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Positive definite state extensions in transformation group C*-algebras

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1. Introduction

In this note, we discuss the state extensions in transformation group C*-algebras by means of positive definite functions. We say these extensions to be positive definite state extensions.

Let A be the C*-algebra $C(X)$ of continuous functions on a compact Hausdorff space X and σ the action of a discrete group G on A induced by an action σ of G on X as homeomorphisms. We denote by $\tilde{A}$ the crossed product C*-algebra $A \rtimes_\sigma G$ of A by G under σ. The C*-algebra $\tilde{A}$ is generated by A, δ_s $(s \in G)$, where δ_s is a function on G satisfying $\delta_s(s)=1$, $\delta_s(t)=0(t \neq s)$. If τ is an automorphism of A which commutes with σ, it induces an automorphism $\tilde{\tau}$ of $\tilde{A}$ such that $\tilde{\tau}=\tau$ on A and $\tilde{\tau}=\mathrm{id}$ on G. If ϕ is a positive definite function on G with $\phi(e)=1$ and μ a σ-invariant state on A, then $\varphi=\sum_s \oplus \phi(s)\mu$ is a state on $\tilde{A}$ ([4], Prop.3.3.6). It is evident that if μ is τ-invariant, then the state φ is $\tilde{\tau}$-invariant. And also when τ is an identity automorphism and μ is the Dirac measure $\delta_x(x \in X)$, then it is known that there exists a bijective correspondence between the set of state extensions of δ_x on $\tilde{A}$ and that of positive definite functions ϕ on G_x with $\phi(e)=1$. The correspondence: $\varphi \leftrightarrow \phi$ is given by $\varphi=\sum_s \oplus \phi(s)\delta_x$ with $\phi(s)=0$ for $s \notin G_x$, where G_x is the isotropy group $\{s \in G | \sigma_s x=x\}$([4], Th.3.3.7).

Now we introduce the notion of a positive definiteness of a function ϕ on G to discuss the positive definite state extensions, which is somewhat stronger than the usual one. We say that a function ϕ on G is positive definite with respect to σ under μ if it satisfies

$$\sum_{i,j} \phi(s_i^{-1}s_j)\mu(\sigma_{s_i^{-1}}(a_i^* a_j)) \geq 0$$

for any finite sets $\{s_i\}_{i \in F}$ in G and $\{a_i\}_{i \in F}$ in A. If a function ϕ on G is positive definite with respect to σ under μ, then it is the usual positive definite function on G and conversely if a positive definite function ϕ on G is supported in G_μ, then ϕ is positive definite with respect to σ under μ. From them follow that if μ is σ-invariant, then a positive definite function on G is positive definite with respect to σ under μ and that a positive definite function on G_x is positive definite with respect to σ under δ_x. We show that if a function ϕ on G with $\phi(e)=1$ is positive definite with respect to σ under μ and is supported in the subgroup $G_\mu=\{s \in G|\mu\circ\sigma_s=\mu\}$, then $\varphi=\sum_s\oplus\phi(s)\mu$ is a state on $\tilde{A}$ (Prop. 2) and that if μ is τ-ergodic, then the $\tilde{\tau}$-invariant state extension of μ is only of this form (Prop. 4). We say an invariant state on a C*-algebra to be ergodic if it is an extreme point in the invariant states on the C*-algebra. Thus we have that there is a bijective correspondence between the set of $\tilde{\tau}$-invariant state extensions of μ on $\tilde{A}$ and that of positive definite functions ϕ on G with $\phi(e)=1$, which is supported in G_μ (Th.4). If τ is an identity automorphism and μ is the Dirac measure δ_x, then from this theorem is implied that there is a bijective correspondence between the set of state extensions of δ_x on $\tilde{A}$ and that of positive definite functions ϕ on G_x with $\phi(e)=1$ ([4], Th.3.3.7).

2. Positive definite state extensions in transformation group C*-algebras

A state φ on $\tilde{A}$ defines a function Φ on G to A^*, the dual space of A, by

$$<a, \Phi(s)>=<a\delta_s, \varphi>, \ (a\in A, s\in G)$$

([4]). If $\mu=\Phi(e)$, then $\mu=\varphi|_A$ and the state φ is an extension of the state μ on A. The state φ on $\tilde{A}$ gives a distribution of a bounded family $\{\mu_s\}$ of linear functionals on A which is denoted by $\varphi=\sum_s\oplus\mu_s$, where $\mu_s=\Phi(s)$. On the other hand, if μ is a σ-invariant state on A, then it is known the following way to consruct a state φ on $\tilde{A}$.

Proposition 1. ([4], Prop.3.3.6) *If ϕ is a positive definite function on G with $\phi(e)=1$ and μ is a σ-invariant state on A, then the functional $\varphi=\sum_s\oplus\phi(s)\mu$ is a state on $\tilde{A}$.*

This method is not available for a state on A which is not necessarily σ-invariant. But it is also known that the state extension φ on $\tilde{A}$ of the Dirac measure δ_x is given by $\varphi=\sum_s\oplus\phi(s)\sigma_x$, where ϕ is a positive definite function on the isotropy group G_x([4], Th.3.3.7). We note that if μ is a σ-invariant state on A, then a positive definite function on G is positive definite with respect to σ under μ and that a positive definite function ϕ on the isotropy group G_x is positive definite with respect to σ under δ_x. Thus these ways of state extensions by means of positive definite functions are generalized as the following

Proposition 2. *Let A, G, σ, $\tilde{A}$ be as above, and μ be a state on A. If a function ϕ on G is positive definite with respect to σ under μ and satisfies $\phi(e)=1$, then the functional $\varphi=\sum_s\oplus\phi(s)\mu$ is a state on $\tilde{A}$.*

Proof. Let $\varphi=\sum_s\oplus\phi(s)\mu$, where ϕ is a function on G with $\phi(e)=1$, which is positive definite with respect to σ under μ. It is evident that φ extends to a bounded linear functional on $\ell^1(G, A)$. Moreover, for a finite sum $x=\Sigma_i a_i\delta_{s_i}$ in $\tilde{A}$, we have

$$\begin{aligned}\varphi(x^*x)&=\varphi\Big(\sum_{i,j}\sigma_{s_i^{-1}}(a_i^*a_j)\delta_{s_i^{-1}s_j}\Big)\\&=\sum_{i,j}\phi(s_i^{-1}s_j)\mu(\sigma_{s_i^{-1}}(a_i^*a_j))\\&\geq 0.\end{aligned}$$

Since φ is positive on the dense $*$-subalgebra of $\tilde{A}$ which consists of all finite sums $\Sigma_i a_i\delta_{s_i}$, where $a_i\in A$ and $s_i\in G$, φ is extended to a state on $\tilde{A}$.

This completes the proof.

Proposition 3. *Let A, G, σ be as above, and μ be a state on A. If a positive definite function ϕ on G is supported in the subgroup $G_\mu=\{s\in G: \mu\circ\sigma_s=\mu\}$, then it is positive definite with respect to σ under μ.*

Proof. Let $S=\{s_i\}_{i\in F}$ be a finite set of G and $\{a_i\}_{i\in F}$ be a finite set of A. We have $S=\bigcup_{k=1}^m S_k$, where S_k is the set of elements in S belonging in the same coset of G_μ. If $S_k=\{s_{k1}, s_{k2},\dots,s_{kl_k}\}$, then there are an s in G and $t_i\in G_\mu$ such that $s_{ki}=st_i$ for $i=1,2,\dots,l_k$. For each k, we have

$$\begin{aligned}\sum_{i,j}\phi(s_{ki}^{-1}s_{kj})\mu(\sigma_{s_{ki}^{-1}}(a_{ki}^*a_{kj}))&=\sum_{i,j}\phi(t_i^{-1}t_j)\mu(\sigma_{s^{-1}}(a_{ki}^*a_{kj}))\\&=\mu\circ\sigma_{s^{-1}}\Big(\sum_{i,j}\phi(t_i^{-1}t_j)a_{ki}^*a_{kj}\Big)\\&\geq 0\end{aligned}$$

In fact, since ϕ is positive definite on G_μ, we have that for every x in X,

$$\Big(\sum_{i,j}\phi(t_i^{-1}t_j)a_{ki}^*a_{kj}\Big)(x)=\sum_{i,j}\phi(t_i^{-1}t_j)\overline{a_{ki}(x)}a_{kj}(x)\geq 0.$$

Hence $\Sigma_{i,j}\phi(t_i^{-1}t_j)a_{ki}^*a_{kj}$ is a positive element of $A=C(X)$. On the other hand, if s_i and s_j are not in the same set S_k, then $s_i^{-1}s_j$ is not in G_μ and $\phi(s_i^{-1}s_j)=0$. Therefore we have

$$\sum_{i,j}\phi(s_i^{-1}s_j)\mu(\sigma_{s_i^{-1}}(a_i^*a_j))=\sum_{k=1}^{m}\sum_{i,j}\phi(s_{ki}^{-1}s_{kj})\mu(\sigma_{s_{ki}^{-1}}(a_{ki}^*a_{kj}))\geq 0.$$

This completes the proof.

Now let τ be an automorphism of A which commutes with the action σ of G on A and $\tilde{\tau}$ the automorphism of $\tilde{A}$ induced by τ. We have that if μ is τ-ergodic, then the $\tilde{\tau}$-invariant state extension φ on $\tilde{A}$ of μ is given by $\varphi=\sum_s\oplus\phi(s)\mu$, where ϕ is a function on G which is positive definite with respect to σ under μ and is supported in G_μ.

Proposition 4. *Let A, G, σ, τ, $\tilde{A}$, $\tilde{\tau}$, be as above. Let μ be a τ-ergodic state on A. Suppose that μ is extended to a $\tilde{\tau}$-invariant state φ on $\tilde{A}$: $\varphi=\sum_s\oplus\mu_s$. Then there is a function ϕ on G such that $\mu_s=\phi(s)\mu$, which is positive definite with respect to σ under μ and is supported in the subgroup G_μ.*

Proof. Since φ is an extension of μ, it follows from Prop.3.3.4 [4] that each μ_s is absolutely continuous with respect to $\mu_e=\mu$. Then the Radon-Nikodym derivative $\mathrm{d}\mu_s/\mathrm{d}\mu(x)$ is τ-invariant. The ergodicity of μ implies that $\mathrm{d}\mu_s/\mathrm{d}\mu(x)$ is a constant $\phi(s)$ for each $s\in G$. Thus we have $\mu_s=\phi(s)\mu$. We show that $\phi(s)$ is positive definite with respect to σ under μ. Take finite sets $\{s_i\}_{i\in F}$ in G and $\{a_i\}_{i\in F}$ in A. Then we have

$$\begin{aligned}0&\leq\varphi\Big(\Big(\sum_i a_i\delta_{s_i}\Big)^*\Big(\sum_i a_i\delta_{s_i}\Big)\Big)\\&=\varphi\Big(\sum_{i,j}\sigma_{s_i^{-1}}(a_i^*a_j)\delta_{s_i^{-1}s_j}\Big)\\&=\sum_{i,j}\mu_{s_i^{-1}s_j}(\sigma_{s_i^{-1}}(a_i^*a_j))\\&=\sum_{i,j}\phi(s_i^{-1}s_j)\mu(\sigma_{s_i^{-1}}(a_i^*a_j)).\end{aligned}$$

Therefore ϕ is positive definite with respect to σ under μ.

Secondly we show that $\phi(s)$ is supported in the subgroup G_μ. In fact, since the state φ is an extension of μ, it also follows from Prop.3.3.4 [4] that μ_s is absolutely continuous with respect to $\mu_e\circ\sigma_{s^{-1}}=\mu\circ\sigma_{s^{-1}}$. Then the Radon-Nikodym derivative $\mathrm{d}\mu_s/\mathrm{d}\mu\circ\sigma_{s^{-1}}(x)=\phi(s)\mathrm{d}\mu/\mathrm{d}\mu\circ\sigma_{s^{-1}}(x)$ is τ-invariant and thus the ergodicity of $\mu\circ\sigma_{s^{-1}}$ implies $\phi(s)\mathrm{d}\mu/\mathrm{d}\mu\circ\sigma_{s^{-1}}(x)$ is a constant for each $s\in G$. Hence if $\phi(s)\neq0$, then $\mathrm{d}\mu/\mathrm{d}\mu\circ\sigma_{s^{-1}}(x)$ is a constant c. Since

$$c=\int c\,\mathrm{d}\mu\circ\sigma_{s^{-1}}(x)=\int\frac{\mathrm{d}\mu}{\mathrm{d}\mu\circ\sigma_{s^{-1}}}(x)\mathrm{d}\mu\circ\sigma_{s^{-1}}(x)=\int\mathrm{d}\mu(x)=1,$$

we have that if $\phi(s)\neq0$, then $\mu=\mu\circ\sigma_{s^{-1}}$. From it follows that $\phi(s)=0$ for $s\notin G_\mu$.

This completes the proof.

Thus we have the following

Theorem 5. *Let* A, G, σ, τ, $\tilde{A}$, $\tilde{\tau}$, *be as above. Let* μ *be a* τ*-ergodic state on* A. *Then there is a bijective correspondence between the set of* $\tilde{\tau}$*-invariant state extensions* φ *of* μ *on* $\tilde{A}$ *and that of positive definite functions* ϕ *on* G *with* $\phi(e)=1$, *supported in the subgroup* G_μ. *The correspondence*: $\varphi \leftrightarrow \phi$ *is given by* $\varphi=\sum_s \oplus \phi(s)\mu$.

Proof. Suppose that φ is a $\tilde{\tau}$-invariant state extension of μ. By Proposition 4, there is a function ϕ on G such that $\mu_s=\phi(s)\mu$, which is positive definite with respect to σ under μ and supported in G_μ. It is enough to show the mapping is bijective. For this, let $\varphi=\sum_s \oplus \phi(s)\mu$, where ϕ is a function on G with $\phi(e)=1$, which is positive definite with respect to σ under μ and is supported in G_μ. It is evident that φ is $\tilde{\tau}$-invariant. By Proposition 2, φ is a state on $\tilde{A}$ and thus we have a $\tilde{\tau}$-invariant state extension φ on $\tilde{A}$ of μ. Injectivity follows from the uniqueness of the decomposition $\varphi=\sum_s \oplus \mu_s$([4]).

This completes the proof.

Corollary 6. *Let* A, G, σ, τ, $\tilde{A}$, $\tilde{\tau}$, *be as above. The* $\tilde{\tau}$*-ergodic state extension of a* τ*-ergodic state* μ *is unique if and only if* G_μ *is trivial.*

Since any positive definite function ϕ on G_μ taking value 1 at 0 is nothing but a state on the group C*-algebra $C^*(G_\mu)$, we have

Corollary 7. *In the correspondence as the above theorem, the* $\tilde{\tau}$*-invariant state* φ *is* $\tilde{\tau}$*-ergodic if and only if* ϕ *is a pure state on* $C^*(G_\mu)$.

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