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Representation of adjoint of composition operator

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Abstract

The adjoint operators of composition operators in Hardy space have nice properties. But in general, we can't know their exact representation until now. In this paper, we give exact representation of adjoint of composition operator in Hardy space.

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1. Introduction

Let D be the unit disc centered at origin in complex plane C and $H(D)$ the space of all holomorphic functions defined in D. If

$$f(z), g(z) \in H(D), f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n, g(z)=\sum_{n=0}^{\infty}\hat{g}(n)z^n,$$

we denote $\|f\|^2=\sum_{n=0}^{\infty}|\hat{f}(n)|^2=\lim_{r\to 1^-}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2 d\theta$ and $H^2=\{f\in H(D): \|f\|<+\infty\}$. If

$f(z),\ g(z)\in H^2$, then we denote $\langle f,\ g\rangle=\sum_{n=0}^{\infty}\hat{f}(n)\bar{\hat{g}}(n)=\lim_{r\to 1^-}\frac{1}{2\pi}\int_0^{2\pi}f(re^{i\theta})\bar{g}(re^{i\theta})d\theta.$

It is well known that the functional $\|\cdot\|$ is the norm on H^2 which makes H^2 a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$. As usual, we associate with each holomorphic $\varphi: D \to D$ the composition operator C_φ by equation :

$$C_\varphi f=f\circ\varphi, \qquad \forall f\in H(D)$$

C_φ is linear and continuous on $H(D)$. Among other things, the goal of many papers is to connect the function-theoretic properties of φ with the operator-theoretic behavior of C_φ. A consequence of a famous Subordination theorem Littlewood ([4]) is that each holomorphic self map φ of D induces a bounded C_φ on H^2.

The following very useful computation characterizes the adjoint of a composition operator

$$\langle f, C_\varphi^* K_a\rangle = \langle C_\varphi f, K_a\rangle = f(\varphi(a)) = \langle f, K_{\varphi(a)}\rangle,$$

so $C_\varphi^* K_a = K_{\varphi(a)}$. As an example of the use of this computation, we note that if $\varphi(\alpha)=\varphi(\beta)$, then:

$$C_\varphi^*(K_\alpha - K_\beta) = K_{\varphi(\alpha)} - K_{\varphi(\beta)} = 0,$$

and C_φ^* has nontrivial kernel. Moreover, if φ is not one-to-one, there is an open set of such pairs α, β which means that the kernel of C_φ^* is infinite dimensional. The adjoint has a very nice representation in terms of the kernel functions, but the absence of a more general representation for the adjoint is a major stumbling block in the theory. On H^2 for φ linear fractional, C_φ^* has been calculated. For a bounded analytic function g, let T_g denote the analytic Toeplitz operator on H^2 given by $T_g(f)(z)=g(z)f(z)$.

Cowen's Adjoint Theorem ([2]. Theorem 2) If $\varphi(z)=(az+b)/(cz+d)$ be a linear fractional self map of D, where ad$-$bc$\neq 0$. Then $\sigma(z)=\frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}$ maps D into itself, $g(z)=\frac{1}{-\bar{b}z+\bar{d}}$ and $h(z)=cz+d$ are in H^∞ and $C_\varphi^* = T_g C_\sigma T_h^*$ in H^2.

In [2] Theorem 2, Cowen requires that $ad-bc=1$, but this is not necessary.

Following the Cowen's work ([2]), in 2000, Cowen ([3]) generalized his result to n variables and in 2003 J. N. Mc Donald ([5]) make some progress about this problem but not solve it completely. Cowen ([1]) pointed out that one of the main difficulties in advancing a general theory of composition operators is that a nice representation of C_φ^* is unknown. In this paper, we give exact representation of adjoint of composition operator in Hardy space.

2. Some lemmas

Because H^2 is Hilbert space, C_φ: $H^2 \to H^2$ is bounded, so C_φ^*: $H^2 \to H^2$ is bounded also. Let $f(z)=\sum_{n=0}^{\infty}\hat{f}_n z^n, g(z)=\sum_{n=0}^{\infty}\hat{g}_n z^n \in H^2, \varphi(z)=\sum_{n=0}^{\infty}\hat{\varphi}_n z^n, \varphi(D)\subseteq D, \hat{f}=(\hat{f}_0,\hat{f}_1,\cdots,\hat{f}_n,\cdots), \hat{g}=(\hat{g}_0,\hat{g}_1,\cdots,\hat{g}_n,\cdots)$, denote

$$\langle \hat{f}, \hat{g}\rangle := \sum_{n=0}^{\infty}\hat{f}_n\bar{\hat{g}}_n = \sum_{n=0}^{\infty}\hat{f}_n\bar{\hat{g}}_n$$

In the following, we compute infinite matrix and vector just like finite dimension case and we can obtain lemma 2. 1 immediately.

Lemma 2.1 Let $h(z)=f(z)g(z)=\sum_{n=0}^{\infty}\hat{h}_n z^n, F=(f_{i,j}), G=(g_{i,j}), H=(h_{i,j})$ are infinite upper triangular matrix where

$$f_{i,j}=\begin{cases}0, & \text{if } j<i,\\ \hat{f}_{j-i}, & \text{if } j\geq i.\end{cases}$$

$$g_{i,j}=\begin{cases}0, & \text{if } j<i,\\ \hat{g}_{j-i}, & \text{if } j\geq i.\end{cases}$$

$$h_{i,j}=\begin{cases}0, & \text{if } j<i,\\ \hat{h}_{j-i}, & \text{if } j\geq i.\end{cases}$$

then $H=FG$.

Lemma 2.2 $\forall n \in N$, $\langle f,\varphi^n\rangle=\langle \tilde{f},e\Phi^n\rangle$ where $\varphi(z)=\sum_{n=0}^{\infty}\hat{\varphi}_n z^n, e=(1,0,\cdots,)$ and $\Phi=(\varphi_{i,j})$ is infinity upper triangular matrix

$$\varphi_{i,j}=\begin{cases}0, & \text{if } j<i,\\ \hat{\varphi}_{j-i}, & \text{if } j\geq i.\end{cases}$$

Proof. $\forall n\in N$, Let $\varphi^n(z)=\sum_{k=0}^{\infty}\hat{\varphi}_{n,k}z^k, \widetilde{\varphi^n}=(\hat{\varphi}_{n,0},\hat{\varphi}_{n,1},\cdots,\hat{\varphi}_{n,k},\cdots)$. By Lemma 2.1, $\widetilde{\varphi^n}=e\Phi^n$, thus $\langle f,\varphi^n\rangle=\langle\tilde{f},\widetilde{\varphi^n}\rangle=\langle\tilde{f},e\Phi^n\rangle$.

Let $\Omega=\{\Phi:\ \varphi\in H^2,\ \Phi$ is infinity matrix defined in Lemma 2.2$\}$ and we define $\|\Phi\|=\|\varphi\|$ in Ω. By Lemma 2.1, Lemma 2.2 we know these definitions are rational and $\|\cdot\|$ is a norm in Ω. If $\|\Phi\|\leq 1$, $z\in D$, computation in the following lemmas is rational.

Lemma 2.3 Let $\varphi(z)=\sum_{n=0}^{\infty}\hat{\varphi}_n z^n$ be an analytic mapping from D to D. Let I be infinite unit matrix, $B=(b_{i,j})$ be infinite upper triangular matrix, $\forall i,j\in N\cup\{0\}$

$$b_{i,j}=\begin{cases}0, & i>j,\\ \dfrac{1}{1-\hat{\varphi}_0\bar{z}}, & i=j,\\ \displaystyle\sum_{p=0}^{k-1}\ \sum_{\sum_{t=1}^{p+1}s_t=k,\ s_t>0(1\leq t\leq p+1)}\frac{\bar{z}^{p+1}\prod_{t=1}^{p+1}\hat{\varphi}_{s_t}}{(1-\hat{\varphi}_0\bar{z})^{p+2}} & j-i=k>0.\end{cases}$$

Then $\forall z\in D$, we have $(I-\Phi\bar{z})^{-1}=B$.

Proof. Let $t_{i,j}=\sum_{k=1}^{\infty}(\delta_i^k-\varphi_{i,k}\bar{z})\cdot b_{k,j}=\sum_{k=i}^{j}(\delta_i^k-\hat{\varphi}_{k-i}\bar{z})\cdot b_{j-k}$. then if $i>j$, $t_{i,j}=0$; if $i=j$, $t_{i,i}=\sum_{k=i}^{i}(\delta_i^k-\hat{\varphi}_{k-i}\bar{z})\cdot b_{j-k}=(1-\hat{\varphi}_0\bar{z})b_0=1$;
if $i<j$,

$$
\begin{aligned}
t_{i,j}&=(\delta_i^i-\hat{\varphi}_{j-i}\bar{z})\cdot b_0+\sum_{k=i+1}^{j-1}(\delta_i^k-\hat{\varphi}_{k-i}\bar{z})\cdot b_{j-k}+(\delta_i^i-\hat{\varphi}_0\bar{z})\cdot b_{j-i}\\
&=\frac{-\hat{\varphi}_{j-i}\bar{z}}{1-\hat{\varphi}_0\bar{z}}+\sum_{k=i+1}^{j-1}(-\hat{\varphi}_{k-i}\bar{z})\sum_{p=0}^{j-k-1}\sum_{s_1+\cdots+s_{p+1}=j-k,s_t>0}\frac{\bar{z}^{p+1}\prod_{t=1}^{p+1}\hat{\varphi}_{s_t}}{(1-\hat{\varphi}_0\bar{z})^{p+2}}\\
&\quad+(1-\hat{\varphi}_0\bar{z})\cdot\sum_{p=0}^{j-i-1}\sum_{s_1+\cdots+s_{p+1}=j-i,s_t>0}\frac{\bar{z}^{p+1}\prod_{t=1}^{p+1}\hat{\varphi}_{s_t}}{(1-\hat{\varphi}_0\bar{z})^{p+2}}\\
&=\frac{-\hat{\varphi}_{j-i}\bar{z}}{1-\hat{\varphi}_0\bar{z}}-\sum_{p=0}^{j-i-2}\sum_{k=i+1}^{j-p-1}\hat{\varphi}_{k-i}\bar{z}\sum_{s_1+\cdots+s_{p+1}=j-k,s_t>0}\frac{\bar{z}^{p+1}\prod_{t=1}^{p+1}\hat{\varphi}_{s_t}}{(1-\hat{\varphi}_0\bar{z})^{p+2}}\\
&\quad+\sum_{p=0}^{j-i-1}\sum_{s_1+\cdots+s_{p+1}=j-i,s_t>0}\frac{\bar{z}^{p+1}\prod_{t=1}^{p+1}\hat{\varphi}_{s_t}}{(1-\hat{\varphi}_0\bar{z})^{p+1}}\\
&=\frac{-\hat{\varphi}_{j-i}\bar{z}}{1-\hat{\varphi}_0\bar{z}}-\sum_{p=0}^{j-i-2}\sum_{s_1+\cdots+s_{p+2}=j-i,s_t>0}\frac{\bar{z}^{p+2}\prod_{t=1}^{p+2}\hat{\varphi}_{s_t}}{(1-\hat{\varphi}_0\bar{z})^{p+2}}\\
&\quad+\sum_{p=0}^{j-i-1}\sum_{s_1+\cdots+s_{p+1}=j-i,s_t>0}\frac{\bar{z}^{p+1}\prod_{t=1}^{p+1}\hat{\varphi}_{s_t}}{(1-\hat{\varphi}_0\bar{z})^{p+1}}\\
&=\frac{-\hat{\varphi}_{j-i}\bar{z}}{1-\hat{\varphi}_0\bar{z}}-\sum_{p=1}^{j-i-1}\sum_{s_1+\cdots+s_{p+1}=j-i,s_t>0}\frac{\bar{z}^{p+1}\prod_{t=1}^{p+1}\hat{\varphi}_{s_t}}{(1-\hat{\varphi}_0\bar{z})^{p+1}}\\
&\quad+\sum_{p=0}^{j-i-1}\sum_{s_1+\cdots+s_{p+1}=j-i,s_t>0}\frac{\bar{z}^{p+1}\prod_{t=1}^{p+1}\hat{\varphi}_{s_t}}{(1-\hat{\varphi}_0\bar{z})^{p+1}}=0
\end{aligned}
$$

That is

$$(I-\Phi\bar{z})B=I,$$

Thus

$$(I-\Phi\bar{z})^{-1}=B.$$

Lemma 2.4 If $n\geq p$, then

$$|\{(s_1,s_2,\cdots,s_p):\sum_{t=1}^{p}s_t=n,s_1,s_2,\cdots,s_p>0\}|=C_{n-1}^{p-1}.$$

Proof.

$$|\{(s_1,s_2,\cdots,s_p):\sum_{t=1}^{p}s_t=n,s_1,s_2,\cdots,s_p>0\}|$$

$$=|\{(s_1,s_2,\cdots,s_p):\sum_{t=1}^{p}s_t=n-p,s_1,s_2,\cdots,s_p\geq 0\}|$$

$$=\frac{1}{(n-p)!}\left[\frac{1}{(1-x)^p}\right]^{(n-p)}\Bigg|_{x=0}$$

$$=\frac{1}{(n-p)!}\frac{(n-1)!}{(p-1)!}=C_{n-1}^{p-1}$$

3. Main result

Theorem 3.1 Let $\varphi(z)=\sum_{n=0}^{\infty}\hat{\varphi}_n z^n$ be an analytic mapping from D to D, then C_φ^* in H^2 can be represented as

$$C_\varphi^* f(z)=\frac{\hat{f}_0}{1-\bar{\hat{\varphi}}_0 z}+\sum_{n=1}^{\infty}\hat{f}_n\sum_{p=0}^{n-1}\sum_{s_1+\cdots+s_{p+1}=n,s_t>0}\frac{z^{p+1}\prod_{t=1}^{p+1}\bar{\hat{\varphi}}_{s_t}}{(1-\bar{\hat{\varphi}}_0 z)^{p+2}}$$

where $\quad f(z)=\sum_{n=0}^{\infty}\hat{f}_n z^n\in H^2.$

Proof. Let $C_\varphi^* f(z)=\sum_{n=0}^{\infty}a_n z^n$, By Lemma 2.2, we have

$$\sum_{n=0}^{\infty}a_n\bar{\hat{g}}_n=\langle\sum_{n=0}^{\infty}a_n z^n,\sum_{n=0}^{\infty}\hat{g}_n z^n\rangle=\langle C_\varphi^* f(z),\ g(z)\rangle$$

$$=\langle f(z),C_\varphi g(z)\rangle=\langle f(z),\ g(\varphi(z))\rangle=\langle f(z),\sum_{n=0}^{\infty}\hat{g}_n\varphi^n(z)\rangle$$

$$=\sum_{n=0}^{\infty}\bar{\hat{g}}_n\langle f(z),\varphi^n(z)\rangle=\sum_{n=0}^{\infty}\bar{\hat{g}}_n\langle\tilde{f},\ e\Phi^n\rangle$$

Compare the left side with the right side of above equality, we have $a_n=\langle\tilde{f},e\Phi^n\rangle$.

$$C_\varphi^* f(z)=\sum_{n=0}^{\infty}a_n z^n=\sum_{n=0}^{\infty}\langle\tilde{f},\ e\Phi^n\rangle z^n$$

$$=\langle\tilde{f},e\sum_{n=0}^{\infty}\Phi^n\bar{z}^n\rangle=\langle\tilde{f},\ e(I-\Phi\bar{z})^{-1}\rangle$$

By Lemma 2.3, we have

$$C_\varphi^* f(z)=\frac{\hat{f}_0}{1-\bar{\varphi}_0 z}+\sum_{n=1}^{\infty}\hat{f}_n\sum_{p=0}^{n-1}\sum_{s_1+\cdots+s_{p+1}=n,s_t>0}\frac{z^{p+1}\prod_{t=1}^{p+1}\bar{\varphi}_{s_t}}{(1-\bar{\varphi}_0 z)^{p+2}}$$

Corollary 3.2 If

$$\varphi(z)=(\alpha z+\beta)/(\gamma z+\delta)$$

be a linear fractional self map of D, where $\alpha\delta-\beta\gamma\neq 0$,

$$\sigma(z)=\frac{\bar{\alpha}z-\bar{\gamma}}{-\bar{\beta}z+\bar{\delta}},\quad g(z)=\frac{1}{-\bar{\beta}z+\bar{\delta}},\quad h(z)=\gamma z+\delta,$$

then $C_\varphi^*=T_g C_\sigma T_h^*$ in H^2.

Proof. (1) If

$$\varphi(z)=\frac{b+cz}{1-az}=(b+cz)\sum_{n=0}^{\infty}a^n z^n=\sum_{n=0}^{\infty}a^n bz^n+\sum_{n=0}^{\infty}a^n cz^{n+1}$$

$$=b+\sum_{n=1}^{\infty}(a^n b+a^{n-1}c)z^n=b+\sum_{n=1}^{\infty}a^{n-1}(ab+c)z^n.$$

By Theorem 3.1 and Lemma 2.4, we have

$$C_\varphi^* f(z)=\frac{\hat{f}_0}{1-\bar{b}z}+\sum_{n=1}^{\infty}\hat{f}_n\sum_{p=0}^{n-1}\sum_{s_1+\cdots+s_{p+1}=n,s_t>0}\frac{z^{p+1}\prod_{t=1}^{p+1}\overline{(ab+c)}\,\bar{a}^{s_t-1}}{(1-\bar{b}z)^{p+2}}$$

$$=\frac{\hat{f}_0}{1-\bar{b}z}+\sum_{n=1}^{\infty}\hat{f}_n\sum_{p=0}^{n-1}\sum_{s_1+\cdots+s_{p+1}=n,s_t>0}\frac{(\bar{a}\bar{b}+\bar{c})^{p+1}\bar{a}^{n-p-1}z^{p+1}}{(1-\bar{b}z)^{p+2}}$$

$$=\frac{\hat{f}_0}{1-\bar{b}z}+\sum_{n=1}^{\infty}\frac{\hat{f}_n z(\bar{a}\bar{b}+\bar{c})}{(1-\bar{b}z)^2}\sum_{p=0}^{n-1}C_{n-1}^p\bar{a}^{n-1-p}\left[\frac{(\bar{a}\bar{b}+\bar{c})z}{1-\bar{b}z}\right]^p$$

$$=\frac{\hat{f}_0}{1-\bar{b}z}+\sum_{n=1}^{\infty}\frac{\hat{f}_n z(\bar{a}\bar{b}+\bar{c})}{(1-\bar{b}z)^2}\left[\bar{a}+\frac{(\bar{a}\bar{b}+\bar{c})z}{1-\bar{b}z}\right]^{n-1}$$

$$=\frac{\hat{f}_0}{1-\bar{b}z}+\sum_{n=1}^{\infty}\frac{\hat{f}_n z(\bar{a}\bar{b}+\bar{c})}{(1-\bar{b}z)^2}\left(\frac{\bar{a}+\bar{c}z}{1-\bar{b}z}\right)^{n-1}$$

If

$$\varphi(z)=\frac{a'z+b'}{c'z+d'}=\frac{\frac{b'}{d'}+\frac{a'}{d'}z}{1-(-\frac{c'}{d'})z},\quad (d'\neq 0)$$

Let $a=-\frac{c'}{d'}$, $b=\frac{b'}{d'}$, $c=\frac{a'}{d'}$, By above equality, we have

$$C_\varphi^* f(z)=\frac{\hat{f}_0}{1-\frac{\bar{b}'}{\bar{d}'}z}+\sum_{n=1}^{\infty}\frac{\hat{f}_n z(-\frac{\bar{c}'}{\bar{d}'}\frac{\bar{b}'}{\bar{d}'}+\frac{\bar{a}'}{\bar{d}'})}{(1-\frac{\bar{b}'}{\bar{d}'}z)^2}\left(\frac{-\frac{\bar{c}'}{\bar{d}'}+\frac{\bar{a}'}{\bar{d}'}z}{1-\frac{\bar{b}'}{\bar{d}'}z}\right)^{n-1}$$

$$=\frac{\hat{f}_0\bar{d}'}{\bar{d}'-\bar{b}'z}+\sum_{n=1}^{\infty}\frac{\hat{f}_n z(-\bar{c}'\bar{b}'+\bar{a}'\bar{d}')}{(\bar{d}'-\bar{b}'z)^2}\left(\frac{-\bar{c}'+\bar{a}'z}{\bar{d}'-\bar{b}'z}\right)^{n-1}$$

(2)Let

$$f(z)=\sum_{n=0}^{\infty}\hat{f}_n z^n,\ g(z)=\sum_{n=0}^{\infty}\hat{g}_n z^n,\ h(z)=\sum_{n=0}^{\infty}\hat{h}_n z^n, T_h^* f=\sum_{n=0}^{\infty}a_n z^n.$$

Then

$$h(z)g(z)=\sum_{n=0}^{\infty}(\sum_{k=0}^{n}\hat{h}_k\hat{g}_{n-k})z^n$$

$$\sum_{n=0}^{\infty}a_n\bar{\hat{g}}_n=\langle T_h^* f(z),\ g(z)\rangle=\langle f(z),\ T_h g(z)\rangle=\langle f(z),\ h(z)g(z)\rangle$$

$$=\sum_{n=0}^{\infty}\hat{f}_n(\sum_{k=0}^{n}\bar{\hat{g}}_k\bar{\hat{h}}_{n-k})=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\hat{f}_n\bar{\hat{g}}_k\bar{\hat{h}}_{n-k}$$

$$=\sum_{k=0}^{\infty}\bar{\hat{g}}_k\sum_{n=k}^{\infty}\hat{f}_n\bar{\hat{h}}_{n-k}=\sum_{n=0}^{\infty}\bar{\hat{g}}_n\sum_{k=n}^{\infty}\hat{f}_k\bar{\hat{h}}_{k-n}$$

Compare the left side with the right side of above equality, we have

$$a_n=\sum_{k=n}^{\infty}\hat{f}_k\bar{\hat{h}}_{k-n}$$

If $h(z)=cz+d,\sigma(z)=\frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}},\ g(z)=\frac{1}{-\bar{b}z+\bar{d}}$, Then $a_n=\hat{f}_n\bar{d}+\hat{f}_{n+1}\bar{c}$,

$$T_g C_\sigma T_h^* f=T_g C_\sigma\left[\sum_{n=0}^{\infty}(\hat{f}_n\bar{d}+\hat{f}_{n+1}\bar{c})z^n\right]$$

$$=\frac{1}{-\bar{b}z+\bar{d}}\sum_{n=0}^{\infty}(\hat{f}_n\bar{d}+\hat{f}_{n+1}\bar{c})\left[\frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}\right]^n$$

$$=\frac{\hat{f}_0\bar{d}}{-\bar{b}z+\bar{d}}+\frac{1}{-\bar{b}z+\bar{d}}\sum_{n=1}^{\infty}\hat{f}_n\left\{\bar{d}\left[\frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}\right]^n+\bar{c}\left[\frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}\right]^{n-1}\right\}$$

$$=\frac{\hat{f}_0\bar{d}}{-\bar{b}z+\bar{d}}+\frac{1}{-\bar{b}z+\bar{d}}\sum_{n=1}^{\infty}\hat{f}_n\frac{z(-\bar{c}\bar{b}+\bar{a}\bar{d})}{\bar{d}-\bar{b}z}\left[\frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}\right]^{n-1}$$

Compare the result in (1) with the result in (2), we come to the conclusion.

Remark : This Corollary is the Theorem 2 in [2].

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