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${}^2D_{p+1}(2)(5\leq p\neq 2^m-1)$ Can be Characterized by Its Order Components

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Abstract

In this paper we prove that if G is a finite group, $M={}^2D_{p+1}(2), p\neq 2^m-1$, G and M have the same order components, then $G\cong M$

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1. Introduction

If G is a finite group, we define the prime graph $\Gamma(G)$ as following: its vertices are the primes dividing the order of G, and two vertices p and q are joined by an edge, if and only if there is an element in G of order pq. We demote the set of all the connected components of graph $\Gamma(G)$ by $T(G)=\{\pi_i(G), \textit{for } i=1, 2\cdots t(G)\}$, where $t(G)$ is the number of connected components of $\Gamma(G)$, and if G is of even order we always assume 2 in π_1. We also demote the set of all the primes dividing n by $\pi(n)$ where n is a natural number, obviously $|G|$ can be expressed as a product of $m_1, m_2,\cdots, m_{t(G)}$, where m_i is a positive integer with $\pi(m_i)=\pi_i$. All m_i, are called the order components of G. Let $OC(G)=\{m_1, m_2, \cdots m_{t(G)}\}$ be the set of order components of G. The order components of non-abelian simple groups having at least two prime graph components have been abtained in [2]

Some simple groups are characterized by their order components, such as a finite simple group with at least three prime graph components [1], sporadic simple groups [2], Suzuki-Ree groups [3], $G_2(q)$[4], $E_8(q)$[5], $PSL_2(q)$[6], ${}^3D_4(q)$[7], $PSU_5(q)$[10], $PSU(3, q)$ for $q>5$[11], ${}^2D_4(q)$[12], ${}^2E_6(q)$[13], $E_6(q)$[14]. In this paper we continue this work and will prove the following

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theorem:

Theorem. Let $M={}^2D_{p+1}(2)$, $p\neq 2^m-1$. If a finite group G has the same order components of M, then $G\cong M$.

2. Preliminary Results

Lemma 1. *If G is a 2-Frobenius group of even order, then $G=ABC$, where AB and BC are Frobenius groups with kernels A and B, complements B and C respectively. Moreover*

(1) $t(\Gamma(G))=2$, $\pi(B)=\pi_2$ *and* $\pi(A)\cup\pi(C)=\pi_1$.

(2) *B and C are cyclic groups satisfying that* $|C|\,|\,|Aut(B)|$.

(3) $OC(G)=\{|A|\cdot|C|, |B|\}$, *and A is nilpotent.*

Lemma 2. [[2] *Lemma* 6] *If $t(G)\geq 2$, H is a π_1 subgroup of G, and $H\lhd G$, then* $\prod_{j=1, j\neq i}^{t(G)} m_j \,|\, (|H|-1)$.

The folloing lemma is a corollary of Theorem A of [15].

Lemma 3. *If G is a simple group with $t(G)\geq 2$, then one of the following holds:*

(1) *G is a Frobenius group or 2-Frobenius group;*

(2) *G has a normal series $1\unlhd H\unlhd K\unlhd G$ such that H is a nilpotent π_1 group, K/H is a non-abelian simple group, any odd order component of G is equal to one of those of K/H, G/K is a π_1 group, and* $|G/K|\,|\,|Out\,(K/H)|$.

Lemma 4. [[16] *Remark*] *The equation $p^m-q^n=1$ holds iff $3^2-2^3=1$, where p, q are primes and m, $n>1$.*

Lemma 5. [17] *Let p be a prime and n be a natural number, $n\geq 2$. Then there exists a prime divisor r of p^n-1 which does not divide p^m-1 for any natural number $m<n$. except $n=6$, $p=2$ and $n=2$, $p+1$ is a power of 2. Such r is called a primitive prime divisor of p^n-1.*

Of course a primitive prime divisor of p^n-1 can't divide p^n+1 or p^m-1 for $1\leq m<2n$.

Lemma 6. [[9] *Theorem* 1] *If $a=(q^{n+1}+1)\prod_{i=1}^{n}(q^{2i}-1)$ where q is a power of a prime, then there is a positive integer f dividing a such that $(f, a/f)=1$. Let $f=\prod p_i^{a_i}$, then one of the following holds:*

(*1*) *If* $10\leq n\leq 17$, *then* $p_i^{a_i}<\frac{q^n-1}{n(q-1)}$ *and* $f>q^{4(n+1)}$;

(*2*) *If* $n\geq 18$, then $p_i^{a_i}<\frac{q^n-1}{n(q-1)}$ *and* $f>q^{6(n+1)}$.

Lemma 7. [[18] *Lemma* 1] *Let $n\geq 6$ be a positive integer, then there exists at least $s(n)$ prime numbers p_i such that $\frac{n+1}{2}<p_i<n$, where $1\leq i\leq s(n)$ and $s(n)$ is one of the following*:

$s(n)=6$	$n\geq 49$
$s(n)=5$	$42\leq n\leq 47$
$s(n)=4$	$38\leq n\leq 41$
$s(n)=3$	$18\leq n\leq 37$
$s(n)=2$	$14\leq n\leq 17$
$s(n)=1$	$6\leq n\leq 13$

Lemma 8. *Let p be an odd prime, e the exponent of 4 modulo p. Suppose that $4^e=1+p^rk$, where p is co-prime to k. If t is a natural number satisfying $t=p^su$, $(p, u)=1$ then $p^{r+s}\|4^{et}-1$.*

Proof. Because $4^{et}-1=(1+p^r\cdot k)^{p^su}-1=\sum_{i=1}^{p^su}\frac{(p^su)!}{(p^su-i)!\,i!}(p^rk)^i$, so $p^{r+s}\|4^{et}-1$.

Lemma 9. *Let $a=\prod_{i=1}^{n}(2^{2i}-1)$. If p is an odd prime number and $p|a$ then $a_p<2^{3n}$. Furthermore, if $p\geq 5$ then $a_p<2^{2n}$, where a_p is the exponent of the power with base the prime p in the standard decomposition of a.*

Proof. Let e be the exponent of 4 modulo p and $4^e=1+p^rk$, $p\nmid k$. Assume that $b=(4^e-1)(4^{2e}-1)\cdots(4^{we}-1)$, where $w=[\frac{n+1}{e}]$. Then

$$a_p=b_p=p^{rw+\Sigma_{j=1}^{\infty}[w/p^j]}\leq p^{rw+w/(p-1)}.$$

Since $p^r\| 4^e-1$, which implies that $p^r\|(2^e-1)$ or $p^r\|(2^e+1)$, so

$$a_p<(2^e+1)^w(2^e+1)^{w/(p-1)}=(2^e+1)^{wp/(p-1)}\leq(3^e+1)^{1.5w}<2^{3n}.$$

Clearly if $p\geq 5$ then $a_p>2^{2n}$.

3. Proof of the Theorem

Proof. If $M={}^2D_{p+1}(2)$ where $p\neq 2^m-1$, then the unique odd order component of M is $m_2=2^p-1$. We will prove the Theorem step by step.

Step 1. G can't be a Frobenius group or 2-Frobenius group.

(1) If G is a Frobenius group with Frobenius kernel H and Frobenius complement K, let $|H|=m_1$, $|K|=m_2$. Since $p\geq 5$, there exists a primitive prime divisor r of $2^{2(p+1)}-1$ and $S_r\in Syl_r(G)$, then $|S_r|\|(2^{p+1}+1)$ and $S_r\trianglelefteq G$. By Lemma 2, $|S_r|\equiv 1(\mathrm{mod}\ m_2)$, which is impossible.

(2) If G is a 2-Frobenius group, then there is a normal series $1 \unlhd H \unlhd K \unlhd G$ such that H is a nilpotent π_1 group, $|K/H|=m_2$, $|G/K|\,|(|K/H|-1)=2^p-2$. So $(2^{p+1}+1)\,||H|$. Similarly to (1), it's impossible.

From (1), (2) and Lemma 3 we know there is a normal series $1 \unlhd H \unlhd K \unlhd G$ such that H is a nilpotent π_1 group, K/H is a non-abelian simple group and $m_2 \in OC(K/H)$, G/K is a π_1 group and $|G/K|\,|\,|Out(K/H)|$.

Step 2. $K/H \not\cong E_7(2)$, $E_7(3)$, $A_2(2)$, $A_2(4)$, ${}^2A_5(2)$, ${}^2E_6(2)$, ${}^2F_4(2)'$ or one of the sporadic simple groups.

Because $p \geq 5$, we have that J_4, ON, Ly, B or HN has an order component of type 2^p-1, $p \neq 2^m-1$, which implies p=5 and $M \cong {}^2D_6(5)$. But the order of any one of J_4, ON, Ly, B or HN can't divide $|{}^2D_6(5)|$. This concludes **Step 2.**

Step 3. $K/H \not\cong A_n$.

Otherwise, $|A_{2^p-1}|\,|\,|A_n|\,|\,|{}^2D_{p+1}(2)|$. By $p \geq 5$, there exist at least three prime numbers p_i satisfying $2^{p-1}<p_i<2^p-1$ by Lemma 7, which of course divide $|A_n|$. But there exist at most two prime divisors of $|{}^2D_{p+1}(2)|$ between 2^{p-1} and 2^p-1, a contradiction.

Step 4. $K/H \not\cong A_n(q)$ or ${}^2A_n(q)$.

(1) If $K/H \cong A_1(q)$, $4|(q+1)$, then $2^p-1=q$ or $(q-1)/2$. Furthermore, $2^p-q=1$ or $2^{p+1}-q=1$, which contradicts to Lemma 4.

(2) If $K/H \cong A_1(q)$, $4|(q-1)$, then $2^p-1=q$ or $(q+1)/2$. Similarly to (1), we have that $2^p-1 \neq q$.

If $2^p-1=(q+1)/2$ then $q=2^{p+1}-3$ and $|K/H|<2^{3(p+1)}$. Assume $q=r^f$. Of course r can't be 2 or 3, so $r \geq 5$, But $5^{2p}>2^{p+1}-3$. Hence $|G/K|<2^{p+1}$ as $|G/K|\,|\,|Out(K/H)|=2f$. If $p+1 \geq 10$, by Lemma 6, then there exists a positive integer f satisfying:

$$f=\prod p_i^{a_i},\ p_i^{a_i}<2^p-1,\ f|\,|G|,\ (f,|G|/f)=1 \text{ and } f>q^{4(p+1)}.$$

Hence $(f, |H|) \neq 1$, let $S_{p'} \in Syl_{p'}(G)$ and p' be a prime satisfying $p'|(f, |H|)$. Now we have that $S_{p'} \unlhd G$ and $S_{p'}$ is a π_1 group, which contradicts to Lemma 2. which is impossible for $p'>5$. By trivial calculation we can show that p' can't be 2, 3, or 5.

Similarly, we can show that $K/H \not\cong A_1(q)$, $2|q$.

(4) If $K/H\cong A_{p'}(q)$, $q-1|\ p'-1$, then $2^p-1=(q^{p'}-1)/(q-1)$. Thus $q^{p'}\geq 2^p$.

If $p'>7$ then $q^{p'(p'+1)/2}>2^{3(p+1)}$, which implies q is a power of 2 by Lemma 9. Suppose $q=2^r$ then $(2^{rp'}-1)/(2^r-1)=2^p-1$, hence $2^{rp'}=2^{r+p}-2^r-2^p+2$, so $r=1$ and $p=p'$. Thus

$$|G/K|\cdot|H|=\frac{2^{p(p+1)/2}\prod_{i=1}^{p+1}(2^i+1)}{2^{p+1}-1}.$$

But there exists a primitive prime divisor of $2^{p+1}-1$ that can't divide $2^{p(p+1)}\prod_{i=1}^{p+1}(2^i+1)$ by Lemma 5, a contradiction. By trivial calculation we can show that p' can't be 2, 3 or 5.

(5) If $K/H\cong A_{p'-1}(q)$ then $(q^{p'}-1)/(q-1)(p', q-1)$. The step is divided into several sub-steps:

(i) If $p'\geq 11$ then similarly to (4) we have that q is a power of 2. Suppose $q=2^r$ then we have $q^3|2^p$ since $p'\geq 11$. Furthermore, $q^{p'}=(2^pq-2^p-q)(p', q-1)+(p', q-1)+1$. Hence $-q(p', q-1)+(p', q-1)+1\equiv 0 (\mathrm{mod}\ q^3)$, which means $q=2$ and $p'=p$. Thus $|G/K|\cdot|H|=2^{p(p+3)/2}\prod_{i=1}^{p+1}(2^i+1)$. On the other hand because of $|G/K|\,|\,|Out(A_{p-1}(2))|=2$, so $(2^{p+1}+1)\,|\,|H|$. Similarly to sub-step (1) in Step 1, we can get a contradiction.

(ii) If $p'=7$ then $q\neq 2$ or 3, so $q^7-1\geq 3\cdot 2^p-3$. Furthermore, $q^{21}\geq 2^{3(p+1)}$. Similarly to (i), we can get a contradiction.

(iii) If $p'=5$ and $(5, q-1)=1$ then $q^4+q^3+q^2+q=2^p-2$. Similarly to (i), we have q can't be a power of 2. So q is odd. Suppose $q=2k+1$. Hence $0\equiv(2k+1)^4+(2k+1)^3+(2k+1)^2+2k+1=2^p-2\equiv 2 (\mathrm{mod}\ 4)$, which is a contradiction. Similarly when $p'=5$ and $(5, q-1)=5$ we can get a contradiction too.

(iv) If $p'=3$ then $\frac{q^3-1}{(q-1)(3, q-1)}=2^p-1$. Hence $q^3-1\leq 3(q-1)(2^p-1)$ and $q^2+q+1\leq 3(2^p-1)$. On the other hand $|A_2(q)|=q^3(q^2-1)(q^3-1)\leq(q^3-1)^3$ since $q\geq 2$. So $|A_2(q)|\leq(q-1)^3(3(2^p-1))^3<(3(2^p-1))^{4.5}$ since $(q-1)^2<q^2+q+1$. Furthermore, $|A_2(q)|<2^{5(p+1)}$. Similarly to (2) we can prove that p can't be greater than 17. By trivial calculation, a contradiction appears.

Similarly we can prove that $K/H\not\cong {}^2A_n(q)$.

Step 5. $K/H\not\cong B_n(q)$ or $C_n(q)$.

(1) If $K/H\cong C_n(q)$, $n=p'$ is a prime and $q=2$, then $2^p-1=2^{p'}-1$, which implies $p=p'$. And because of $|G/K|\ |\ |Out(K/H)|=|Out(C_p(2))|=1$. So $|H|=\frac{|G|}{|K/H|}=2^p\cdot(2^{p+1}+1)$. Similarly to Step 1(1), we can get a contradiction.

(2) If $K/H \cong C_n(q)$, $n=p'$ is prime and $q=3$ then $2^p-1=(3^{p'}-1)/2$. Furthermore, $2^{p+1}-3^{p'}=1$, which contradicts to Lemma 4.

(3) If $K/H \cong C_n(q)$, $2\leq n=2^{m'}$ then $\frac{q^n+1}{(2,\, q-1)}=2^p-1$. By trivial calculation we have q can't be a power of 2 or 3. Therefore $q^n=2^{p+1}-3$. If $n\geq 4$ then $q^{n^2}=(2^{p+1}-3)^n\geq(2^{p+1}-3)^4$. Hence $q^{n^2}>2^{2(p+1)}$, which contradicts to Lemma 9. If $n=2$ then $q^2=2^{p+1}-3$. Thus $q^2-1=2^{p+1}-4$. Furthermore, $0\equiv(q+1)\cdot(q-1)=2^{p+1}-4\equiv 4 \pmod 8$, which is impossible.

Similarly we can prove that $K/H \not\cong B_n(q)$.

Step 6. $K/H \not\cong D_n(q)$.

(1) If $K/H \cong D_{p'}(5)$, $p'\geq 5$, then $(5^{p'}-1)/4=2^p-1$. Thus $5^{p'}=2^{p+2}-3>2^{p+1}$. Hence $5^{p'(p'-1)}>2^{4(p+1)}$, which contradicts to Lemma 9.

(2) Similarly to sub-step (2) in Step 5, we can show that K/H can't be $D_{p'}(3)$ or $D_{p'+1}(3)$.

Step 7. $K/H \not\cong E_n(q)$, $F_4(q)$, $G_2(q)$, ${}^2E_6(q)$, ${}^2F_4(q)$ or ${}^2G_2(q)$.

If $K/H \cong E_6(q)$ then $(q^6+q^3+1)/(3,\, q-1)=2^p-1$ and so that $q^9>2^{p+1}$. Thus $q^{36}>2^{4(p+1)}$, which implies q is a power of 2 by Lemma 9. If $q=2^r$ then similarly to sub-sub-step (i) in sub-step (5) of Step, we can get a contradiction.
By the same reasoning, we can show that K/H can't be $E_8(q)$ or $F_4(q)$, $G_2(q)$, ${}^2E_6(q)$, ${}^2F_4(q)$ or ${}^2G_2(q)$.

Step 8. $K/H \not\cong {}^2B_2(q)$, $q=2^{2k+1}$.

If $K/H \cong {}^2B_2(q)$, $q=2^{2k+1}$ then $2^p-1=q\pm\sqrt{2q}+1$ or $q-1$. In the case of $2^p-1=q\pm\sqrt{2q}+1$, one has that $0\equiv q^{2k+1}\pm q^{k+1}=2^p-2\equiv 2 \pmod 4$, a contradiction. So $q-1=2^p-1$. Hence $q=2^p$ and

$$|G/K|\cdot|H|=\frac{2^{p^2-p}(2^p+1)(2^{p+1}+1)\prod_{i=1}^{p-1}(2^{2i}-1)}{2^{2p}+1}.$$

By the reasoning to Step 4(4) we can get a contradiction.

Step 9. From Step 1 to Step 8 and Lemma 3 we have that K/H is one of ${}^2D_n(q)$.

(1) If $K/H\cong{}^2D_n(3)$, $n=2^k+1$ isn't a prime, then $(3^{n-1}+1)/2=2^p-1$, so $3^{n-1}+3=2^{p+1}$, which is impossible.

(2) If $K/H\cong{}^2D_{p'}(3)$, $5<p'\neq 2^k+1$, then $(3^{p'}+1)/4=2^p-1$. Hence $3^{p'}>2^{p+1}$. Furthermore, $3^{p'(p'-1)}>2^{4(p+1)}$, which contradicts to Lemma 6.

(3) If $K/H\cong{}^2D_n(q)$, $4\leq n=2^k$ then $(q^n+1)/(2, q-1)=2^p-1$. It's easy to see that q can't be a power of 2, which we have $q^n=2^{p+1}-3$ from. If $n>4$ then $q^{n(n-1)}>2^{3(p+1)}$, which contradicts to Lemma 9. If $n=4$ then $q^{n(n-1)}>2^{2(p+1)}$, which implies q is a power of 3 by Lemma 9, similarly to (1), a contradiction.

(4) If $K/H\cong{}^2D_{p'+1}(2)$, $p'=2^k-1$, $k\geq 2$ then $2^p-1=2^{p'}+1$ or $2^{p'+1}+1$, which is impossible. Similarly to (1) and (2) we have that $K/H\not\cong{}^2D_{p'}(3)$, $p'=2^k+1$, $k\geq 2$.

Now we have that $K/H\cong{}^2D_{p'+1}(2)$, $p'\neq 2^m-1$. Thus $2^{p'}-1=2^p-1$. It follows that $p=p'$, $G/K=1$ and $H=1$, which means $G\cong M$. This is the end of the proof.

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