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On a Gaugeability for Non-local Feynman-Kac Functionals

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Abstract

An analytic characterization of gaugeability (or conditional gaugeability) for non-local Feynman-Kac functionals is given by a large deviation approach.

1. Introduction

Let $\mathbf{M}=(\Omega,X_t,P_x,\zeta)$ be a transient m-symmetric Hunt process on a locally compact separable metric space X. Here m is a positive Radon measure on X with full support and ζ is the life time of X_t. Denote by $R(x,y)$ the Green function of $\mathbf{M}$ and P_x^y the probability of Doob's $R(\cdot,y)$-transformed process starting from x. Let A_t^μ be the continuous additive functional of $\mathbf{M}$ with a signed Revuz measure $\mu=\mu^+-\mu^-$. We say that μ is gaugeable (resp. conditionally gaugeable) if

$$\sup_{x\in X}E_x(\exp(A_\zeta^\mu))<\infty \quad \left(\text{resp.} \sup_{(x,y)\in X\times X\backslash d}E_x^y(\exp(A_{\zeta^y}^\mu))\right)<\infty. \tag{1.1}$$

Here ζ^y is the lifetime of the conditioned process P_x^y.

In [13], the author obtained using a large deviation method and time change operator, the analytic characterization of gaugeability and conditionally gaugeability for a general potential given by a measure μ as follows; under suitable conditions for μ and $\mathbf{M}$, the gaugeability (1. 1) is equivalent with

$$\inf\left\{\mathcal{E}(u,u)+\int_X\tilde{u}^2d\mu^-\right\}>1, \tag{1.2}$$

where $(\mathcal{E},\mathcal{F})$ is the Dirichlet form generated by $\mathbf{M}$, $\tilde{u}$ is a quasi continuous m-version of $u\in\mathcal{F}$ and the inf is calculated on $u\in\mathcal{F}$ satisfying $\int_X\tilde{u}^2d\mu^+=1$. The characterization (1.2) was very useful in determining whether μ being gaugeable or conditionally gaugeable in many concrete cases, as illustrated by several examples given in his paper.

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Let us consider the following non-local type of additive functional of the form,

$$A_t^{\mu,F} := A_t^{\mu}+A_t^F, \quad A_t^F=\sum_{s\le t}F(X_{s-},X_s). \tag{1.3}$$

Here $F(x,y)$ is a symmetric ($F(x,y)=F(y,x)$) bounded Borel function on $X\times X$ which vanishes along the diagonal. In the past years, the Dirichlet form and the associated semigroup perturbed by (1.3) have been studied by many authors with some related applications ([5], [8], [10], [14]).

The gauge theorem for (1.3) was first studied in [10] on symmetric stable processes and generalized on symmetric Markov processes ([3], [5]). In particular, Chen [2] recently give the analytic characterization of gaugeability (or conditional gaugeability) for (1.3) on general symmetric Markov processes by using a pure jump Girsanov transform and his gauge theorems. The main result was as follows; under suitable Kato class conditions for μ and F,

$$\sup_{x\in X}E_x(\exp(A_\zeta^{\mu,F}))<\infty \tag{1.4}$$

is equivalent with

$$\inf\left\{\mathscr{E}(u,u)-\int_{X\times X\setminus d}\tilde{u}(x)\tilde{u}(y)G(x,y)N(x,dy)\mu_H(dx)+\int_X\tilde{u}(x)^2(\mu^-+\mu_{G^+})(dx)\right\}>1, \tag{1.5}$$

where the inf is calculated on $u\in\mathscr{F}$ satisfying $\int_X\tilde{u}(x)^2(\mu^++\mu_{G^+})(dx)=1$, and the definitions of G and μ_{G^+} are appeared in §2 of this paper.

An objective of this paper is to derive the characterization (1.5) for the gaugeability (1.4) by using a large deviation method and time change operator. Thus, we extend the analytic characterization of gaugeability for local Feynman-Kac functionals in [13] to non-local cases.

Unlike the local cases, we can not consider the time change operator related to (1.3) directly. However, some properties of the subprocess by the jump type Girsanov transform introduced in § 3 will make it possible to use the time change operator. In § 4, the main theorem is given, in which, some regularity assumptions on Green functions as in [13] are removed.

2. Preliminaries

In what follows, let $(p_t)_{t\ge0}$ be the transition semigroup, $p_tf(x)=E_x(f(X_t))$, and $(R_\alpha)_{\alpha\ge0}$ the resolvent of $\mathbf{M}$, $R_\alpha f(x)=E_x(\int_0^\infty e^{-\alpha t}f(X_s)ds)$. Let $R_\alpha(x,y)$, $(\alpha\ge0)$ be the α-resolvent density, $R_\alpha(x,y)=\int_0^\infty e^{-\alpha t}p_t(x,y)dt$, where $p_t(x,y)$ is the transition density of $(p_t)_{t\ge0}$. Then $R_0(x,y)=:$

$R(x,y)$ is the Green function of $\mathbf{M}$. For any given smooth measure μ, define $R_\alpha\mu(x)=\int_X R_\alpha(x,y)\mu(dy)$. For $z\in X$, denote by

$$R^z(x,y):=\frac{R(x,y)R(y,z)}{R(x,z)},\quad x,y\in X\setminus\{z\}$$

the Green function of Doob's $R(\cdot,z)$-transformed process and P_x^z the probability starting from x. Let $A_t^\mu=A_t^{\mu^+}-A_t^{\mu^-}$ be a continuous additive functional with signed smooth measure $\mu=\mu^+-\mu^-$ as its Revuz measure. Denote by $|A_t^\mu|=A_t^{\mu^+}+A_t^{\mu^-}$ and $|\mu|=\mu^++\mu^-$. Let $F(x,y)=F^+(x,y)-F^-(x,y)$ be a symmetric ($F(x,y)=F(y,x)$) bounded Borel function on $X\times X$ that vanishes along the diagonal and denote by $|F(x,y)|=F^+(x,y)+F^-(x,y)$.

Definition 2.1 (*i*) *A signed smooth measure μ on X is said to be in the Kato class $\mathscr{K}(R)$ if*

$$\lim_{t\downarrow 0}\sup_{x\in X}E_x(|A_t^\mu|)=0.$$

(*ii*) *A signed smooth measure μ is said to be in the class $\mathscr{K}_\infty(R)$ if for any $\varepsilon>0$, there exists a compact set $K=K(\varepsilon)$ of finite $|\mu|$-measure and a constant $\delta=\delta(\varepsilon)>0$ such that for all measurable set $B\subset K$ with $|\mu|(B)<\delta$,*

$$\sup_{x\in X}\int_{(K\setminus B)^c}R(x,y)|\mu|(dy)<\varepsilon.$$

(*iii*) *A signed smooth measure μ is said to be in the class $\mathscr{S}_\infty(R)$ if for any $\varepsilon>0$, there exists a compact set $K=K(\varepsilon)$ of finite $|\mu|$-measure and a constant $\delta=\delta(\varepsilon)>0$ such that for all measurable set $B\subset K$ with $|\mu|(B)<\delta$,*

$$\sup_{x,z\in X}\int_{(K\setminus B)^c}R^z(x,y)|\mu|(dy)<\varepsilon.$$

Definition 2.2 *Let (N,H) be the Lévy system for* $\mathbf{M}$ *and μ_H be the Revuz measure of the positive continuous additive functional H.*

(*i*) *A bounded function F on $X\times X\setminus d$ is said to be in the class $\mathscr{J}(R)$ if the measure*

$$\mu_{|F|}(dx):=N|F|(x)\mu_H(dx),\quad N|F|(x)=\int_X|F(x,y)|N(x,dy)$$

belongs to the class $\mathscr{K}(R)$.

(*ii*) *A bounded function F on $X\times X\setminus d$ is said to be in the class $\mathscr{J}_\infty(R)$ if the measure $\mu_{|F|}$ belongs to the class $\mathscr{K}_\infty(R)$.*

(*iii*) *A bounded function F on $X\times X\setminus d$ is said to be in the class $\mathscr{A}_\infty(R)$ if for any $\varepsilon>0$,* *there exists a compact set $K=K(\varepsilon)$ of finite $\mu_{|F|}$-measure and a constant $\delta=\delta(\varepsilon)>0$ such that for all measurable set $B\subset K$ with $\mu_{|F|}(B)<\delta$,*

$$\sup_{x,z\in X}\int_{((K\backslash B)\times(K\backslash B))^c} R^2(x,y)|F(y,w)|N(y,dw)\mu_H(dy)<\varepsilon$$

and the measure $\mu_{|F|}$ belongs to $\mathscr{S}_\infty(R)$.

We note that the relations $\mathscr{S}_\infty(R)\subset\mathscr{K}_\infty(R)\subset\mathscr{K}(R)$ (and consequently $\mathscr{A}_\infty(R)\subset\mathscr{J}_\infty(R)\subset\mathscr{J}(R)$) which are proved in [5].

Now let us make the following assumptions on $\mathbf{M}$ throughout this paper.

($\mathbf{A}_1$) (Irreducibility) If a Borel set B satisfies $\chi_B p_t f=p_t(\chi_B f)$ for any $f\in L^2(X;\ m)\cap B_b(X)$ and $t>0$, then $m(B)=0$ or $m(X\backslash B)=0$. Here $B_b(X)$ is the space of bounded Borel functions on X.

($\mathbf{A}_2$) (Strong Feller Property) $p_t(B_b(X))\subset C_b(X)$, where $C_b(X)$ is the space of bounded continuous functions on X.

Let us denote by λ_p the spectral radius of the semigroup $(p_t)_{t\geq 0}$ as an operator on $L^p(X,m)$,

$$\lambda_p=-\lim_{t\to\infty}\frac{1}{t}\log\|p_t\|_{p,p},\quad 1\leq p\leq\infty,$$

where $\|\cdot\|_{p,p}$ is the operator norm form L^p to L^p. Then the following theorem is known as a corollary of Donsker-Varadhan type's large deviation principle.

Theorem 2.3 ([13]) *For any $\varepsilon>0$, let us suppose that there exists a compact set K such that $\sup_{x\in X}R_1\chi_{K^c}(x)\leq\varepsilon$. Then, we have*

$$\lambda_\infty=\inf_{u\in\mathscr{F},\|u\|_2=1}\mathscr{E}(u,u). \tag{2.1}$$

Note that the equation (2.1) implies the p-independence of the spectral radius λ_p because the right hand side of (2.1) is equal to λ_2. On the other hand, Sato [9] showed that the exponential boundedness of the life time ζ of $\mathbf{M}$ is equivalent with the condition $\lambda_\infty>1$. So, by combining this fact with the p-independence of λ_p, it holds that

Corollary 2.4 ([13]) *For any $\varepsilon>0$, let us suppose that there exists a compact set K such that $\sup_{x\in X}R_1\chi_{K^c}(x)\leq\varepsilon$. Then*

$$\sup_{x\in X}E_x(\exp(\zeta))<\infty \text{ if and only if } \lambda_2>1. \tag{2.2}$$

3. Subprocess by jump type Girsanov transform

For $F(x,y)=F^+(x,y)-F^-(x,y)$ in §2, we put

$$G(x,y)=G^+(x,y)-G^-(x,y)=\left(e^{F^+(x,y)}-1\right)-\left(e^{F^-(x,y)}-1\right).$$

Define the associated additive functional of G by

$$A_t^G=A_t^{G_+}-A_t^{G_-}=\sum_{s\le t}G^+(X_{s-},X_s)-\sum_{s\le t}G^-(X_{s-},X_s). \tag{3.1}$$

Then its dual predictable projection, namely, $\hat{A}_t^G=\hat{A}_t^{G_+}-\hat{A}_t^{G_-}$ is written as

$$\hat{A}_t^G=\hat{A}_t^{G_+}-\hat{A}_t^{G_-}=\int_0^t NG_+(X_s)dH_s-\int_0^t NG_-(X_s)dH_s, \tag{3.2}$$

where

$$NG_\pm(x)=\int_X N(x,dy)G^\pm(x,y)$$

and (N,H) is the Lévy system of **M**. We denote by $\mu_G=\mu_{G^+}-\mu_{G^-}$ the Revuz measure of the continuous additive functional $\hat{A}_t^G=\hat{A}_t^{G_+}-\hat{A}_t^{G_-}$.

Lemma 3.1 *For a signed smooth measure μ and the symmetric bounded functions F,G defined as above,*

$$S_t:=\exp\left(A_t^F-A_t^{\mu_-}-\hat{A}_t^{G_+}\right)$$

is a supermartingale multiplicative functional.

Proof. Note that $A_t^G-\hat{A}_t^G$ is a P_x-martingale for every $x\in X$. So it follows from Doléans-Dade exponential formula that

$$\begin{aligned}Q_t:&=e^{A_t^G-\hat{A}_t^G}\prod_{s\le t}(1+G(X_{s-},X_s))e^{-G(X_{s-},X_s)}\\&=e^{-\hat{A}_t^G}\prod_{s\le t}(1+G(X_{s-},X_s))\\&=e^{A_t^F-\hat{A}_t^G}\end{aligned} \tag{3.3}$$

is a supermartingale multiplicative functional. Hence, we have for $t\ge0$,

$$E_x(S_t)=E_x\left(Q_t\exp\left(-\left(A_t^{\mu_-}+\hat{A}_t^{G_-}\right)\right)\right)\le E_x(Q_t)\le1.$$

The proof is completed □

Let $\mathbf{M}^S=(\Omega,X_t,P_x^S,\zeta)$ be the transformed process of $\mathbf{M}$ by the jump type Girsanov transform S_t. That is, the semigroup of $\mathbf{M}^S$, the so called Feynman-Kac semigroup is given by

$$p_t^S f(x)=E_x^S(f(X_t))=E_x(S_t f(X_t)),$$

and also its resolvent is given by

$$R_\alpha^S f(x)=E_x\Big(\int_0^\infty e^{-\alpha t}S_t f(X_t)dt\Big).$$

Let us consider the symmetric bilinear form $(\mathscr{E}^S,\mathscr{F}^S)$ on $L^2(X;m)$ defined by

$$\mathscr{E}^S(u,v)=\mathscr{E}(u,v)+\int_X \tilde{u}(x)\tilde{v}(x)(\mu^-+\mu_{G^+})(dx)$$
$$-\int_{X\times X}\tilde{u}(x)\tilde{v}(y)G(x,y)N(x,dy)\mu_H(dx)$$
$$\mathscr{F}^S=\{u\in\mathscr{F}:\ \tilde{u}\in L^2(X;|\mu^-|+|\mu_G|)\}$$

Lemma 3.2 *Assume that* $\mu^-\in\mathscr{K}(R)$ *and* $F\in\mathscr{J}(R)$. *Then*

(*i*) $\{p_t^S\}$ *is a strongly continuous semigroup of symmetric bounded operators on* $L^2(X;m)$. *That is,* $(\mathscr{E}^S,\mathscr{F}^S)$ *is a lower semibounded closed symmetric bilinear form on* $L^2(X;m)$.

(*ii*) *For any* $p\geq 1$, *there exist constants* α, $\beta>0$ *such that for all* $t\geq 0$,

$$\sup_{x\in X}E_x((S_t)^p)\leq \alpha e^{\beta t}.$$

(*iii*) *For any* $p\geq 1$, $\lim_{t\to 0}\sup_{x\in X}E_x(|V_t|^p)=0$. *Here* $V_t=A_t^F-A_t^{\mu^-}-\hat{A}_t^{G^+}$. *In particular,*

$$\lim_{t\to 0}\sup_{x\in X}E_x(|S_t-1|^2)=0.$$

Proof. It is easy to see that if $F_1,F_2\in\mathscr{J}(R)$ and c is a constant, then cF_1 and F_1+F_2 belong to $\mathscr{J}(R)$. Furthermore, if $F\in\mathscr{J}(R)$, then

$$G_p:=e^{2pF}-1$$

also belongs to $\mathscr{J}(R)$ by the boundedness of the function $x\to x^{-1}(e^x-1)$ on $[-a,a]$ such that $|F(x,y)|<a$ for any $x,y\in X$. Now, the statement (i) is the consequence of Theorem 3.2 in [14].

(ii) It is well-known that the local Feynman-Kac functional exp (A_t^ν) associated with a measure $\nu\in\mathscr{K}(R)$ is bounded in the sense that there exist constants $\alpha_1,\beta_1>0$ such that for any $t>0$,

$$\sup_{x\in X} E_x(\exp(A_t^F)) \leq \alpha_1 e^{\beta_1 t} \tag{3.4}$$

(Lemma 3 in [6]). Now, let us define the additive functional $A_t^{G_p}$ and its dual predictable projection $\hat{A}_t^{G_p}$ for the function G_p as the same way of (3.1) and (3.2). On the same reason of (3.3),

$$\exp\left(2pA_t^F - \hat{A}_t^{G_p}\right)$$

is a supermartingale multiplicative functional. So applying the Schwartz inequality, we have

$$\begin{aligned} E_x((S_t)^p) &\leq E_x(\exp(pA_t^F)) \\ &\leq \left[E_x\left(\exp\left(2pA_t^F - \hat{A}_t^{G_p}\right)\right)\right]^{1/2}\left[E_x\left(\exp\left(\hat{A}_t^{G_p}\right)\right)\right]^{1/2} \\ &\leq \left[E_x\left(\exp\left(\hat{A}_t^{G_p}\right)\right)\right]^{1/2}. \end{aligned} \tag{3.5}$$

Since $G_p \in \mathscr{J}(R)$, there exist constants $\alpha_1, \beta_1 > 0$ such that for any $t > 0$,

$$\sup_{x\in X} E_x\left(\exp\left(\hat{A}_t^{G_p}\right)\right) \leq \alpha_1 e^{\beta_1 t}$$

on account of (3.4). Thus, (ii) is proved by (3.5).

(iii) Note that for $x > 1$, x^{2p} is dominated up to a constant c by e^x. Thus, by the Schwartz inequality, we have

$$\begin{aligned} E_x(V_t^p) &= E_x(V_t^p;\ V_t > 1) + E_x(V_t^p;\ V_t \leq 1) \\ &\leq E_x(V_t^{2p};\ V_t > 1)^{1/2} P_x(V_t > 1)^{1/2} + E_x(V_t) \\ &\leq c^{1/2} E_x(S_t)^{1/2} E_x(V_t)^{1/2} + E_x(V_t) \end{aligned}$$

which implies that $\lim_{t\to 0}\sup_{x\in X} E_x((V_t)^p) = 0$. Moreover, by noting $|e^x - 1| \leq |x|(e + e^x)$, we have

$$\begin{aligned} E_x(|S_t - 1|^2) &\leq E_x(V_t^2(e + S_t)^2) \\ &\leq eE_x(V_t^2) + 2eE_x(V_t^4)^{1/2} E_x(S_t^2)^{1/2} \\ &\quad + E_x(V_t^4)^{1/2} E_x(S_t^4)^{1/2}. \end{aligned}$$

Now, the proofs are completed. □

Lemma 3.3 *Assume that* $\mu^- \in \mathscr{S}_\infty(R)$ *and* $F \in \mathscr{A}_\infty(R)$. *Then the Green function* $R^S(x,y)$ *of* $\mathbf{M}^S$ *is comparable to* $R(x,y)$ *in the sense that*

$$R^S(x,y) \leq kR(x,y)$$

for some constant $k>1$.

Proof. By the assumptions of this lemma, the measures $\mu_{|F|}$ and μ_{G^+} belong to $\mathcal{S}_\infty(R)$. Therefore, the gaugeability and conditional gaugeability for the additive functional $A_t^F-A_t^{\mu^-}-\hat{A}_t^{G^+}$ are equivalent each other on account of Theorem 3.10 in [5]. Moreover, since

$$E_x(S_\zeta)\leq 1$$

for any $x\in X$ by virtue of (3.3), the conditional gauge integral $E_x^y(S_{\zeta^y})$ is to be bounded. So, we have $R^S(x,y)=E_x^y(S_{\zeta^y})R(x,y)\leq kR(x,y)$ for some constant $k>1$. □

Proposition 3.4 *Assume that* $\mu^-\in\mathcal{S}_\infty(R)$ *and* $F\in\mathcal{A}_\infty(R)$. *Then the transformed process* $\mathbf{M}^S$ *is an* m*-symmetric Markov process satisfying the assumptions* (A_1) *and* (A_2).

Proof. Let r_t be the reversal operator on Ω; for $\omega\in\Omega$ and $t<\zeta(\omega)$, $X_s(r_t\omega)=X_{t-s}(\omega)$ for $s\leq t$ and $X_s(r_t\omega)=\Delta$ (the trap point of $\mathbf{M}$) for $s>t$. Since F is symmetric, it is easy to check that $S_t(r_t\omega)=S_t(\omega)$, P_m-a.e.. Hence we have for any measurable functions f,g

$$\begin{aligned}(p_t^Sf,g)_m &= E_m(S_t(\omega)f(X_t(\omega))g(X_0(\omega)))\\ &= E_m(S_t(r_t\omega)f(X_t(r_t\omega))g(X_0(r_t\omega)))\\ &= E_m(S_t(\omega)f(X_0(\omega))g(X_t(\omega)))\\ &= (f,p_t^Sg)_m\end{aligned}$$

which implies that $\mathbf{M}^S$ is m-symmetric. The irreducibility of $\mathbf{M}^S$ is immediately derived by the positivity of S_t up to the lifetime ζ. For proving the strong Feller property, the standard methods by Theorem 3.9 in [1] are still useful. More pricisely, for any $f\in B_b(X)$ and $t>0$, we can write

$$p_t^Sf(x)=E_x(e^{V_t-V_\varepsilon}f(X_t))+E_x(e^{V_t-V_\varepsilon}(e^{V_\varepsilon}-1)f(X_t)). \tag{3.7}$$

Here $V_t=A_t^F-A_t^{\mu^-}-\hat{A}_t^{G^+}$. The first term on the right-hand side of (3.7), $p_\varepsilon(p_{t-\varepsilon}^Sf)\ (x)$ is bounded continuous by virtue of Lemma 3.2 (i) and the assumption (A_2). For the second term, it is dominated up to

$$\|f\|_\infty\Big(E_x\Big(|S_\varepsilon-1|^2\Big)\Big)^{1/2}\sup_{0\leq s\leq t}(E_x(S_s^2))^{1/2}$$

which tends to 0 uniformly as $\varepsilon\to 0$ by Lemma 3.2 (ii) and (iii). Hence $(p_t^S)_{t\geq 0}$ is a strong Feller semigroup. □

Lemma 3.5 *Assume that* $\mu^- \in \mathcal{S}_\infty(R)$ *and* $F \in \mathcal{A}_\infty(R)$. *Then*

$$\inf_{u\in\mathcal{F}^S, \|u\|_2=1} \mathcal{E}^S(u,u) = \inf_{u\in\mathcal{F}, \|u\|_2=1} \mathcal{E}^S(u,u). \tag{3.8}$$

Proof. It is well-known that for any positive smooth measure ν of $\mathbf{M}$,

$$\int_X \tilde{u}^2 d\nu \leq \|R\nu\|_\infty \mathcal{E}(u,u)$$

for any $u \in \mathcal{F}$([12]). Since $\mu^- \in \mathcal{S}_\infty(R)$ and $F \in \mathcal{A}_\infty(R)$, we have $\int_X \tilde{u}^2 d\mu^- < \infty$, $\int_X \tilde{u}^2 d\mu_{|G|} < \infty$ and consequently, it follows from the Schwartz inequality

$$\int_{X\times X} \tilde{u}(x)\,\tilde{v}(y)|G(x,y)|N(x,dy)\mu_H(dx) \leq \int_X \tilde{u}^2 d\mu_{|G|} < \infty$$

for any $u \in \mathcal{F}$. Therefore

$$\mathcal{E}^S(u,v) = \mathcal{E}(u,v) + \int_X \tilde{u}(x)\,\tilde{v}(x)(\mu^- + \mu_{G^+})(dx)$$
$$- \int_{X\times X} \tilde{u}(x)\,\tilde{v}(y)G(x,y)N(x,dy)\mu_H(dx)$$

is well defined for any $u \in \mathcal{F}$ which implies (3.8). □

4. Analytic characterization and time change operator

In this section, we are going to derive the analytic characterization of gaugeability (1.5) for the non-local Feynman-Kac functional (1.3) by using the time change operator associated with the additive functional (4.1) below. The original idea is due to [13] but some regularity assumptions on Green functions will be removed in our present section.

For notation convenience, we denote

$$\hat{A}_t^{(\mu+G)_+} := \hat{A}_t^{\mu_+} + \hat{A}_t^{G_+}.$$

Let $\{\tau_t(\mu+G)_+\}_{t\geq 0}$ be the right continuous inverse of the positive continuous additive functional

$$\hat{A}_t^{(\mu+G)_+}, \tag{4.1}$$

that is, $\tau_t(\mu+G)_+ = \inf\{s>0 : \hat{A}_s^{(\mu+G)_+} > t\}$, where $(\mu+G)_+ := \mu^+ + \mu_{G^+}$ stands for the smooth measure of $\hat{A}_t^{(\mu+G)_+}$ as the Revuz correspondence. Let $E = E^{(\mu+G)_+}$ be the fine support of $(\mu + G)_+$ in the sense that

$$E = \{x \in X : P_x^S(\tau_0(\mu+G)_+ = 0) = 1\}.$$

The time changed process $(\check{\mathbf{M}}^S, (\mu+G)_+) := (\check{X}_t, P_x^S, \check{\zeta})$ of $\mathbf{M}^S$ with respect to $\hat{A}_t^{(\mu+G)_+}$ is then defined by

$$\check{X}_t = X_{\tau_t(\mu+G)_+}.$$

The transition semigroup and the resolvent of $(\check{\mathbf{M}}^S, (\mu+G)_+)$ are given by

$$\check{p}_t^{(\mu+G)_+}\varphi(x) = E_x^S(\varphi(X_{\tau_t(\mu+G)_+})), \quad x \in E$$

and

$$\check{R}_\alpha^{(\mu+G)_+}\varphi(x) = E_x^S\left(\int_0^\infty e^{-\alpha \hat{A}_t^{(\mu+G)_+}}\varphi(X_t)d\hat{A}_t^{(\mu+G)_+}\right), \quad x \in E$$

respectively. It follows from Theorem 6.2.1 in [7] that the time changed process $(\check{\mathbf{M}}^S, (\mu+G)_+)$ is a $(\mu+G)_+$-symmetric Hunt process on E with lifetime

$$\check{\zeta} = \hat{A}_\zeta^{(\mu+G)_+}. \tag{4.2}$$

We assume that E equals to the topological support of $(\mu+G)_+$; $E = \text{supp}\,[(\mu+G)_+]$. Set

$$H_E^S u(x) = E_x^S\left(u(X_{\sigma_E^S})\right),$$

where $\sigma_E^S = \inf\{t>0 : X_t \in E\}$. Then the corresponding Dirichlet form $(\check{\mathcal{E}}^S, \check{\mathcal{F}})$ on $L^2(E; (\mu+G)_+)$ of the time changed process $(\check{\mathbf{M}}^S, (\mu+G)_+)$ is represented as follows:

$$\check{\mathcal{E}}^S(\varphi,\varphi) = \mathcal{E}^S(H_E^S u, H_E^S u),$$

for $\varphi \in \check{\mathcal{F}}$, $\varphi = u$ $(\mu+G)_+$-a.e. on E, $u \in \mathcal{F}_e$. Here

$$\check{\mathcal{F}} = \{\varphi \in L^2(E; (\mu+G)_+) : \varphi = u \ (\mu+G)_+\text{-a.e. on } E\}$$

for some u belongs to the extended Dirichlet space $\mathcal{F}_e$ of $(\mathcal{E}^S, \mathcal{F})$ (Theorem 6.2.1 in [7]).

Theorem 4.1 *Assume that $\mu \in \mathcal{S}_\infty(R)$ and $F \in \mathcal{A}_\infty(R)$. Then the time changed process $(\check{\mathbf{M}}^S, (\mu+G)_+)$ satisfies the assumptions (A_1) and (A_2). Moreover, there exists a compact set K such that $\sup_{x\in E} \check{R}_1^{(\mu+G)_+}\chi_{K^c}(x) \le \varepsilon$, for any $\varepsilon > 0$.*

Proof. Let $\check{R}^{(\mu+G)_+}(x,dy)$ be the 0-resolvent kernel of $(\check{\mathbf{M}}^S, (\mu+G)_+)$. Since $\mu^+, \mu_{G^+} \in \mathcal{S}_\infty(R) \subset \mathcal{K}_\infty(R)$, we have for any $\varepsilon_1 > 0$ and some constant $k > 1$

$$\begin{aligned}\check{R}_1^{(\mu+G)_+}\chi_{K^c}(x) &\leqq \int_{K^c}\check{R}^{(\mu+G)_+}(x,dy)\\ &= \int_{K^c}R^S(x,y)(\mu+G)_+(dy)\\ &\leq k\int_{K^c}R(x,y)(\mu+G)_+(dy)\\ &\leq k\varepsilon_1 :=\varepsilon\end{aligned}$$

by Lemma 3.3 and the definition of the time changed process. Now, we note that the rest of the proofs for this theorem is equivalent to show that the time change operator relative to the measure $(\mu+G)_+$ inherits the properties (A_1) and (A_2) of $\mathbf{M}^S$ by virtue of Proposition 3.4.

(A_1): In general, the irreducibility is preserved by time change of strictly increasing positive continuous additive functional. So, the time changed process $(\check{\mathbf{M}}^S, m+(\mu+G)_+)$ by additive functional corresponding to $m+(\mu+G)_+$ is to be irreducible. Now let us consider the time changed process of $(\check{\mathbf{M}}^S, m+(\mu+G)_+)$ by additive functional

$$C_t=\int_0^t r(\check{X}_s)ds, \qquad r=\frac{d(\mu+G)_+}{d(m+(\mu+G)_+)}.$$

Then this time changed process is nothing but $(\check{\mathbf{M}}^S,(\mu+G)_+)$ and for $\varphi\in B_b(E)$,

$$\check{R}_0^{(\mu+G)_+}\varphi(x)=\check{R}_0^{m+(\mu+G)_+}(\varphi r)(x) \tag{4.3}$$

where the right hand side of (4.3) denotes the resolvent of $(\check{\mathbf{M}}^S,m+(\mu+G)_+)$. Suppose that $(\check{\mathbf{M}}^S, (\mu+G)_+)$ is not irreducible. Then by the definition of irreducibility, there exists $B\in B(E_r)$, $E_r=\mathrm{supp}[r]$ with $(\mu+G)_+(B)>0$ such that

$$\check{R}_0^{(\mu+G)_+}\chi_B(x)=0 \qquad (\mu+G)_+\text{-a.e. on } E_r\backslash B.$$

On the other hand, since $m(B)+(\mu+G)_+(B)>0$,

$$\check{R}_0^{(\mu+G)_+}\chi_B(x)=\check{R}_0^{m+(\mu+G)_+}(\chi_B r)(x)>0 \qquad (\mu+G)_+\text{-a.e.}$$

by the irreducibility of $(\check{\mathbf{M}}^S,m+(\mu+G)_+)$. This is a contradiction.

(A_2): For $\varphi\in B_b(E)$ and $\alpha\geq 0$, we have

$$\begin{aligned}&\check{R}_\alpha^{(\mu+G)_+}\varphi(x)\\ &=E_x^S\left(\int_0^\infty e^{-\alpha\hat{A}_t^{(\mu+G)_+}}\varphi(X_t)d\hat{A}_t^{(\mu+G)_+}\right)\\ &=E_x^S\left(\int_0^t e^{-\alpha\hat{A}_s^{(\mu+G)_+}}\varphi(X_s)d\hat{A}_s^{(\mu+G)_+}\right)+p_t^S\left(\check{R}_\alpha^{(\mu+G)_+}\varphi\right)(x).\end{aligned} \tag{4.4}$$

The first term of the right-hand side of (4.4) is dominated up to

$$\|\varphi\|_\infty E_x(S_t^2)^{1/2} E_x\Big((\hat{A}_t^{(\mu+G)_+})^2\Big)^{1/2}$$

which goes to 0 as $t \to 0$ by virtue of Lemma 3.2 (ii) and (iii). Hence we have $\check{R}_\alpha^{(\mu+G)_+}\varphi(x) \in C_b(E)$ because the last term of the right-hand side of (4.4) is in $C_b(E)$ on account of Lemma 3.2 (i). Now, the proofs are completed. □

Theorem 4.2 *Assume that* $\mu \in \mathcal{S}_\infty(R)$ *and* $F \in \mathcal{A}_\infty(R)$. *Then*

$$\sup_{x \in X} E_x(\exp(A_\zeta^{\mu,F})) < \infty$$

if and only if

$$\inf\Big\{ \mathcal{E}(u,v) + \int_X \tilde{u}(x)\tilde{v}(x)(\mu^- + \mu_{G^+})(dx)$$
$$- \int_{X \times X} \tilde{u}(x)\tilde{v}(y)G(x,y)N(x,dy)\mu_H(dx)\Big\} > 1$$

where the inf is calculated on $u \in \mathcal{F}$ *satisfying* $\int_X \tilde{u}(x)^2(d\mu^+ + d\mu_{G^+})(dx) = 1$.

Proof. Note that it follows from Corollary 2.4, Lemma 3.5 and Theorem 4.1 that

$$\sup_{x \in X} E_x^S(\exp(\hat{A}_\zeta^{(\mu+G)_+})) < \infty$$

if and only if

$$\inf\Big\{ \check{\mathcal{E}}^S{}_,(u,u) :\ u \in \check{\mathcal{F}}, \int_E \tilde{u}^2 d(\mu+G)_+ = 1\Big\} > 1. \tag{4.5}$$

By using the same argument of Theorem 2.4 in [13], one can easily check that (4.5) is equivalent with

$$\begin{aligned}
&\inf\Big\{ \mathcal{E}^S(u,u) :\ u \in \mathcal{F}, \int_X \tilde{u}^2 d(\mu+G)_+ = 1\Big\} \\
&= \inf\Big\{ \mathcal{E}(u,u) + \int_X \tilde{u}^2(x)(\mu^- + \mu_{G^+})(dx) \\
&\quad - \int_{X \times X} \tilde{u}(x)\tilde{u}(y)G(x,y)N(x,dy)\mu_H(dx) :\ \tilde{u} \in \mathcal{F}, \int_X \tilde{u}^2 d(\mu+G)_+ = 1\Big\} > 1.
\end{aligned} \tag{4.6}$$

On the other hand, let $\mathbf{M}^Q = (\Omega, X_t, P_x^Q, \zeta)$ be the transformed process of $\mathbf{M}$ by Q_t in (3.3) and E^Q be the expectation related to $\mathbf{M}^Q$. Since for any $t \geq 0$

$$S_t = Q_t \exp\left(-\bar{A}_t^{(\mu+G)_-}\right),$$

$\mathbf{M}^Q$ is the subprocess of $\mathbf{M}^S$ killed at rate $(\mu+G)_-$. So, the same argument of Theorem 2.10 in [5] implies that

$$E_x^S\left(\exp\left(\bar{A}_\zeta^{(\mu+G)_+}\right)\right)<\infty \Longleftrightarrow E_x^Q\left(\exp\left(\bar{A}_\zeta^{(\mu+G)}\right)\right)<\infty. \tag{4.7}$$

Moreover, for each fixed $k>0$, we have

$$\begin{aligned} &E_x^Q\left(k\wedge\exp\left(\bar{A}_\zeta^{(\mu+G)}\right)\right)\\ &=E_x\left(\int_0^\zeta\left[k\wedge\exp\left(\bar{A}_{s-}^{(\mu+G)}\right)\right](-dQ_s)\right)+E_x\left(k\wedge Q_\zeta\exp\left(\bar{A}_\zeta^{(\mu+G)}\right)\right)\end{aligned} \tag{4.8}$$

by virtue of 62 in [11]. The first term of the right hand side of (4.8) is equal to

$$-E_x\left(\int_0^\zeta\left[k\wedge\exp\left(A_{s-}^{\mu,F}\right)\right]d\left(A_t^G-\bar{A}_t^G\right)\right)$$

which is zero for each fixed $k>0$. Since for any $t\geq 0$,

$$Q_t\exp\left(\bar{A}_t^{(\mu+G)}\right)=\exp\left(A_t^{\mu,F}\right),$$

we have

$$E_x^Q\left(\exp\left(\bar{A}_\zeta^{(\mu+G)}\right)\right)=E_x\left(\exp\left(A_\zeta^{\mu,F}\right)\right) \tag{4.9}$$

by passing $k\to\infty$. Now, combining (4.7) and (4.9) with (4.6), we reach the conclusion.

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