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Gauss-type curvatures and tubes for polyhedral surfaces*

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Abstract

In this article we consider analogies for polyhedral surfaces in $\mathbf{R}^3$ with respect to Theorema Egregium and Weyl's volume formula of tube in classical surface theory.

Theorema Egregium of Gauss is a well known, famous theorem in the classical surface theory. Roughly speaking, it says that the Gauss curvature defined extrinsicly coincides with some intrinsic value. An extrinsic definition of Gauss curvature for polytopes was given in [4] by D. Hilbert and S. Cohn-Vossen. In the case of the pölytopes, A. D. Alexandrov discussed an analogous theorem in [1] and [2]. In the case of general polyhedra T. Banchoff discussed it in [3], but his definition of extrinsic curvature is different from the original one in [4].

In this article we first define the extrinsic curvature of (not necessarily convex) polyhedra, following the direction indicated in [4], and prove Theorema Egregium for polyhedra in $\mathbf{R}^3$. Next we consider an analogy of Weyl's volume formula for polyhedra in $\mathbf{R}^3$ by using another type of curvature. The general dimensional case is treated in [5] and [6].

A polyhedral surface P is a PL 2-dimensional manifold, which we assume triangulated by Euclidean simplices σ^d, such that P is locally embedded in $\mathbf{R}^3$ around for any interior vertex v. We say an interior vertex v has *property* (*), if there is a unit vector n such that for all faces f_i of P incident to v, the normal vectors n_i on the same side of P satisfy $\angle(n_i,n)<\frac{\pi}{2}$.

We define the intrinsic curvature $K^i(v)$ at a vertex v of P as follows:

$$K^i(v):=\sum_{d=0}^{2}(-1)^d\sum_{\sigma^d\ni v}\beta^d(v,\sigma^d),$$

where $\beta^d(v,\sigma^d)$ denotes the outer angle at v of the d-simplex σ^d. The outer angle is defined

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by ($\pi-$inner angle of σ^d)$/2\pi$ if $d=2$, formally defined by $\frac{1}{2}$ if $d=1$ and by 1 if $d=0$. It holds that $K^i(v)=(2\pi-\Sigma_j\alpha_j)/2\pi$, where α_j denotes the inner angle at v of the face f_j incident to v.

Next we define an extrinsic curvature for a polyhedral surface P. For each vertex $v\in P$, denote the faces around v by $\{f_i\}$ $(i=1,\cdots,k)$ and their normal vectors on the same side of P by $\{n_i\}$, where the indices are numbered in the anti-clockwise order looking from the side of the normal direction. Let $c_{j,j+1}$ be the shortest arc from n_j to n_{j+1} on the unit sphere $S^2(1)$. We define the closed broken geodesic arc c by $c:=c_{1,2}\cup c_{2,3}\cup\cdots\cup c_{k,1}$ and define $K^e(v,n)$ by the signed area of the domain enclosed by c with respect to the base point n. Denote the oriented spherical triangle with vertices n, n_j, $n_{j+1}(n\to n_j\to n_{j+1}\to n)$ by $T(j)$. Then $K^e(v,n)$ is given by

$$K^e(v,n):=\frac{1}{2\pi}\sum_j(-1)^{\delta(j)}Area(T(j)),$$

where $\delta(j)$ is 1 if the orientation $n\to n_j\to n_{j+1}\to n$ of the spherical triangle $T(j)$ is clockwise, and 0 otherwise. If a polyhedron P has property(*), then the value of $K^e(v,n)$ clearly does not depend on the choice of n; in this case we write it $K^e(v)$ and call the extrinsic curvature of v. Now we get the following theorem.

Theorem 1 (Theorema Egregium for polyhedra). *For any polyhedral surface P in $\mathbf{R}^3$ and an interior vertex v of P with property* (*) *it holds that $K^i(v)=K^e(v)$.*

Remark. (i) The above definition of $K^e(v)$ seems complicated, but it is natural at least when we consider the example of the vertex v with faces around it, of normal vectors $n_1=(b,-a,c)$, $n_2=(a,-b,c)$, $n_3=(a,b,c)$, $n_4=(b,a,c)$, $n_5=(-b,a,c)$, $n_6=(-a,b,c)$, $n_7=(-a,-b,c)$, $n_8=(-b,-a,c)$ $(0<a<c<b<1)$.
(ii) There is a polyhedron P and a vertex $v\in P$ such that for any vector $n\in S^2(1)$ it holds $K(v)\neq K^e(v,n)$ [5]. This shows the necessity of the additional property(*).

Lemma. *If a vertex v of a polyhedral surface P has 3 faces around it, then $K^i(v)=K^e(v)$.*

Proof of Lemma. Let us assume first that P is convex around v. Let f_i be the faces around v, n_i the unit nomal vector of f_i and α_i the inner angle of f_i at v $(i=1,2,3)$. Let θ_i be the inner angle at n_i of the spherical triangle $\Delta(n_1,n_2,n_3)$ on $S^2(1)$. Lemma follows easily from the area formula $Area(\Delta)=\Sigma_i^3\theta_i-\pi$ and $\theta_i=\pi-\alpha_i=2\pi\beta_i$. In fact, it holds that

$$K^e(v):=\frac{1}{2\pi}(\sum_{i=0}^{3}\theta_i-\pi)=\frac{1}{2\pi}(\sum_{i=1}^{3}2\pi\beta_i-\pi)=1-\frac{3}{2}+\sum_{i=1}^{3}\beta_i=K^i(v).$$

Next let's assume that P is not convex around v. We may assume, without loss of generality, that the inner angle α_3 of f_3 is larger than π. Define the spherical triangle $\Delta(n_1,n_2,n_3)$ and θ_i as above. Note that in this case the orientation $n_1\rightarrow n_2\rightarrow n_3\rightarrow n_1$ is clockwise and $\alpha_1=\theta_1, \alpha_2=\theta_2,;\ \alpha_3-\pi=\theta_3$. From these considerations the conclusion follows.

Proof of Theorem 1. Let $f_1,\cdots,f_k$ be the faces around an interior vertex v. From the property (*) we can take $n\in S^2(1)$ and two planes $pl^{\pm}$ which are orthogonal to n and go through the points $v\pm\varepsilon n$ respectively, such that there are not vertices of f_i between pl^+ and pl^-. Here, ε is a positive number, conveniently small. Let hs^+ (resp. hs^-) be the half space with boundary pl^+ (resp. pl^-) not containing v and f^+_{pl} (resp. f^-_{pl}) be the connected component of $pl^+\backslash P$ (resp. $pl^-\backslash P$) containing $v+\varepsilon n$ (resp. $v-\varepsilon n$). Let w_i be the point of $f_i\cap f_{i+1}$ in pl^+, or pl^-, according to the case. Consider the polyhedra $(P\cap hs^+)\cup f^+_{pl}$ and $(P\cap hs^-)\cup f^-_{pl}$ with vertices $w_1,\cdots,w_n$. Note that $K^e(w_j)=(-1)^{s(j)}Area(T(j))$, so $K^e(v)=\Sigma_j K^e(w_j)$. We can easily check that $\beta(v,f_i)=\beta(w_i, f_i\cap hs^{\pm})+\beta(w_{i-1}, f_i\cap hs^{\pm})$ if f_i does not intersect both hs^+ and hs^-, and $\beta(v,f_i)=\beta(w_i, f_i\cap hs^{\pm})+\beta(w_{i-1},f_i\cap hs^{\pm})+\frac{1}{2}$ if f_i intersects both hs^+ and hs^-. The number of faces which intersect both hs^+ and hs^- is even and denoted by $2m$. Note that the number of the connected components of $P\cap hs^+\cap L(v)$ is $2m$, where $L(v)$ denotes the link of P at v. We can easily get that $\Sigma_j\beta(w_j,pl^{\pm})=-m+1$. Then it holds that

$$\sum_j K^i(w_j)=\sum_j\left(1-\frac{3}{2}+\beta(w_j,f_j)+\beta(w_j,f_{j+1})+\beta(w_j,pl^{\pm})\right)=-\frac{n}{2}+\sum_j\beta(v,f_j)+1.$$

Therefore we get Theorem 1 from previous Lemma.

The Weyl's volume formula of tube for a closed surface M in $\mathbf{R}^3$ [7] says that for any positive number r small enough, it holds $VolN_r(M)=2rAreaM+\frac{2r^3}{3}\int_M Kdv$, where $N_r(M)$ denotes the r neighborhood of M and K denotes Gauss curvature on M. Let P be a polyhedral surface in $\mathbf{R}^3$. We will consider the problem: What is the volume formula for polyhedra P? The difficulty is to define $\tilde{N}_r(P)$ and $\tilde{K}(v)$ such that $Vol\ \tilde{N}_r(M)=2rAreaM+\frac{2r^3}{3}\Sigma_{v\in P}\tilde{K}(v)$.

Let us define the curvature $\tilde{K}(v)$ at a vertex v of P with three faces f_j around it. Denote by n_j the unit normal direction of f_j $(j=1,2,3)$ on the same side of P. Let T_j be the tangent plane of $S^2(1)$ at n_j and T_j^+ be the closed half space with boundary T_j and containig $S^2(1)$. The indices are always numbered in anti-clockwise order, looking from the side of normal directions. Put $T^+:=\cap_j T_j^+$. Let H_j be the plane through the origin containing two normal vectors different to n_j, and H_j^+ (resp. H_j^-) be the half space with boundary H_j and not containing (resp. containing) n_j. Put $q:=\cap^3_{j=1}T_j$. Define the unit vector n by $n=q\|q\|^{-1}$. Let L_j be the plane trough n, n_j and the origin and D_j be the domain bounded by L_{j+1}, L_{j+2} and containing the great arc $n_{j+1}n_{j+2}$. We define $\tilde{K}(v)$ by

$$\tilde{K}(v):=\sum_{j=1}^{3}(-1)^{\delta(j)}Area\{(T_j\cup T_{j+1})\cap(T^+\cap H_{j+2}^+\cap D_{j+2})\},$$

where $\delta(j)$ is 1 if the orientation $n\to n_j\to n_{j+1}\to n$ of spherical triangle nn_jn_{j+1} is clockwise, and 0 otherwise.

Let P be an oriented polyhedral surface in $\mathbf{R}^3$. We consider a neighborhood $\tilde{N}_r(P)$ of P which is given by the parallel translation of each face with constant speed. It is exactly defined as follows. Let f be a face of P and n_f be the unit normal direction of f. Let pl_f be the plane containing the face f. Define a parallel plane by $pl_f(t):=pl_f+tn_f$. Let $\{f'\}$ be the set of faces with $f\cap f'\neq\emptyset$. Let $hs_{f'}(t)$ be the half space with boundary $pl_{f'}(t)$ such that $hs_{f'}\supset f'$ if $f\cap pl_{f'}(t)=\emptyset$, $hs_{f'}\not\supset f'$ if $f\cap pl_{f'}(t)\neq\emptyset$. Put $\varphi_f(t):=\bigcap_{\{f'\}}(pl_f(t)\cap hs_{f'}(t))$, and $P(t):=\bigcup_{f\in F}\varphi_f(t)$, where F denotes the set of all faces of P. Define $\tilde{N}_r(P)$ by $\tilde{N}_r(P):=\bigcup_{-r\le t\le r}P(t)$. Now we get the following theorem.

Theorem 2 (Weyl type volume formula for polyhedra). *Let P be a 2-dimensional closed orientable polyhedral surface in $\mathbf{R}^3$ such that each vertex has just three faces around it. For any positive number r small enough, it holds that*

$$Vol\tilde{N}_r(M)=2rAreaM+\frac{2r^3}{3}\sum_{v\in P}\tilde{K}(v).$$

Remark. (i) The above definition of $\tilde{K}(v)$ seems complicated, but it is natural at least when we consider the example of the vertex v with faces around it, of normal vectors $n_1=(a,0,b)$, $n_2=(0,b,a)$, $n_3=(0,-b,a)$ $(0<a\ll b<1)$.
(ii) If $q\in\bigcap_{j=1}^3H_j^+$, then $\tilde{K}(v)=Area\{(\bigcup_{j=1}^3T_j)\cap T^+\cap(\bigcap_{j=1}^3H_j^+)\}$. If P is convex around v, then $K^e(v)=Area(S^2\cap(\bigcap_{j=1}^3H_j^+))$. Hence it seems that $\tilde{K}(v)$ has some relation to $K^e(v)$ and the usual Gauss curvature.
(iii) The above formula does not hold without the assumption that any vertex have precisely 3 faces around it.

Proof of Theorem 2. We use the notations and the constructions considered for the definition of $\tilde{N}_r(P)$. Let n_f be the unit normal vector of the face f on a side and $\bar{n}_f$ the one on the other side of f. Let Σ_f (resp. $\bar{\Sigma}_f$) be the direct product set $f\times s_f$ (resp. $f\times\bar{s}_f$) where s_f (resp. $\bar{s}_f$) denotes the line-segment rn_f (resp. $r\bar{n}_f$). Put $N_f:=\bigcup_{0\le t\le r}\varphi_f(t)$, $\bar{N}_f:=\bigcup_{-r\le t\le 0}\varphi_f(t)$ and $e(r):=pl_f(r)\cap pl_{f'}(r)$. Let H_e (resp. $H_{v,e}$) be the plane orthogonal to e and through the midpoint

m_e of e (resp. through v), and $H^+_{v,e}$ be the half space with $\partial H^+_{v,e}=H^+_{v,e}$ and not containing e. For each edge e around f, take a triangle $t_{e,f}(r)\subset H_e$ with vertices m_e, m_e+rn_f, a, where a denotes the nearest point on $e(r)$ to m_e. Put $v(r):=\bigcap_{f\ni v}\varphi_f(r)$. Take the tetrahedron $T_{v,e,f}(r)$ such that the triangle $t_{v,e,f}(r)$ on $pl_f(r)$ with vertices $v+rn_f, b, v(r)$ is its base and the fourth vertex coincides with v, where b denotes the nearest point on $e(r)$ to v. Note that when v_1, v_2 are two vertices of e, we can assume that $T_{v_1,e,f}(r)\cap T_{v_2,e,f}(r)=\emptyset$, because r is small enough. For any face f it holds that

$$N_f=(\Sigma_f \bigsqcup_{e\in f}(t_{e,f}(r)\times e))\bigsqcup_{v\in f} T_{v,e,f}(r),$$

where the first symbol $\sqcup$ denotes either the union if $f\cap pl_{f'}(t)=\emptyset$, or the setminus if $f\cap pl_{f'}(t)\neq\emptyset$, and the second symbol $\sqcup$ denotes either the union if $f\cap pl_{f'}(t)=\emptyset$ and $v(r)\in H^+_{v,e}$, or $f\cap pl_{f'}(t)\neq\emptyset$ and $v(r)\notin H^+_{v,e}$, or denotes the setminus otherwise. We obtain the same relation for $\bar{N}_f$ by using $-r$ instead of r. Indeed, if it holds that $f\cap pl_{f'}(t)=\emptyset$ and $v(r)\in H^+_{v,e}$ at $v\in f$, then that $f\cap pl_{f'}(t)\neq\emptyset$ and $v(-r)\notin H^+_{v,e}$ on the other side. Note that the volume of disjoint union of Σ_f and $\bar{\Sigma}_f$ is equal to $2rArea(f)$. Then we get from the above considerations that

$$Vol(N_f\cup\bar{N}_f)=2rArea(f)+\sum_{(v,e)\in f}(-1)^{\Delta(v,e,f)}\frac{1}{3}rArea(t_{v,e,f}(r)\cup t_{v,e,f}(-r)),$$

where $\Delta(v,e,f)=0$ if $\sqcup$ denotes the union, $\Delta(v,e,f)=1$ if $\sqcup$ denotes the setminus. From the definition of $\tilde{K}(v)$ it holds that

$$\tilde{K}(v)=\sum_{(e,f)\ni v}(-1)^{\Delta(v,e,f)}\frac{2r}{3}Area(t_{v,e,f}(r))$$

Hence it follows the conclusion of Theorem 2.

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