$.

The following lemmas can be shown immediately.

Lemma 1.

$$\theta_j\left(\frac{\ell_1}{\ell_0},\dots,\frac{\ell_j}{\ell_0}\right)=\frac{(-1)^{j-1}}{j}\left(\frac{\ell_1}{\ell_0}\right)^j+(\text{lower in } \ell_0^{-1}).$$

Lemma 2. *Let $t=x_0+\delta e^{\sqrt{-1}\vartheta}$ be on the chain Δ_ϑ. If δ is sufficiently small, we get*

$$\arg\left(\frac{\ell_1(t)}{\ell_0(t)}\right)\sim\pi+\arg\left(\frac{z_{00}z_{11}-z_{10}z_{01}}{z_{10}^2}\right)-\vartheta.$$

Proof of Theorem. Put $\arg\varepsilon=\nu$. Since x_0 is a singular point of both $V_\varepsilon(t)$ and $\hat{V}(t)$, the direction ϑ should be chosen so that the integrals over Δ_ϑ converge. Assume that $|\varepsilon|$ is sufficiently small. Then by Lemma 1, we have

$$e^{\sum_{j=1}^{m}(a_j-\varepsilon^{-m-1+j}a_{m+1})\theta_j\left(\frac{\ell_1}{\ell_0},\dots,\frac{\ell_j}{\ell_0}\right)}\sim e^{-\varepsilon^{-1}a_{m+1}\frac{(-1)^{m-1}}{m}\left(\frac{\ell_1}{\ell_0}\right)^m}$$

as $t\rightarrow x_0$. Thus for the convergence of the integral of $V_\varepsilon(t)\hat{f}(t)$, we must take ϑ so that

$$\frac{\pi}{2}+2k_1\pi<\arg\left(-\varepsilon^{-1}a_{m+1}\frac{(-1)^{m-1}}{\mathrm{m}}\left(\frac{\ell_1(t)}{\ell_0(t)}\right)^m\right)<\frac{3}{2}\pi+2k_1\pi$$

holds for an integer k_1. Since we may assume that $\delta=|t-x_0|$ is small, this inequality is equivalent to (8) by virtue of Lemma 2. Again by Lemma 1 we have

$$e^{\sum_{j=1}^{m+1}a_j\theta_j\left(\frac{\ell_1}{\ell_0},\dots,\frac{\ell_j}{\ell_0}\right)}\sim e^{a_{m+1}\frac{(-1)^m}{m+1}\left(\frac{\ell_1}{\ell_0}\right)^{m+1}}$$

as $t\rightarrow x_0$, and hence, for the convergence of the integral of $\hat{V}(t)\hat{f}(t)$, ϑ should taken as

$$\frac{\pi}{2}+2k_2\pi<\arg\left(a_{m+1}\frac{(-1)^m}{m+1}\left(\frac{\ell_1}{\ell_0}\right)^{m+1}\right)<\frac{3}{2}\pi+2k_2\pi$$

for an integer k_2. By Lemma 2 this inequality is equivalent to (9).

By virtue of (4), to prove the convergence (6) it is sufficient to show that $V_\varepsilon(t)$ is bounded on Δ_θ uniformly in ε and t (cf. [R], [Z]). We may assume that $|\varepsilon|$ and $|t-x_0|$ are sufficiently small. Then we have

$$\begin{aligned}
&\ell_0^{-\varepsilon^{-m-1}a_{m+1}}(\ell_0+\varepsilon\ell_1+\cdots+\varepsilon^{m+1}\ell_{m+1})^{\varepsilon^{-m-1}a_{m+1}}\\
&\quad=\left(1+\varepsilon\frac{\ell_1}{\ell_0}+\cdots+\varepsilon^{m+1}\frac{\ell_{m+1}}{\ell_0}\right)^{\varepsilon^{-m-1}a_{m+1}}\\
&\quad\sim\left(1+\varepsilon\frac{\ell_1}{\ell_0}\right)^{\varepsilon^{-m-1}a_{m+1}},\\
&\ell_0^{a_0}e^{\sum_{j=1}^{m}(a_j-\varepsilon^{-m-1+j}a_{m+1})\theta_j\left(\frac{\ell_1}{\ell_0},\dots,\frac{\ell_j}{\ell_0}\right)}\\
&\quad\sim e^{-\varepsilon^{-m-1}a_{m+1}\sum_{j=1}^{m}\varepsilon^j\frac{(-1)^{j-1}}{j}\left(\frac{\ell_1}{\ell_0}\right)^j}.
\end{aligned}$$

Thus, by setting

$$y=\varepsilon\frac{\ell_1}{\ell_0},$$

we get

$$V_\varepsilon(t)\sim\bar{V}_\varepsilon(t):=(1+y)^{\varepsilon^{-m-1}a_{m+1}}e^{-\varepsilon^{-m-1}a_{m+1}\sum_{j=1}^{m}\frac{(-1)^{j-1}}{j}y^j} \tag{10}$$

We set

$$\begin{aligned}
&\varphi=\arg(\varepsilon^{-m-1}a_{m+1})=-(m+1)\nu+\arg a_{m+1},\\
&\tau=\arg y=\arg\varepsilon+\arg\left(\frac{\ell_1}{\ell_0}\right),\\
&\eta=|y|.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\varepsilon^{-m-1}a_{m+1}=|\varepsilon^{-m-1}a_{m+1}|(\cos\varphi+\sqrt{-1}\sin\varphi),\\
&\frac{(-1)^{j-1}}{j}y^j=\frac{(-1)^{j-1}}{j}\eta^j(\cos j\tau+\sqrt{-1}\sin j\tau),
\end{aligned}$$

hence

$$|\bar{V}_\varepsilon(t)|=e^{|\varepsilon^{-m-1}a_{m+1}|\phi(y)},$$

where we set

$$\phi(y):=\cos\varphi\left(\log|1+y|-\sum_{j=1}^{m}\frac{(-1)^{j-1}}{j}\eta^j\cos j\tau\right)$$
$$-\sin\varphi\left(\arg(1+y)-\sum_{j=1}^{m}\frac{(-1)^{j-1}}{j}\eta^j\sin j\tau\right).$$

We shall show $\phi(y)\leq 0$. Setting $\phi(y)=\phi(\eta, \tau)$, we see $\phi(0, \tau)=0$, so that it is sufficient to show that $\phi_\eta(\eta, \tau)\leq 0$. Applying the sine formula and the second cosine formula to the triangle surrounded by the complex numbers 1, y and $1+y$, we obtain

$$\log|1+y|=\frac{1}{2}\log(1+2\eta\cos\tau+\eta^2),$$

$$\arg(1+y)=\arctan\frac{\eta\sin\tau}{1+\eta\cos\tau}.$$

Thus we have

$$\frac{\partial}{\partial\eta}\phi(\eta, \tau)=\cos\varphi\left(\frac{\cos\tau+\eta}{1+2\eta\cos\tau+\eta^2}-\sum_{j=1}^{m}(-1)^{j-1}\eta^{j-1}\cos j\tau\right)$$
(11)
$$-\sin\varphi\left(\frac{\sin\tau}{1+2\eta\cos\tau+\eta^2}-\sum_{j=1}^{m}(-1)^{j-1}\eta^{j-1}\sin j\tau\right).$$

Here we recall the generating functions of Chebyshev functions of the first and the second kind ([E]):

(12)
$$\frac{\cos\tau+\eta}{1+2\eta\cos\tau+\eta^2}=\sum_{j=1}^{\infty}(-\eta)^{j-1}\cos j\tau,$$

(13)
$$\frac{\sin\tau}{1+2\eta\cos\tau+\eta^2}=\sum_{j=1}^{\infty}(-\eta)^{j-1}\sin j\tau.$$

From (12) we obtain

$$(\cos\tau+\eta)-(1+2\eta\cos\tau+\eta^2)\sum_{j=1}^{m}(-1)^{j-1}\eta^{j-1}\cos j\tau$$
$$=(1+2\eta\cos\tau+\eta^2)\sum_{j=m+1}^{\infty}(-1)^{j-1}\eta^{j-1}\cos j\tau,$$

in which the right hand side consists of the terms in η^k $(k\geq m)$, while the left hand side consists of the terms in $\eta^k\,(0\leq k\leq m+1)$. Hence both sides contain only the terms in η^m and η^{m+1}. Thus we get

$$(\cos\tau+\eta)-(1+2\eta\cos\tau+\eta^2)\sum_{j=1}^{m}(-1)^{j-1}\eta^{j-1}\cos j\tau$$
$$=(-1)^m\{\eta^m\cos(m+1)\tau+\eta^{m+1}\cos m\tau\},$$

and hence

$$\frac{\cos\tau+\eta}{1+2\eta\cos\tau+\eta^2}-\sum_{j=1}^{m}(-1)^{j-1}\eta^{j-1}\cos j\tau$$
$$=\frac{(-1)^m\{\eta^m\cos(m+1)\tau+\eta^{m+1}\cos m\tau\}}{1+2\eta\cos\tau+\eta^2}.$$

In a similar way we have

$$\frac{\sin\tau}{1+2\eta\cos\tau+\eta^2}-\sum_{j=1}^{m}(-1)^{j-1}\eta^{j-1}\sin j\tau$$
$$=\frac{(-1)^m\{\eta^m\sin(m+1)\tau+\eta^{m+1}\sin m\tau\}}{1+2\eta\cos\tau+\eta^2}.$$

Putting these into (11), we obtain

$$\frac{\partial}{\partial\eta}\phi(\eta,\,\tau)=\frac{(-1)^m\{\eta^m\cos(\varphi+(\mathrm{m}+1)\tau)+\eta^{m+1}\cos(\varphi+m\tau)\}}{1+2\eta\cos\tau+\eta^2}. \tag{14}$$

By using Lemma 2 we see that

$$\begin{aligned}\tau&\sim\nu+\pi+\arg\left(\frac{z_{00}z_{11}-z_{10}z_{01}}{z_{10}^2}\right)-\vartheta\\&=-\frac{\varphi}{m+1}+\pi+(\sigma-\vartheta),\end{aligned} \tag{15}$$

and hence

$$\varphi+(m+1)\tau\sim(m+1)\pi+(m+1)(\sigma-\vartheta). \tag{16}$$

By the assumption (9) we obtain from (16) that

$$-\frac{\pi}{2}+2k_2\pi+(m+1)\pi<\varphi+(m+1)\tau<\frac{\pi}{2}+2k_2\pi+(m+1)\pi,$$

from which we get

(17) $$\mathrm{sgn}(\cos(\varphi+(m+1)\tau))=(-1)^{m-1}.$$

Again by (15) we have

(18) $$\varphi+m\tau\sim-\nu+\frac{1}{m+1}\arg_{m+1}+m\pi+m(\sigma-\vartheta).$$

By the assumption (8) we obtain from (18) that

$$\frac{\pi}{2}+2k_1\pi+m\pi<\varphi+m\tau<\frac{3}{2}\pi+2k_1\pi+m\pi,$$

from which we get

(19) $$\mathrm{sgn}(\cos(\varphi+m\tau))=(-1)^{m-1}.$$

Since $1+2\eta\cos\tau+\eta^2\geq 0$, we obtain from (14), (17) and (19) that

$$\frac{\partial}{\partial\eta}\phi(\eta,\ \tau)<0,$$

which shows the boundedness of $|\bar{V}_\varepsilon(t)|$. This completes the proof.

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Yoshishige Haraoka
Department of Mathematics,
Faculty of Science,
Kumamoto University,
Kumamoto 860-8555 Japan