Kumamoto J. Math.
Vol. 18, 17-23 March (2005)

A note on the meromorphic $\mathscr{O}(X)$-convexity

Dedicated to Professor Hideaki Kazama on the occasion of his sixtieth birthday

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(Received January 31, 2005)

Abstract

An open set D of a reduced Stein space X is meromorphically $\mathscr{O}(X)$-convex if and only if D is the union of an increasing sequence $\{D_\nu\}_{\nu=1}^{\infty}$ of open sets of X such that D_ν is $\mathscr{Q}_X(D_\nu)$-convex for every $\nu \in \mathbf{N}$.

1. Introduction

In this paper we prove that an open set D of a reduced Stein space X is meromorphically $\mathscr{O}(X)$-convex if and only if D is the union of an increasing sequence $\{D_\nu\}_{\nu=1}^{\infty}$ of open sets of X such that D_ν is $\mathscr{Q}_X(D_\nu)$-convex for every $\nu \in \mathbf{N}$, where $\mathscr{Q}_X(D_\nu)$ denotes the family of the functions φ on D_ν of the form $\varphi=(f/g)|_{D_\nu}$ such that f, $g \in \mathscr{O}(X)$, $g \not\equiv 0$ on any irreducible component of X and $g \neq 0$ on D_ν (see Theorem 4.1).

By the similar argument we also prove that an open set D of $\mathbf{C}^n$ is rationally convex if and only if D is the union of an increasing sequence $\{D_\nu\}_{\nu=1}^{\infty}$ of open sets of $\mathbf{C}^n$ such that each D_ν is convex with respect to the rational functions which are holomorphic on D_ν (see Theorems 4.2 and 4.3).

2. Preliminaries

Throughout this paper all complex spaces are supposed to be *reduced* and *second countable.* Let X be a complex space and D an open set of X. We denote by $\mathscr{Q}_X(D)$ the family of the functions φ on D of the form $\varphi=(f/g)|_D$ such that f, $g \in \mathscr{O}(X)$, $g \not\equiv 0$ on any irreducible component of X and $g \neq 0$ on D. Since every strong Poincaré problem is solvable in $\mathbf{C}^n$, we have that $\mathscr{Q}_{\mathbf{C}^n}(D)=\mathscr{M}(\mathbf{C}^n) \cap \mathscr{O}(D)$ for every open set D of $\mathbf{C}^n$.

Let X be a complex space and let $\mathscr{F} \subset \mathscr{O}(X)$. Then X is said to be *meromorphically $\mathscr{F}$-convex* if for every compact set K of X the *meromorphically convex hull* $\tilde{K}_{\mathscr{F}}:=\{x \in X \mid f(x) \in f(K)$ for every $f \in \mathscr{F}\}$ of K with respect to $\mathscr{F}$ is compact. An open set D of X is said to be *meromorphically $\mathscr{F}$-convex* if D is meromorphically $\mathscr{F}|_D$-convex, that is, for every compact

Mathematics Subject Classification (2000) : 32E20, 32E30, 41A20.

set K of D the set $\tilde{K}_{\mathscr{F}} \cap D$ is compact. If X is a Stein space, then an open set D of X is meromorphically $\mathcal{O}(X)$-convex if and only if for every compact set K of D we have that $\tilde{K}_X \subset D$, where $\tilde{K}_X := \tilde{K}_{\mathcal{O}(X)}$ (see Theorem 12 of Abe [1]).

Let $z_1, z_2, \ldots, z_n$ be the coordinates of $\mathbf{C}^n$. We denote by $\mathbf{C}[z_1, z_2, \ldots, z_n]$ and by $\mathbf{C}(z_1, z_2, \ldots, z_n)$ the set of polynomial functions on $\mathbf{C}^n$ and the set of rational functions on $\mathbf{C}^n$ respectively. Let K be a compact set of $\mathbf{C}^n$. The set $\tilde{K}_{\mathbf{C}[z_1, z_2, \ldots, z_n]}$ is said to be the *rationally convex hull* of K, which coincides with the set of the points $x \in \mathbf{C}^n$ such that if $h \in \mathbf{C}(z_1, z_2, \ldots, z_n)$ is holomorphic near K, then h is also holomorphic near x and $|h(x)| \leq \| h \|_K$ (see Stolzenberg [11, p. 262] or Lemma 2.4 of Gamelin [4, p. 69]). An open set D of $\mathbf{C}^n$ is said to be *rationally convex* if D is meromorphically $\mathbf{C}[z_1, z_2, \ldots, z_n]$-convex. Since we have that $\tilde{K}_{\mathbf{C}[z_1, z_2, \ldots, z_n]} = \tilde{K}_{\mathbf{C}^n}$ for every compact set K of $\mathbf{C}^n$, an open set D of $\mathbf{C}^n$ is rationally convex if and only if D is meromorphically $\mathcal{O}(\mathbf{C}^n)$-convex (see Lemma 2 of Abe [1]). If an open set D of $\mathbf{C}^n$ is $\mathcal{Q}_{\mathbf{C}^n}(D)$-convex, then D is rationally convex in $\mathbf{C}^n$. The converse however is not true if $n \geq 2$ (see Abe [2]).

Let $\mathscr{R}(D) := \mathbf{C}(z_1, z_2, \ldots, z_n) \cap \mathcal{O}(D)$ for every open set D of $\mathbf{C}^n$. If an open set D of $\mathbf{C}^n$ is $\mathscr{R}(D)$-convex, then D is $\mathcal{Q}_{\mathbf{C}^n}(D)$-convex. The converse however is not true if $n \geq 2$. As an example, let $D := \mathbf{C}^n \setminus S$, where S is an irreducible transcendental hypersurface of $\mathbf{C}^n$. Then D is $\mathcal{Q}_{\mathbf{C}^n}(D)$-convex and is not $\mathscr{R}(D)$-convex.

Let X be a complex space. Let $f_\mu, g_\mu \in \mathcal{O}(X)$ and let $g_\mu \not\equiv 0$ on any irreducible component of X for $\mu = 1, 2, \ldots, m$. Let $h_\mu := f_\mu / g_\mu$ for $\mu = 1, 2, \ldots, m$. Let $Z_1, Z_2, \ldots, Z_m$ be open sets of $\mathbf{C}$. Let G be an open set of $X \setminus A$, where $A := \{g_1 g_2 \cdots g_m = 0\}$. Let $W := G \cap \{x \in X \setminus A \mid h_\mu(x) \in Z_\mu$ for every $\mu = 1, 2, \ldots, m\}$ and assume that $W \Subset G$. Then the open set W is said to be a *meromorphic polyhedron* of X. A meromorphic polyhedron W of $\mathbf{C}^n$ is said to be a *rational polyhedron* of $\mathbf{C}^n$ if the functions $f_1, f_2, \ldots, f_m, g_1, g_2, \ldots, g_m$ are chosen to be polynomial.

3. Lemmas

We use the notation in Sect. 2 for the meromorphic or rational polyhedron W in the following lemmas. Let $\Delta := \{t \in \mathbf{C} \mid |t| < 1\}$.

Lemma 3.1. *If X is a Stein space or an irreducible complex space, then every meromorphic polyhedron W of X is $\mathcal{Q}_X(W)$-convex.*

Proof. Let K be an arbitrary compact set of W. Assume that $\tilde{K}_{\mathcal{Q}_X(W)}$ is not compact. Then there exist a sequence $\{p_\nu\}_{\nu=1}^\infty \subset \tilde{K}_{\mathcal{Q}_X(W)}$ and $p_0 \in \partial W$ such that $\lim_{\nu \to \infty} p_\nu = p_0$ in G. There exists an index μ_0 such that $c := h_{\mu_0}(p_0) \in \partial Z_{\mu_0}$. Since $f_{\mu_0} / g_{\mu_0} = h_{\mu_0} \not\equiv c$ on W, we have that $f_{\mu_0} - c g_{\mu_0} \not\equiv 0$ on W. We consider the case when X is Stein. Let $\{X_i\}_{i \in I}$ be the set of irreducible components of X. Let $I' := \{i \in I \mid f_{\mu_0} - c g_{\mu_0} \equiv 0$ on $X_i\}$, $I'' := I \setminus I'$ and $X'' := \bigcup_{j \in I''} X_j$. Then $W \subset X''$. Take a point $\xi_i \in X_i \setminus X''$ for every $i \in I'$. Since $X'' \cup \{\xi_i \mid i \in I'\}$ is an analytic set of a Stein space X, there exists $v \in \mathcal{O}(X)$ such that $v = f_{\mu_0} - c g_{\mu_0}$ on X'' and $v(\xi_i) = 1$ for every $i \in I'$.

Then $v \not\equiv 0$ on any irreducible component of X. Let $l := g_{\mu_0}/v$ on X. In the case when X is an irreducible complex space, let $l := g_{\mu_0}/(f_{\mu_0} - cg_{\mu_0})$ on X. In both cases we have that $l \in \mathcal{O}_X(W)$ and that $l = 1/(h_{\mu_0} - c)$ on W. Therefore we have that $\lim_{\nu \to \infty} |l(p_\nu)| = +\infty$. On the other hand $|l(p_\nu)| \leq \| l \|_K$ for every $\nu \in \mathbf{N}$. It is a contradiction. It follows that $\hat{K}_{\mathcal{O}_X(W)}$ is compact. Thus we proved that W is $\mathcal{O}_X(W)$-convex. □

Lemma 3.2. *Every rational polyhedron W of $\mathbf{C}^n$ is $\mathcal{R}(W)$-convex.*

Proof. Applying the argument in the proof of Lemma 3.1 in the case when X is irreducible, we obtain the assertion. □

Lemma 3.3. *Let K be a compact set of $\mathbf{C}^n$ and E an open set of $\mathbf{C}^n$ such that $K \subset E \Subset \mathbf{C}^n$ and $\hat{K}_{\mathbf{C}^n} \cap \partial E = \emptyset$. Then there exists a rational polyhedron W of $\mathbf{C}^n$ with $Z_1 = Z_2 = \cdots = Z_m = \Delta$ and $f_1 = f_2 = \cdots = f_m = 1$ such that $\hat{K}_{\mathbf{C}^n} \subset W \Subset E$.*

Proof. We use the method of the proof of Lemma 2[1] of Abe-Furushima [3] or of Lemma 5 of Abe [1]. Take an arbitrary point $p \in \partial E$. Since $p \notin \hat{K}_{\mathbf{C}^n}$, there exists $u^{(p)} \in \mathbf{C}[z_1, z_2, \ldots, z_n]$ such that $u^{(p)}(p) \notin u^{(p)}(K)$. Then there exist $a_p \in \mathbf{C}$ and $\varepsilon_p > 0$ such that $u^{(p)}(p) \in \{t \in \mathbf{C} \mid 0 < |t - a_p| < \varepsilon_p\}$ and $u^{(p)}(K) \subset \{t \in \mathbf{C} \mid |t - a_p| > \varepsilon_p\}$. Let $g^{(p)} := (u^{(p)} - a_p)/\varepsilon_p$, $U_p := \{g^{(p)} \neq 0\}$, $V_p := \{x \in U_p \mid |1/g^{(p)}(x)| > 1\}$ and $W_p := \{x \in U_p \mid |1/g^{(p)}(x)| < 1\}$. Then $g^{(p)} \in \mathbf{C}[z_1, z_2, \ldots, z_n]$, $p \in V_p$, $K \subset W_p$, $\overline{W}_p \subset U_p$ and $V_p \cap \overline{W}_p = \emptyset$. Since ∂E is compact, there exist finitely many points $p_1, p_2, \ldots, p_m \in \partial E$ such that $\partial E \subset \bigcup_{\mu=1}^m V_{p_\mu}$. Let $g_\mu := g^{(p_\mu)}$ for $\mu = 1, 2, \ldots, m$. Let $A := \{g_1 g_2 \cdots g_m = 0\}$, $G := E \setminus A$ and $W := G \cap \{x \in \mathbf{C}^n \setminus A \mid |1/g_\mu(x)| < 1 \text{ for every } \mu = 1, 2, \ldots, m\}$. It is easy to verify that $W = G \cap (\bigcap_{\mu=1}^m W_{p_\mu}) \Subset G$. Then W is a rational polyhedron of $\mathbf{C}^n$ and $\hat{K}_{\mathbf{C}^n} \subset W \Subset E$. □

Lemma 3.4. *Let X be a complex space. Let $\mathcal{F}$ be a subfamily of $\mathcal{O}(X)$ such that if $f \in \mathcal{F}$ and $c > 0$, then $cf \in \mathcal{F}$. Let K be a compact set of X and E an open set of X such that $K \subset E \Subset X$ and $\hat{K}_{\mathcal{F}} \cap \partial E = \emptyset$. Then there exist finitely many $h_1, h_2, \ldots, h_m \in \mathcal{F}$ such that $K \subset W \Subset E$, where $W := E \cap \{x \in X \mid |h_\mu(x)| < 1$ for every $\mu = 1, 2, \ldots, m\}$.*

Proof. Take an arbitrary point $p \in \partial E$. Since $p \notin \hat{K}_{\mathcal{F}}$, there exists $h^{(p)} \in \mathcal{F}$ such that $|h^{(p)}(p)| > \|h^{(p)}\|_K$. Multiplying a positive constant we may assume that $|h^{(p)}(p)| > 1 > \|h^{(p)}\|_K$. Then $V_p := \{x \in X \mid |h^{(p)}(x)| > 1\}$ is an open neighborhood of p. Since ∂E is compact, there exist finitely many points $p_1, p_2, \ldots, p_m \in \partial E$ such that $\partial E \subset \bigcup_{\mu=1}^m V_{p_\mu}$. Let $h_\mu := h^{(p_\mu)}$ for every $\mu = 1$,

[1]The proof of Lemma 2 of Abe-Furushima [3] contains an inadequate argument. For the corrected proof see Lemma 10 of Abe [1].

2, ..., m. Let $W:=E\cap\{x\in X\mid |h_\mu(x)|<1$ for every $\mu=1, 2, ..., m\}$. Then we have that $K\subset W\Subset E$. □

Lemma 3.5. *Let W be a rational polyhedron of $\mathbf{C}^n$ with $Z_1=Z_2=\cdots=Z_m=\Delta$ and let D be an open set of $\mathbf{C}^n$ such that $W\subset D\subset \mathbf{C}^n\setminus A$. Then for every compact set K of W we have that $\hat{K}_{\mathscr{R}(D)}\subset W$.*

Proof. The map $\psi:=(h_1,h_2,...,h_m,z_1,z_2,...,z_n): \mathbf{C}^n\setminus A\to\mathbf{C}^{m+n}$ is injective and regular. Since the map $(h_1,h_2,...,h_m): W\to\Delta^m$ is proper (see E.51f of Kaup-Kaup [5, p. 226]), the induced map $\psi_{W,\Delta^m\times\mathbf{C}^n}: W\to\Delta^m\times\mathbf{C}^n$ is also proper. It follows that $\psi_{W,\Delta^m\times\mathbf{C}^n}: W\to\Delta^m\times\mathbf{C}^n$ is a closed holomorphic embedding. Let K be an arbitrary compact set of W. Take an arbitrary point $x\in\hat{K}_{\mathscr{R}(D)}$. Since $|h_\mu(x)|\le\|h_\mu\|_K<1$ for $\mu=1, 2, ..., m$, we have that $\psi(x)=(h_1(x),h_2(x),...,h_m(x),x)\in\Delta^m\times\mathbf{C}^n$. Assume that $\psi(x)\notin\psi(W)$. Since $\psi(W)\cup\{\psi(x)\}$ is an analytic set of a Stein manifold $\Delta^m\times\mathbf{C}^n$, there exists $\alpha\in\mathscr{O}(\Delta^m\times\mathbf{C}^n)$ such that $\alpha=0$ on $\psi(W)$ and $\alpha(\psi(x))=1$. There exists a polynomial function β on $\mathbf{C}^{m+n}$ such that $|\alpha-\beta|<1/2$ on $\psi(K\cup\{x\})$. Then $|\beta\circ\psi|<1/2$ on K and $|\beta(\psi(x))|>1/2$. Since $\beta\circ\psi$ is a polynomial of $h_1, h_2, ..., h_m, z_1, z_2, ..., z_n$, there exist $u\in\mathbf{C}[z_1, z_2, ..., z_n]$ and a monic monomial v of $g_1, g_2, ..., g_m$ such that $\beta\circ\psi=u/v$ on $\mathbf{C}^n\setminus A$. Since $u/v\in\mathscr{R}(D)$, we have that $|\beta(\psi(x))|\le\|\beta\circ\psi\|_K<1/2$. It is a contradiction. It follows that $\psi(x)\in\psi(W)$. Since ψ is injective, we have that $x\in W$. Thus we proved that $\hat{K}_{\mathscr{R}(D)}\subset W$. □

Lemma 3.6. *If an open set D of $\mathbf{C}^n$ is $\mathscr{R}(D)$-convex, then for every compact set K of D every connected component of $\hat{K}_{\mathscr{R}(D)}$ intersects K.*

Proof. Assume that there exists a connected component L of $\hat{K}_{\mathscr{R}(D)}$ such that $L\cap K=\emptyset$. Let $\mathscr{S}$ be the family of the subsets of $\hat{K}_{\mathscr{R}(D)}$ which contain L and are simultaneously open and closed in $\hat{K}_{\mathscr{R}(D)}$. Since $\hat{K}_{\mathscr{R}(D)}$ is compact, we have that $L=\bigcap_{S\in\mathscr{S}}S$ (see Narasimhan [6, p. 234] or Remmert [9, p. 304]). Since K is a compact set of $\hat{K}_{\mathscr{R}(D)}$ which does not intersect L, there exist finitely many $S_1, S_2, ..., S_N\in\mathscr{S}$ such that $(\bigcap_{i=1}^N S_i)\cap K=\emptyset$. Then $L'':=\bigcap_{i=1}^N S_i\in\mathscr{S}$. The set $L':=\hat{K}_{\mathscr{R}(D)}\setminus L''$ is also open and closed in $\hat{K}_{\mathscr{R}(D)}$. We have that $L'\cup L''=\hat{K}_{\mathscr{R}(D)}$, $L'\cap L''=\emptyset$ and $K\subset L'$. Since L' and L'' are compact, we can take an open set E of D such that $L'\subset E\Subset D\setminus L''$. Then $\hat{K}_{\mathscr{R}(D)}\cap\partial E=\emptyset$. By applying Lemma 3.4 for $X=D$ and $\mathscr{F}=\mathscr{R}(D)$, there exist finitely many $h_1, h_2, ..., h_m\in\mathscr{R}(D)$ such that $K\subset W\Subset E$, where $W:=\{x\in E\mid |h_\mu(x)|<1$ for every $\mu=1, 2, ..., m\}$. Then W is a rational polyhedron of $\mathbf{C}^n$ with $Z_1=Z_2=\cdots=Z_m=$

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[2]If the polynomials f and g are chosen to be relatively prime, then the function f/g cannot be holomorphic in any neighborhood of a point $p\in\mathbf{C}^n$ such that $g(p)=0$ (see Theorem 1.3.2 of Rudin [10]). Therefore we have that $\mathscr{R}(D)=\{(f/g)|_D \mid f, g\in\mathbf{C}[z_1,z_2,...,z_n]$ and $g\neq 0$ on $D\}$ for every open set D of $\mathbf{C}^n$.

Δ such that $A \cap D = \emptyset$.[2] By Lemma 3.5 we have that $\hat{K}_{\mathscr{R}(D)} \subset W$. It follows that $\hat{K}_{\mathscr{R}(D)} \subset D \setminus L''$ and therefore $L'' = \emptyset$. Since $\emptyset \neq L \subset L''$, it is a contradiction. □

Lemma 3.7. *If an open set D of $\mathbf{C}^n$ is $\mathscr{R}(D)$-convex, then every connected component of D is also $\mathscr{R}(D)$-convex.*

Proof. Let C be a connected component of D. Let K be a compact set of C. Since D is $\mathscr{R}(D)$-convex, the set $\hat{K}_{\mathscr{R}(D)}$ is compact. Assume that $P := \hat{K}_{\mathscr{R}(D)} \setminus C \neq \emptyset$ and take a point $x_0 \in P$. Let L be a connected component of $\hat{K}_{\mathscr{R}(D)}$ containing x_0. Since P is closed and open in $\hat{K}_{\mathscr{R}(D)}$, we have that $L \subset P$. It follows that $L \cap K = \emptyset$. It contradicts Lemma 3.6. □

4. Results

We have the following characterization of a meromorphically $\mathcal{O}(X)$-convex open set of a Stein space X.

Theorem 4.1. *Let X be a Stein space and D an open set of X. Then the following two conditions are equivalent.*

(1) *D is meromorphically $\mathcal{O}(X)$-convex.*

(2) *D is the union of an increasing sequence $\{D_\nu\}_{\nu=1}^{\infty}$ of open sets of X such that D_ν is $\mathcal{O}_X(D_\nu)$-convex for every $\nu \in \mathbf{N}$.*

Proof. (1) $\Rightarrow$ (2). Take a sequence $\{K_\nu\}_{\nu=1}^{\infty}$ of compact sets of D such that $\bigcup_{\nu=1}^{\infty} K_\nu = D$ and $K_\nu \subset \mathring{K}_{\nu+1}$ for every $\nu \in \mathbf{N}$. For every compact set K of D we have that $\tilde{K}_X \subset D$ (see Theorem 12 of Abe [1]). There exists a meromorphic polyhedron W of X such that $\tilde{K}_X \subset W \Subset D$ (see Corollary 6 of Abe [1]). Therefore by induction there exists a sequence $\{W_\nu\}_{\nu=1}^{\infty}$ of meromorphic polyhedra of X such that $K_\nu \cup \overline{W}_{\nu-1} \subset W_\nu \Subset D$ for every $\nu \in \mathbf{N}$, where $W_0 := \emptyset$. Then we have that $\bigcup_{\nu=1}^{\infty} W_\nu = D$ and $W_\nu \Subset W_{\nu+1}$ for every $\nu \in \mathbf{N}$. By Lemma 3.1 the open set W_ν is $\mathcal{O}_X(W_\nu)$-convex for every $\nu \in \mathbf{N}$.

(2) $\Rightarrow$ (1). There exists an increasing sequence $\{D_\nu\}_{\nu=1}^{\infty}$ of open sets of X such that $\bigcup_{\nu=1}^{\infty} D_\nu = D$ and D_ν is $\mathcal{O}_X(D_\nu)$-convex for every $\nu \in \mathbf{N}$. Take an arbitrary compact set K of D. There exists $N \in \mathbf{N}$ such that $K \subset D_N$. Since D_N is meromorphically $\mathcal{O}(X)$-convex (see Abe [2]), we have that $\tilde{K}_X \subset D_N \subset D$ (see Theorem 12 of Abe [1]). It follows that D is meromorphically $\mathcal{O}(X)$-convex. □

By the similar argument we also prove the following Theorem 4.2 which characterizes a rationally convex open set of $\mathbf{C}^n$.

Theorem 4.2. *Let D be an open set of $\mathbf{C}^n$. Then the following three conditions are equivalent.*

(1) *D is rationally convex in* $\mathbf{C}^n$.
(2) *D is the union of an increasing sequence $\{D_\nu\}_{\nu=1}^\infty$ of open sets of* $\mathbf{C}^n$ *such that D_ν is $\mathscr{R}(D_\nu)$-convex for every* $\nu \in \mathbf{N}$.
(3) *D is the union of an increasing sequence $\{D_\nu\}_{\nu=1}^\infty$ of open sets of* $\mathbf{C}^n$ *such that D_ν is $\mathcal{O}_{\mathbf{C}^n}(D_\nu)$-convex for every* $\nu \in \mathbf{N}$.

Proof. (1) $\Rightarrow$ (2). Take a sequence $\{K_\nu\}_{\nu=1}^\infty$ of compact sets of D such that $\bigcup_{\nu=1}^\infty K_\nu = D$ and $K_\nu \subset \mathring{K}_{\nu+1}$ for every $\nu \in \mathbf{N}$. For every compact set K of D we have that $\tilde{K}_{\mathbf{C}^n} \subset D$ (see Theorem 12 of Abe [1]). By Lemma 3.3 there exists a rational polyhedron W such that $\tilde{K}_{\mathbf{C}^n} \subset W \Subset D$. Therefore by induction there exists a sequence $\{W_\nu\}_{\nu=1}^\infty$ of rational polyhedra such that $K_\nu \cup \overline{W}_{\nu-1} \subset W_\nu \Subset D$ for every $\nu \in \mathbf{N}$, where $W_0 := \emptyset$. Then we have that $\bigcup_{\nu=1}^\infty W_\nu = D$ and $W_\nu \Subset W_{\nu+1}$ for every $\nu \in \mathbf{N}$. By Lemma 3.2 the open set W_ν is $\mathscr{R}(W_\nu)$-convex for every $\nu \in \mathbf{N}$.
(2) $\Rightarrow$ (3). Clear. □
(1) $\Leftrightarrow$ (3). The assertion is by Theorem 4.1.

We also have the following Theorem 4.3 which characterizes a connected rationally convex open set of $\mathbf{C}^n$.

Theorem 4.3. *Let D be a connected open set of* $\mathbf{C}^n$. *Then the following three conditions are equivalent.*

(1) *D is rationally convex in* $\mathbf{C}^n$.
(2) *D is the union of an increasing sequence $\{D_\nu\}_{\nu=1}^\infty$ of connected open sets of* $\mathbf{C}^n$ *such that D_ν is $\mathscr{R}(D_\nu)$-convex for every* $\nu \in \mathbf{N}$.
(3) *D is the union of an increasing sequence $\{D_\nu\}_{\nu=1}^\infty$ of connected open sets of* $\mathbf{C}^n$ *such that D_ν is $\mathcal{O}_{\mathbf{C}^n}(D_\nu)$-convex for every* $\nu \in \mathbf{N}$.

Proof. (1) $\Rightarrow$ (2). Take a sequence $\{K_\nu\}_{\nu=1}^\infty$ of connected compact sets of D such that $\bigcup_{\nu=1}^\infty K_\nu = D$ and $K_\nu \subset \mathring{K}_{\nu+1}$ for every $\nu \in \mathbf{N}$. By the proof of Theorem 4.2 there exists a sequence $\{W_\nu\}_{\nu=1}^\infty$ of rational polyhedra such that $K_\nu \cup \overline{W}_{\nu-1} \subset W_\nu \Subset D$ for every $\nu \in \mathbf{N}$, where $W_0 := \emptyset$. Let D_ν be the connected component of W_ν containing K_ν for every $\nu \in \mathbf{N}$. By Lemmas 3.2 and 3.7 the open set D_ν is $\mathscr{R}(W_\nu)$-convex and therefore $\mathscr{R}(D_\nu)$-convex. Replacing $\{D_\nu\}_{\nu=1}^\infty$ by a subsequence we also have that $D_\nu \Subset D_{\nu+1}$ for every $\nu \geq 1$.
(2) $\Rightarrow$ (3). Clear.
(3) $\Rightarrow$ (1). The assertion is by Theorem 4.1.

In Oka [8] a domain D in $\mathbf{C}^n$ is said to be *rationnellement convexe* (rationally convex) if D

is $\mathscr{R}(D)$-convex or D can be approximated from the interior by domains D_ν which are $\mathscr{R}(D_\nu)$-convex (see also Nishino [7, p. 99]). By the proof of Theorem 4.2 our definition of the rational convexity for a connected open set of $\mathbf{C}^n$ is equivalent to the one due to Oka [8].

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