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Willmore surfaces in S^3 and minimal surfaces in S_1^4

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Abstract

We shall see that for a space-like minimal surface $\tilde{S}$ in the 4-dimensional de Sitter space S_1^4 such that the shape operator with respect to some light-like normal vector field is nowhere zero, there exists a Willmore surface S in S^3 without any umbilical point such that the image of S by the conformal Gauss map is congruent with $\tilde{S}$ in S_1^4. In addition, we shall show that a Willmore surface S in S^3 without any umbilical point is the inverse image of a minimal surface in $\mathbf{R}^3$ by a stereographic projection if and only if the image of S by the conformal Gauss map is a space-like minimal surface in S_1^4 such that the curvature with respect to the metric induced by the conformal Gauss map is identically equal to one.

1. Introduction

Let M be an oriented smooth two-dimensional manifold. Then for each immersion $\iota: M \to S^3$ of M into S^3, we set

$$W(\iota) := \int_M (H^2+1)dA,$$

where H is the mean curvature of M with respect to ι and dA is the area element of M with respect to the metric g induced by ι. We call W the *Willmore functional*. An immersion ι is called *Willmore* if the first variation of W for ι is zero, where we consider variations of ι with compact support. If ι is Willmore, then the image $\iota(M)$ is called a *Willmore surface*. The Euler-Lagrange equation for Willmore immersions is given by the following ([7]):

$$\Delta H + 2(H^2 - K + 1)H = 0, \tag{1}$$

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where Δ and K are the Laplacian and the curvature of M with respect to g, respectively. From (1), we see that a minimal surface in S^3 is Willmore. Referring to [7], we see that the inverse image of a minimal surface in $\mathbf{R}^3$ by a stereographic projection is Willmore.

Suppose that M has no umbilical point with respect to an immersion ι. In [3], Bryant pointed out that ι is Willmore if and only if the conformal Gauss map γ_ι of ι is a space-like minimal immersion of M into the 4-dimensional de Sitter space S_1^4. In the present paper, we shall see that for a space-like minimal immersion γ of M into S_1^4 such that the shape operator of M with respect to some light-like normal vector field is nowhere zero, there exists a Willmore immersion ι of M into S^3 such that the conformal Gauss map of ι is equal to γ or $-\gamma$. In addition, we shall show that for a Willmore immersion $\iota: M \to S^3$, $\iota(M)$ is the inverse image of a minimal surface in $\mathbf{R}^3$ by a stereographic projection if and only if the conformal Gauss map γ_ι of ι is a space-like minimal immersion such that the curvature of M with respect to the metric induced by γ_ι is identically equal to one.

Remark. In [3], Bryant showed that if M is homeomorphic to S^2, then the image of M by a Willmore immersion is the compactification of the inverse image by a stereographic projection of a complete minimal surface in $\mathbf{R}^3$ with finite total curvature and with embedded, flat ends.

2. The conformal Gauss map

In [3], Bryant introduced and studied the conformal Gauss map of an immersion $\iota: M \to S^3$. The purpose of this section is to describe properties of the conformal Gauss map.

2.1. Preliminaries

Let $\langle\ ,\ \rangle$ be the Lorentzian inner product on $\mathbf{R}^5$, i. e., an indefinite inner product on $\mathbf{R}^5$ defined by

$$\langle x,y\rangle := -x^{(0)}y^{(0)} + x^{(1)}y^{(1)} + x^{(2)}y^{(2)} + x^{(3)}y^{(3)} + x^{(4)}y^{(4)}$$

for two vectors $x := (x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)})$, $y := (y^{(0)}, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)})$ of $\mathbf{R}^5$. We say that a vector $x \in \mathbf{R}^5$ is *space-like* if $\langle x,x\rangle > 0$, *time-like* if $\langle x,x\rangle < 0$, and *light-like* or *null* if x is not zero but satisfies $\langle x,x\rangle = 0$. A time-like or light-like vector $x \in \mathbf{R}^5$ is said to be *future directed* if $x^{(0)} > 0$. Let L^+ be the set of the future directed light-like vectors of $\mathbf{R}^5$. For two vectors $x,y \in L^+$, we write $x \sim y$ if x and y are linearly dependent. We see that $\sim$ is an equivalence relation in L^+ and that $L^+/\sim$ may be identified with the 3-dimensional sphere. In addition, a subset $\{x \in L^+ ;\ x^{(0)} = 1\}$ of L^+, which is a set of representatives of the equivalence classes, may be identified with the 3-dimensional unit sphere S^3, which is the 3-dimensional sphere equipped with the standard metric.

Let G be the group of the linear transformations of $\mathbf{R}^5$ which preserve the Lorentzian inner product $\langle\ ,\ \rangle$, the orientation of $\mathbf{R}^5$ and the future direction. Then each element X of G induces a conformal transformation t_X of S^3 which preserves the orientation of S^3: if for $y \in S^3$, $t_X(y)$ denotes a unique intersection of $L^+ \cap \{x^{(0)}=1\}$ with a line through the origin of $\mathbf{R}^5$ determined by the light-like vector $X(y)$, then t_X is a conformal transformation of S^3 which preserves the orientation of S^3. Conversely, for each conformal transformation t of S^3 which preserves the orientation of S^3, there exists a unique element X of G satisfying $t_X=t$. Therefore G may be identified with the group of the conformal transformations of S^3 which preserve the orientation of S^3. The group G may be considered as a connected Lie group isomorphic to the identity component of $SO(4,1)$.

2.2. The conformal Gauss map and the induced metric

Let M be an oriented smooth two-dimensional manifold and $\iota: M \to S^3$ an immersion of M into S^3. For each $X \in G$, let $e_{4,X}$ be a smooth vector field along M normal to S^3 at $t_X(\iota)$ and satisfying $e_{4,X} \in L^+$ and $\langle e_{4,X}, t_X(\iota)\rangle = -1$, and let $e_{3,X}$ be a smooth unit normal vector field on M with respect to $t_X \circ \iota: M \to S^3$ such that for each $p \in M$ and an ordered basis (v_1, v_2) of $T_p(M)$ which gives the orientation of M,

$$(t_X(\iota(p)), d(t_X \circ \iota)(v_1), d(t_X \circ \iota)(v_2), e_{3,X}(p), e_{4,X}(p))$$

is an ordered basis of $T_{t_X(\iota(p))}(\mathbf{R}^5)$ which gives the orientation of $\mathbf{R}^5$. Let H_X be the mean curvature of M with respect to $t_X \circ \iota: M \to S^3$. For the identity transformation $\mathrm{id} \in G$, we set $e_4 := e_{4,\mathrm{id}}$, $e_3 := e_{3,\mathrm{id}}$ and $H := H_{\mathrm{id}}$. Let $\gamma_{\iota,X}$ be a smooth map of M into the 4-dimensionalde Sitter space $S_1^4 := \{x \in \mathbf{R}^5;\ \langle x, x\rangle = 1\}$ defined by $\gamma_{\iota,X} := e_{3,X} + H_X \cdot (t_X \circ \iota)$. We set $\gamma_\iota := \gamma_{\iota,\mathrm{id}}$. Then $\gamma_{\iota,X} = \gamma_{t_X \circ \iota}$ holds. Let $\mathrm{Reg}(M,\iota)$ be the set of the non-umbilical points of M with respect to $\iota: M \to S^3$. Let p be an element of $\mathrm{Reg}(M,\iota)$ and U_p a neighborhood of p in $\mathrm{Reg}(M,\iota)$. Let (u,v) be local coordinates on U_p such that each of $\partial/\partial u$ and $\partial/\partial v$ is in a principal direction with respect to $\iota: M \to S^3$ at each point of U_p. Let k_1 and k_2 be principal curvature functions on M with respect to $\iota: M \to S^3$ such that k_1 and k_2 correspond to $\partial/\partial u$ and $\partial/\partial v$, respectively. Suppose $k_1 > k_2$. The following hold:

$$d\gamma_\iota\left(\frac{\partial}{\partial u}\right) = -k_1 d\iota\left(\frac{\partial}{\partial u}\right) + H d\iota\left(\frac{\partial}{\partial u}\right) + H_u \iota = -\frac{k_1 - k_2}{2} d\iota\left(\frac{\partial}{\partial u}\right) + H_u \iota, \tag{2}$$

$$d\gamma_\iota\left(\frac{\partial}{\partial u}\right) = -k_2 d\iota\left(\frac{\partial}{\partial u}\right) + H d\iota\left(\frac{\partial}{\partial u}\right) + H_u \iota = -\frac{k_1 - k_2}{2} d\iota\left(\frac{\partial}{\partial u}\right) + H_v \iota. \tag{3}$$

Let g be the metric induced by ι and K the curvature of $M$with respect to g. We set

$$\varepsilon := \frac{k_1 - k_2}{2} = \sqrt{H^2 - K + 1}.$$

Then we see that $\gamma_\iota|_{\mathrm{Reg}(M,\iota)}$ is a conformal immersion of $\mathrm{Reg}(M,\iota)$ into S_1^4 such that $\tilde{g} := \varepsilon^2 g$ is the metric induced by $\gamma_\iota|_{\mathrm{Reg}(M,\iota)}$. If p is an umbilical point of M with respect to ι, then for local coordinates (u,v) on a neighborhood of p, we may show that $(d\gamma_\iota)_p(\partial/\partial u)$ and $(d\gamma_\iota)_p(\partial/\partial v)$ are light-like. The map $\gamma_\iota : M \to S_1^4$ is called the *conformal Gauss map* of the immersion ι.

Remark. Let $\iota_0 : M \to \mathbf{R}^n$ be an immersion of M into $\mathbf{R}^n (n \geqq 3)$ and H_0 the mean curvature vector of M with respect to ι_0. Let g_0 be the metric induced by ι_0, and K_0 and dA_0 the Gaussian curvature and the area element of M with respect to g_0, respectively. We set $\varepsilon_0 := \sqrt{|H_0|^2 - K_0}$. Then $\varepsilon_0^2 dA_0$ is invariant under conformal transformations of $\mathbf{R}^n \cup \{\infty\}$ ([4] for a general $n \geqq 3$, [2], [8] for $n=3$). In the following, we suppose that ι_0 is an immersion into $\mathbf{R}^3$. Let p_0 be a point of S^3 and $\pi : S^3 \setminus \{p_0\} \to \mathbf{R}^3$ a stereographic projection from p_0. Then π is the restriction of an inversion of $\mathbf{R}^4 \cup \{\infty\}$ on S^3. Therefore for the immersion $\iota : M \to S^3$ satisfying $\pi \circ \iota = \iota_0$, noticing that the length of the mean curvature vector of M with respect to $\mathrm{id} \circ \iota : M \to \mathbf{R}^4 (\mathrm{id} : S^3 \to \mathbf{R}^4$ is a natural embedding) is given by $\sqrt{H^2+1}$, we obtain $\varepsilon^2 dA = \varepsilon_0^2 dA_0$, where dA is the area element of M with respect to g. Noticing that π is conformal, we see that $\tilde{g}$ is conformal to g_0 and we obtain

$$\tilde{g} = \varepsilon^2 g = \varepsilon_0^2 g_0. \tag{4}$$

In the following, suppose $\mathrm{Reg}(M,\iota) = M$.Then $\mathrm{Reg}(M, t_X \circ \iota) = M$ holds and $\gamma_{\iota,X}$ is a conformal immersion for any $X \in G$. Noticing the above remark, we see that $\varepsilon^2 dA$ is invariant under conformal transformations of S^3. Therefore we see that the metric induced by $\gamma_{\iota,X}$ does not depend on the choice of $X \in G$. In addition, we shall show

$$\gamma_{\iota,X} = X \circ \gamma_\iota. \tag{5}$$

For each $X \in G$, there exist a smooth vector field $X(e_3)_+$ along M and a smooth function a_X on M satisfying

$$X(e_3) = X(e_3)_+ + a_X X(\iota), \qquad X(e_3)_+ \in S_1^4 \cap \{x^{(0)} = 0\}.$$

Therefore we obtain

$$X(\gamma_\iota) = X(e_3) + H \cdot X(\iota) = X(e_3)_+ + (a_X + H) X(\iota).$$

We set $e_- := (-1,0,0,0,0)$ and $b_X := \langle e_-, X(\iota) \rangle$. Then $X(\iota) = b_X t_X(\iota)$ holds. Therefore noticing $e_{3,X} = X(e_3)_+$ in $\mathbf{R}^5$, we obtain

$$X(\gamma_\iota)=e_{3,x}+(a_x+H)b_x t_x(\iota). \tag{6}$$

If F is a conformal map of M into S_1^4 represented as $F:=e_{3,x}+f\cdot(t_x\circ\iota)$, where f is a smooth function on M, then F must be identically equal to $\gamma_{\iota,x}$, i.e.,$f\equiv H_x$ must hold. Therefore noticing (6), we obtain $\gamma_{\iota,x}=X\circ\gamma_\iota$ and $H_x=(a_x+H)b_x$.

2.3. The mean curvature vector with respect to the conformal Gauss map

We set

$$\nu:=\frac{1}{2}\left(\frac{H_u^2}{A^2\varepsilon^2}+\frac{H_v^2}{B^2\varepsilon^2}+H^2\right)\iota-\frac{H_u}{A^2\varepsilon}d\iota\left(\frac{\partial}{\partial u}\right)+\frac{H_v}{B^2\varepsilon}d\iota\left(\frac{\partial}{\partial v}\right)+He_3+e_4, \tag{7}$$

where (u,v) are local coordinates on an open set U of M such that each of $\partial/\partial u$ and $\partial/\partial v$ is in a principal direction with respect to $\iota: M\to S^3$ at each point of U, and

$$A:=\sqrt{g\left(\frac{\partial}{\partial u},\frac{\partial}{\partial u}\right)},\qquad B:=\sqrt{g\left(\frac{\partial}{\partial v},\frac{\partial}{\partial v}\right)}.$$

Then we see the following: ν is an L^+-valued function; ι and ν may be considered as normal vector fields on M with respect to $\gamma_\iota: M\to S_1^4$; $\langle\nu,\iota\rangle=-1$ holds. We shall compute the trace of the shape operator of M with respect to each of ι and ν. From (2) and (3), we obtain

$$\overline{\nabla}_{\partial/\partial u}\iota=-\frac{1}{\varepsilon}d\gamma_\iota\left(\frac{\partial}{\partial u}\right)+\frac{H_u}{\varepsilon}\iota,\qquad \overline{\nabla}_{\partial/\partial v}\iota=\frac{1}{\varepsilon}d\gamma_\iota\left(\frac{\partial}{\partial v}\right)-\frac{H_v}{\varepsilon}\iota,$$

where $\overline{\nabla}$ is the covariant differentiation with respect to the Levi-Civita connection of $(\mathbf{R}^5,\langle,\rangle)$. Therefore we see that the trace of the shape operator of M with respect to ι is identically equal to zero. If we set

$$\overline{\nabla}_{\partial/\partial u}\nu:=c_{11}d\gamma_\iota\left(\frac{\partial}{\partial u}\right)+c_{12}d\gamma_\iota\left(\frac{\partial}{\partial v}\right)+c_{13}\gamma_\iota+c_{14}\nu+c_{10}\iota,$$

then we obtain

$$\begin{aligned}c_{11}=&\frac{H_{uu}}{A^2\varepsilon^2}+\frac{H_u}{A\varepsilon}\left(\frac{1}{A\varepsilon}\right)_u+\frac{H_v}{B^2\varepsilon^2}(\log A\varepsilon)_v\\&+\frac{Hk_1}{\varepsilon}+\frac{1}{\varepsilon}-\frac{1}{2\varepsilon}\left(-\frac{H_u^2}{A^2\varepsilon^2}-\frac{H_v^2}{B^2\varepsilon^2}+H^2\right);\end{aligned} \tag{8}$$

if we set

$$\overline{\nabla}_{\partial/\partial v}\nu:=c_{21}d\gamma_\iota\left(\frac{\partial}{\partial u}\right)+c_{22}d\gamma_\iota\left(\frac{\partial}{\partial v}\right)+c_{23}\gamma_\iota+c_{24}\nu+c_{20}\iota,$$

then we obtain

$$
\begin{aligned}
c_{22}=&\frac{H_{vv}}{B^2\varepsilon^2}+\frac{H_v}{B\varepsilon}\left(\frac{1}{B\varepsilon}\right)_v+\frac{H_u}{A^2\varepsilon^2}(\log B\varepsilon)_u \\
&-\frac{Hk_2}{\varepsilon}-\frac{1}{\varepsilon}+\frac{1}{2\varepsilon}\left(-\frac{H_u^2}{A^2\varepsilon^2}-\frac{H_v^2}{B^2\varepsilon^2}+H^2\right).
\end{aligned} \tag{9}
$$

From (8) and (9), we see that the trace of the shape operator of M with respect to ν is equal to $-(\tilde{\Delta}H+2H)$, where $\tilde{\Delta}$ is the Laplacian on M with respect to $\tilde{g}$.

Let $\tilde{h}$ be the second fundamental form of M with respect to $\gamma_\iota: M\to S_1^4$. Then for two tangent vectors v_1, v_2 at a point of M and a smooth normal vector field ξ on M with respect to γ_ι,

$$\tilde{g}(\tilde{A}_\xi(v_1),\ v_2)=\langle\tilde{h}(v_1,v_2),\xi\rangle \tag{10}$$

holds, where $\tilde{A}_\xi$ is the shape operator of M with respect to ξ. Therefore by $\langle\nu,\iota\rangle=-1$ together with (10), we obtain

$$\tilde{\Delta}\gamma_\iota+2\gamma_\iota=\frac{1}{A^2}\tilde{h}\left(\frac{\partial}{\partial u},\frac{\partial}{\partial u}\right)+\frac{1}{B^2}\tilde{h}\left(\frac{\partial}{\partial v},\frac{\partial}{\partial v}\right)=(\tilde{\Delta}H+2H)\iota. \tag{11}$$

From(11), we see that $(\tilde{\Delta}H+2H)\iota$ is the twice of the mean curvature vector $\tilde{H}$ of M with respect to $\gamma_\iota: M\to S_1^4$. In addition, noticing (1) and $(H^2-K+1)\,\tilde{\Delta}=\Delta$, we see that $\gamma_\iota: M\to S_1^4$ is minimal if and only if $\iota: M\to S^3$ is Willmore.

Remark. We see that $\tilde{H}$ is light-like.

Remark. We see that $\gamma_\iota: M\to S_1^4$ is minimal if and only if $\gamma_\iota^{(k)}$ is an eigenfunction of $\tilde{\Delta}$ such that the corresponding eigenvalue is equal to 2. This is an analogue of Takahashi's theorem ([6]).

3. Construction of a Willmore surface in S^3 from a space-like minimal surface in S_1^4

Let $\gamma: M\to S_1^4$ be a space-like immersion of M into S_1^4 such that the mean curvature vector $\tilde{H}$ of M with respect to γ is represented by a smooth S^3-valued function ι on M up to a constant at each point of M. Then we see that the trace of the shape operator $\tilde{A}_\iota$ of M with respect to ι is identically equal to zero. Let $\tilde{g}$ be the metric induced by γ and V_1, V_2 smooth unit vector fields on an open set U of M perpendicular to each other with respect to $\tilde{g}$ at any point of U. Then there exists a smooth function l on U satisfying

$$\langle\overline{\nabla}_{V_1}\iota,\tilde{e}_1\rangle=-l,\qquad \langle\overline{\nabla}_{V_2}\iota,\tilde{e}_2\rangle=l,$$

where $\tilde{e}_i:=d\gamma(V_i)$ for $i=1,2$. Suppose that l is nowhere zero and positive-valued. In addition, suppose

$$\langle\overline{\nabla}_{V_1}\iota,\tilde{e}_2\rangle=\langle\overline{\nabla}_{V_2}\iota,\tilde{e}_1\rangle=0. \tag{12}$$

Let e_i be a smooth vector field along M satisfying

$$\langle e_i,e_i\rangle=\frac{1}{\varepsilon^2},\qquad e_i\in\{x^{(0)}=0\},\qquad \frac{1}{\varepsilon}\tilde{e}_i-e_i=c_i\iota,$$

where $\varepsilon:=1/l$. Then noticing $\langle\iota,\gamma\rangle=0$, $\langle\iota,\iota\rangle=0$, (12) and that ι is S^3-valued, we obtain

$$\overline{\nabla}_{V_1}\iota=-e_1,\qquad \overline{\nabla}_{V_2}\iota=e_2. \tag{13}$$

Therefore we may consider ι as an immersion of M into S^3 such that the metric g induced by ι is represented as $g=(1/\varepsilon^2)\,\tilde{g}$. We represent γ as $\gamma:=e_3+H\iota$, where e_3 is a smooth vector field along M satisfying $e_3\in S_1^4\cap\{x^{(0)}=0\}$ and H is a smooth function on M. Then we see that e_3 is a unit normal vector field on M with respect to $\iota:M\to S^3$. By $\gamma=e_3+H\iota$ together with (13), we obtain

$$d\gamma(V_1)=\overline{\nabla}_{V_1}e_3-He_1+V_1(H)\iota,\qquad d\gamma(V_2)=\overline{\nabla}_{V_2}e_3+He_2+V_2(H)\iota.$$

Since $d\gamma(V_i)=\varepsilon(e_i+c_i\iota)$, we obtain

$$\overline{\nabla}_{V_1}e_3=-(H+\varepsilon)d\iota(V_1),\qquad \overline{\nabla}_{V_2}e_3=-(H-\varepsilon)d\iota(V_2).$$

We set $k_1:=H+\varepsilon$, $k_2:=H-\varepsilon$. Then we see that k_1 and k_2 are principal curvature functions on M with respect to $\iota:M\to S^3$ and therefore we see that $H=(k_1+k_2)/2$ is the mean curvature of M with respect to $\iota:M\to S^3$. Therefore referring to Section 2, we obtain

Theorem 3.1. *Let M be an oriented smooth two-dimensional manifold and $\gamma:M\to S_1^4$ a space-like immersion of M into S_1^4 such that the mean curvature vector $\tilde{H}$ of M with respect to γ is represented by a smooth S^3-valued function ι on M up to a constant at each point of M. Let e_3 be a smooth vector field along M satisfying $e_3\in S_1^4\cap\{x^{(0)}=0\}$ and $\gamma=e_3+H\iota$ for some smooth function H on M. Suppose that the shape operator $\tilde{A}_\iota$ of M with respect to ι is nowhere zero. Then the following hold:*

(a) *ι may be considered as an immersion of M into S^3;*
(b) *e_3 is a smooth unit normal vector field on M with respect to $\iota:M\to S^3$;*
(c) *H is the mean curvature of M with respect to ι;*

(d) $\mathrm{Reg}(M,\iota)=M$ *holds, i.e., M has no umbilical point with respect to* ι;

(e) *the metric* g *induced by* ι *is conformal to the metric* $\tilde{g}$ *induced by* γ: *if* k_1, k_2 *are distinct two principal curvature functions on M with respect to* ι *and if we set* $\varepsilon:=(k_1-k_2)/2$, *then* $\tilde{g}=\varepsilon^2 g$ *holds*;

(f) *the conformal Gauss map* γ_ι *of* ι *is equal to* γ *or* $-\gamma$.

In particular, we obtain

Corollary 3.2. *Let M be an oriented smooth two-dimensional manifold and* $\gamma: M \to S_1^4$ *a space-like minimal immersion of M into* S_1^4. *Let* ι *be a smooth normal vector field on M with respect to* γ *satisfying* $\iota \in S^3$. *Suppose that the shape operator* $\tilde{A}_\iota$ *of M with respect to* ι *is nowhere zero. Then* $\iota: M \to S^3$ *is a Willmore immersion of M into* S^3 *satisfying* $\gamma_\iota=\gamma$ *or* $-\gamma$.

Remark. Since γ is space-like, we see that there exist at most two Willmore immersions of M into S^3 such that the conformal Gauss map of each of them is equal to γ or $-\gamma$. Suppose that there exist such distinct two immersions ι_1, ι_2. Then $\{\gamma_{\iota_1},\gamma_{\iota_2}\}=\{\gamma,-\gamma\}$ holds. In [3], one of ι_1 and ι_2 is called the *Willmore dual* of the other.

4. A space-like minimal surface in S_1^4 such that the curvature is identically equal to one

We shall prove

Theorem 4.1. *Let M be an oriented smooth two-dimensional manifold and* $\iota: M \to S^3$ *a Willmore immersion of M into* S^3 *such that M has no umbilical point with respect to* ι. *Then* $\iota(M)$ *is the inverse image of a minimal surface in* $\mathbf{R}^3$ *by a stereographic projection if and only if the conformal Gauss map* γ_ι *of* ι *is a minimal immersion such that the curvature of M with respect to the metric* $\tilde{g}$ *induced by* γ_ι *is identically equal to one.*

In order to prove Theorem 4.1, we need lemmas.

Lemma 4.2. *Let M be an oriented smooth two-dimensional manifold and* $\gamma: M \to S_1^4$ *a space-like minimal immersion of M into* S_1^4 *such that the curvature of M with respect to the metric* $\tilde{g}$ *induced by* γ *is identically equal to one. Suppose that there exists a smooth normal vector field* ι *on M with respect to* γ *satisfying* $\iota \in S^3$ *and the condition that the shape operator* $\tilde{A}_\iota$ *is nowhere zero. Then there exists a vector* $\nu_0 \in L^+$ *such that* $\gamma(M)$ *is contained in a hyperplane* $\{x \in \mathbf{R}^5; \langle x,\nu_0\rangle=0\}$ *of* $\mathbf{R}^5$.

Proof. Let $\tilde{R}$ (respectively, $\tilde{R}'$) be the curvature tensor field on $(M,\tilde{g})$ (respectively, $(S_1^4,\langle,\rangle)$). Then the equation of Gauss is given by

$$\begin{aligned}&\langle \tilde{R}'(X,Y)Z,W\rangle-\tilde{g}(\tilde{R}(X,Y)Z,W)\\&=\langle \tilde{h}(X,Z),\ \tilde{h}(Y,W)\rangle-\langle \tilde{h}(X,W),\ \tilde{h}(Y,Z)\rangle,\end{aligned} \tag{14}$$

where X, Y, Z, W are smooth vector fields on an open set U of $M$and $\tilde{h}$ is the second fundamental form of M with respect to γ. Let V_1 and V_2 be smooth vector fields on U satisfying $\tilde{g}\,(V_i,V_j)=\delta_{ij}\,(i,j\in\{1,2\})$. If we set $X:=V_1$, $Y:=V_2$, $Z:=V_2$, $W:=V_1$ in (14), then noticing

$$\langle \tilde{R}'(V_1,V_2)V_2,V_1\rangle=\tilde{g}(\tilde{R}\,(V_1,V_2)\ V_2,V_1)=1,$$

we obtain

$$\langle \tilde{h}\,(V_1,V_1),\ \tilde{h}\,(V_2,V_2)\rangle=\langle \tilde{h}\,(V_1,V_2),\ \tilde{h}\,(V_1,V_2)\rangle. \tag{15}$$

Let ν be a smooth normal vector field on M with respect to γ satisfying $\nu\in L^+$ and $\langle\nu,\iota\rangle=-1$. Then we may set $\tilde{h}(V_i,V_j):=a_{ij}\iota+b_{ij}\nu$. Then $a_{ij}=a_{ji}$ and $b_{ij}=b_{ji}$ hold. In addition, since γ is minimal, we obtain $a_{11}+a_{22}=0$, $b_{11}+b_{22}=0$. If we set

$$a_1:=a_{11}=-a_{22},\quad a_2=a_{12}=a_{21},\quad b_1:=b_{11}=-b_{22},\quad b_2=b_{12}=b_{21},$$

then (15) is represented as

$$a_1b_1+a_2b_2=0. \tag{16}$$

Since $\tilde{A}_\iota$ is nowhere zero, we suppose that at each point of U, V_i is an eigenvector of $\tilde{A}_\iota$. Then noticing $\langle\nu,\iota\rangle=-1$ and (10), we obtain $b_2\equiv0$. Since we suppose that $\tilde{A}_\iota$ is nowhere zero, we obtain $b_1\neq0$. Therefore from (16), we obtain $a_1\equiv0$. In the following, we set $a:=a_2$, $b:=b_1$. Let (u,v) be local coordinates on U satisfying $\partial/\partial u=AV_1$ and $\partial/\partial v=BV_2$, where A and B are smooth, positive-valued functions. Then the following hold:

$$\overline{\nabla}_{\partial/\partial u}\nu=\frac{A}{B}a\frac{\partial}{\partial v}+c_1\nu,\qquad \overline{\nabla}_{\partial/\partial v}\nu=\frac{B}{A}a\frac{\partial}{\partial u}+c_2\nu, \tag{17}$$

where c_1 and c_2 are smooth functions. By (17), we obtain

$$\begin{aligned}\overline{\nabla}_{\partial/\partial v}\overline{\nabla}_{\partial/\partial u}\nu=&\,a\Big(\frac{A}{B}\Gamma^1_{22}+\frac{B}{A}c_1\Big)\frac{\partial}{\partial u}+\Big\{\Big(\frac{A}{B}a\Big)_v+\frac{A}{B}a\Gamma^2_{22}\Big\}\frac{\partial}{\partial v}\\&+\{-ABab+(c_1)_v+c_1c_2\}\nu-ABa\gamma,\end{aligned} \tag{18}$$

$$\overline{\nabla}_{\partial/\partial u}\overline{\nabla}_{\partial/\partial v}\nu=\left\{\left(\frac{B}{A}a\right)_u+\frac{B}{A}a\Gamma^1_{11}\right\}\frac{\partial}{\partial u}+a\left(\frac{B}{A}\Gamma^2_{11}+\frac{A}{B}c_2\right)\frac{\partial}{\partial v} \\ +\{ABab+(c_2)_u+c_1c_2\}\nu-ABa\gamma, \tag{19}$$

where $\Gamma^k_{ij}(i,j,k=1,2)$ are the Christoffel's symbols of the Levi-Civita connection of $(M,\tilde{g})$ with respect to local coordinates (u,v). From (18), (19) and $\overline{\nabla}_{\partial/\partial v}\overline{\nabla}_{\partial/\partial u}\nu=\overline{\nabla}_{\partial/\partial u}\overline{\nabla}_{\partial/\partial v}\nu$, we obtain

$$\left(\frac{B}{A}a\right)_u+\left(\frac{B}{A}\Gamma^1_{11}-\frac{A}{B}\Gamma^1_{22}-\frac{B}{A}c_1\right)a=0, \tag{20}$$

$$\left(\frac{A}{B}a\right)_v+\left(\frac{A}{B}\Gamma^2_{22}-\frac{B}{A}\Gamma^2_{11}-\frac{A}{B}c_2\right)a=0, \tag{21}$$

$$2ABab-(c_1)_v+(c_2)_u=0. \tag{22}$$

The following hold:

$$\Gamma^1_{11}=(\log A)_u,\quad \Gamma^2_{11}=-\frac{AA_v}{B^2},\quad \Gamma^1_{22}=-\frac{BB_u}{A^2},\quad \Gamma^2_{22}=(\log B)_v. \tag{23}$$

If $a\neq 0$, then by (20), (21) and (23), we obtain $(c_1)_v=(c_2)_u$. This contradicts (22). Therefore we obtain $a\equiv 0$. Then from (17), we obtain $\overline{\nabla}_{\partial/\partial u}\nu=c_1\nu$ and $\overline{\nabla}_{\partial/\partial v}\nu=c_2\nu$. Since $(c_1)_v=(c_2)_u$, there exists a smooth function f satisfying $f_u=c_1$ and $f_v=c_2$. Then we see that $\overline{\nabla}_{\partial/\partial u}(e^{-f}\nu)$ and $\overline{\nabla}_{\partial/\partial v}(e^{-f}\nu)$ are identically zero. Therefore $\nu_0:=e^{-f}\nu$ is constant. Hence we have proved Lemma 4.2. □

Lemma 4.3. ([3]) *Let $\iota: M\to S^3$ be an immersion of M into S^3 and ν as in* (7). *Then for each point $p\in M$, a unique intersection of $S^3=L^+\cap\{x^{(0)}=1\}$ with a line through the origin of $\mathbf{R}^5$ determined by $\nu(p)$ is contained in a sphere in S^3 tangent to M at $\iota(p)$ such that the mean curvature is equal to $H(p)$.*

For the proof of Lemma 4.3, see [3, pp. 33].

Proof of Theorem 4.1. Let $\iota_0: M\to\mathbf{R}^3$ be a minimal immersion of M into $\mathbf{R}^3$ such that M has no umbilical point with respect to ι_0. Then on a neighborhood U_p of each point p of M, there exist isothermal coordinates (u,v) and a smooth, positive-valued function A satisfying

(a) the metric g_0 induced by ι_0 is represented as $A^2(du^2+dv^2)$ on U_p;
(b) each of $\partial/\partial u$ and $\partial/\partial v$ is in a principal direction with respect to ι_0 at each point of U_p;
(c) the Gaussian curvature K_0 of M with respect to g_0 is represented as $K_0=-1/A^4$ on U_p.

Let $\pi: S^3\setminus\{p_0\}\to\mathbf{R}^3$ be a stereographic projection and $\iota: M\to S^3$ the immersion satisfying $\pi\circ\iota=\iota_0$. Then by (4), we see that the metric $\tilde{g}$ induced by the conformal Gauss map of ι is represented as $\tilde{g}=-K_0g_0$. Then $\tilde{g}=A^{-2}(du^2+dv^2)$ holds on U_p. Let $\tilde{K}$ be the curvature of M with respect to $\tilde{g}$. Then $\tilde{K}=-\tilde{\Delta}\log(1/A)$ holds. Since $\tilde{\Delta}=-K_0^{-1}\Delta_0=A^4\Delta_0$, where Δ_0 is the Laplacian on M with respect to g_0, we obtain $\tilde{K}=A^4\Delta_0\log A$. In addition, since $\Delta_0\log A=-K_0=1/A^4$, we obtain $\tilde{K}\equiv 1$.
Let $\iota: M\to S^3$ be a Willmore immersion of M into S^3 such that M has no umbilical point with respect to ι, and suppose that the conformal Gauss map γ_ι of ι is a minimal immersion such that the curvature of M with respect to the metric $\tilde{g}$ induced by γ_ι is identically equal to one. Then by Lemma 4.2, we see that there exists a vector $\nu_0\in L^+$ such that $\gamma_\iota(M)$ is contained in the hyperplane of $\mathbf{R}^5$ determined by ν_0. Then by Lemma 4.3, we see that for any point $p\in M$, a sphere in S^3 tangent to M at $\iota(p)$ such that the mean curvature is equal to $H(p)$ contains the point x_0 of S^3 determined by ν_0. Let π_0 be the stereographic projection from x_0. Then $\pi_0\circ\iota$ is a minimal immersion. Hence we obtain Theorem 4.1. □

Remark. Let $\iota: M\to S^3$ be a minimal immersion such that M has no umbilical point with respect to ι. Then on a neighborhood U_p of each point p of M, there exist isothermal coordinates (u,v) and a smooth, positive-valued function A satisfying the following: (a) the metric g induced by ι is represented as $A^2(du^2+dv^2)$ on U_p; (b) each of $\partial/\partial u$ and $\partial/\partial v$ is in a principal direction with respect to ι at each point of U_p; (c) the curvature K of M with respect to the metric g is represented as $K=1-1/A^4$ on U_p. The metric $\tilde{g}$ induced by γ_ι is represented as $\tilde{g}=A^{-2}(du^2+dv^2)$. Then the curvature $\tilde{K}$ of $M$with respect to $\tilde{g}$ is represented as $\tilde{K}=\tilde{\Delta}\log A$. Since

$$\tilde{\Delta}=(1-K)^{-1}\Delta=A^4\Delta,\qquad \Delta\log A=-K=-1+1/A^4,$$

we obtain $\tilde{K}=1-A^4<1$.

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