Kumamoto J. Math.
Vol. 18, 25-29 March (2005)

A note on domains of holomorphy in two dimensional normal Stein spaces

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(Received January 31, 2005)

1. Introduction

1. Let X be a complex analytic space and D a domain in X. We say that D is a *domain of holomorphy* if there are no open sets U and V in X with the following properties:

(i) $\emptyset \neq V \subset U \cap D$

(ii) U is connected and $U \not\subset D$

(iii) For every holomorphic function f on D there is a holomorphic function g on U such that $f=g$ in V.

Then we shall study the following

Problem 1. *If D is a domain of holomorphy in a Stein space X, then is D Stein ?*
On this problem, the following facts are well-known.

Theorem 1. (1) *A domain D in a Stein manifold M is Stein if and only if D is a domain of holomorphy in M* (see Cartan-Thullen [2] and F. Docquier and H. Grauert [4]).

(2) *Let D be a domain of holomorphy in a Stein space X with the singular locus* Sing X. *Then D is locally Stein at every point $p \in \partial D \setminus$*Sing X, *that is, there is an open neighborhood U of p such that $U \cap D$ is Stein* (see F. Docquier and H. Grauert [4]).

(3) *Let X be a Stein space with isolated singularities and D a domain in X. If D is locally Stein at every point $p \in \partial D$, then D is Stein* (see Andreotti-Narasimhan [1]).

(4) *Let D be a domain of holomorphy in a two-dimensional normal Stein space X. Assume that D is (locally) simply connected near every singular point x_0 of X on the boundary*

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∂D. *Then D is Stein* (see Furushima [6])

(5) *There is a domain of holomorphy D in a normal Stein space of dimension $n \geq 3$ such that D is not Stein* (see Grauert-Remmert [7], Scheja [11] and Ueda [13]).

Remark 1. *We have no example of two dimensional domain of holomorphy which is not Stein.*

From Theorem 1-(3), one sees

Proposition 1. *A domain of holomorphy D in a two-dimensional normal Stein space X is Stein if D is locally Stein at every point $p \in \partial D \cap \mathrm{Sing}\, X$.*

Finally the problem is stated as follow:

Problem 2. *Let D be a domain of holomorphy in a two-dimensional normal Stein space X. Then is D locally Stein at $p \in \partial D \cap Sing\ X$?*

2. A domain of holomorphy in a two dimensional Stein cone

2. Let $\pi: L \longrightarrow R$ be the negative line bundle over a compact Riemann surface R of genus $g \geq 0$ with the negative section Σ_0. Then Σ_0 is analytically contractible to one point, that is, there is a proper surjective holomorphic mapping $\varphi: L \longrightarrow X_g$ of L to a normal Stein space X_g with singularity $x_g = \varphi(\Sigma_0)$ such that $\varphi: L - \Sigma_0 \cong X_g - \{x_g\}$ (biholomorphic). The Stein surface X_g is called the two-dimensional *Stein cone* of genus $g \geq 0$. It is well known that X_g is smooth if and only if $g=0$ and the self-intersection number $(\Sigma_0^2)_L = -1$.

Now let $D \subset X_g$ be a domain of holomorphy in the two-dimensional Stein cone X_g of genus $g \geq 0$.

Proposition 2. *D is Stein if $x_g \notin \partial D$*

Proof. If $x_g \notin \partial D$, then D is locally Stein by Theorem 1-(2). Thus D is Stein by Theorem 1-(3). □

3. Thus we may assume that $x_g \in \partial D$. Let $\varphi: L \longrightarrow X_g$ and $\pi: L \longrightarrow R$ be as above. We set $\tilde{D} := \varphi^{-1}(D) \subset L$, $\tilde{D}_t := \tilde{D} \cap \pi^{-1}(t)$ and $\Gamma_t := \varphi\ (\tilde{D}_t)$ for $t \in R$. Then the one-dimensional analytic subset $\Gamma_t\ (\cong \tilde{D}_t)$ of D is considered as a plane domain containing the origin $0 \in \mathbf{C}$ if $\Gamma_t \neq \emptyset$.

Then we have

Proposition 3. *Assume that $\Gamma_{t_0} = \emptyset$ for some $t_0 \in R$. Then D is Stein.*

Proof. We put $\ell_0 := \varphi(\pi^{-1}(t_0))$. Then we have $D \subset X_g - \ell_0$ by assumption. Then the complement $X_g - \ell_0$ is a Stein manifold by Simha [10]. Since D is a domain of holomorphy in a Stein manifold, it is Stein by the theorem of Docquier and Grauert [4]. □

Proposition 4. *If the genus $g=0$, then D is Stein.*

Proof. If $g=0$, then X_0 has at most a cyclic quotient singularity at x_0, that is, $X_0 \cong \mathbf{C}^2/G$, where $G=\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{n-1} \end{pmatrix}$ and $\zeta^n=1$ for some integer $n \geq 1$. Let $\psi: \mathbf{C}^2 \longrightarrow X_0=\mathbf{C}^2/G$ be the quotient mapping. Since D is a domain of holomorphy, so is $\psi^{-1}(D) \subset \mathbf{C}^2$, hence it is Stein. Since ψ is a proper finite surjective holomorphic mapping, the image $D=\psi(\psi^{-1}(D))$ is also Stein by Narasimhan [9]. □

4. Our main theorem is the following:

Theorem 2. *Let D is a domain of holomorphy in the Stein cone X_g. Assume that $\Gamma_t \subset D$ is simply connected for any $t \in R$. Then D is Stein.*

Proof. We may assume that $g \geq 1$ and $\Gamma_t \neq \emptyset$ for any $t \in \Gamma$ by Proposition 3 and 4. We set $X_g^* := X_g - x_g$ and let $\sigma: \tilde{X}_g^* \longrightarrow X_g^*$ be the universal covering of X_g^*.

(Claim) $\tilde{X}_g^*$ is a Stein (cf. [3]).

In fact, take the universal covering $\tau: \hat{L} \longrightarrow L$ of the negative line bundle L. Since the universal covering of R is biholomorphic to the plane $\mathbf{C}$ (resp. a unit disc Δ) if $g=1$ (resp. $g \geq 2$), $\hat{L}$ is biholomorphic to either $\mathbf{C}^2$ or $\Delta \times \mathbf{C}$. Thus $\hat{L}$ is a two-dimensional Stein manifold. Since $\tau^{-1}(\Sigma_0)$ is a one-dimensional analytic subset of $\hat{L}$, the complement $M := \hat{L} - \tau^{-1}(\Sigma_0)$ is also Stein. Since $\tilde{X}_g^*$ is simply connected, there is an unramified holomorphic covering map $\gamma: \tilde{X}_g^* \longrightarrow M$. Since M is Stein, so is $\tilde{X}_g^*$ by K. Stein [12]. □

We continue the proof of Theorem 2. One has $\Gamma_t \cap \Gamma_{t'} = \emptyset$ for any $t \neq t' \in R$. Any connected component $\hat{\Gamma}_t^{(i)}$ $(i=0, 1, 2, \ldots)$ of $\sigma^{-1}(\Gamma_t)$ is biholomorphic to Γ_t since Γ_t is simply connected. Take a point $p \in D$ and a small neighborhood $U := U(p) \subset D$ such that $\sigma^{-1}(U) = \bigcup_k \tilde{U}_k$ (the disjoint union) and $\tilde{U}_k \overset{\sigma}{\cong} U$ (biholomorphic) for $0 \leq k \in \mathbf{Z}$. Take a connected component $\tilde{U}_0$ of $\sigma^{-1}(U)$. Then there is a connected component $\tilde{\Gamma}_t^{(0)}$ of $\sigma^{-1}(\Gamma_t)$ such that $\tilde{\Gamma}_t^{(0)} \cap \tilde{U}_0 \neq \emptyset$. We set $T := \{t \in R : \tilde{\Gamma}_t^{(0)} \cap \tilde{U}_0 \neq \emptyset\} \neq \emptyset$. Let $\tilde{D}^*$ be the connected component of $\sigma^{-1}(D)$ containing the family $\{\tilde{\Gamma}_t^{(0)}\}_{t \in T}$. Then $\tilde{D}^*$ is biholomorphic to $D \subset X_g^*$. Since D is a domain of holomorphy, so is $\tilde{D}^*$. Since $\tilde{X}_g^*$ is a Stein manifold, D^* is Stein. This completes the proof. □

Finally we propose the following:

Problem 3. *Let $\Omega \subset X$ be an domain of holomorphy in a Stein space X and $\mu: \tilde{\Omega} \longrightarrow \Omega$ an* *infinite unramified covering. If $\tilde{\Omega}$ is a Stein manifold, then is Ω Stein ?*

Remark 2. (1) *In the case where X is a Stein manifold, it is known that any domain Ω whose infinite unramified covering space $\tilde{\Omega}$ is Stein is also Stein by Kerner* [8].

(2) *If Problem 2 is affirmative in general, then from the proof of Theorem 4 one sees that Problem 1 is affirmative.*

Finally the author would like to thank Prof. M. Furushima for the encouragement.

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