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The Gaussian curvature of a parallel curved surface

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Abstract

We shall represent the Gaussian curvature of a canonical parallel curved surface in terms of a generating pair. In addition, we shall study a relation between generating pairs such that the corresponding canonical parallel curved surfaces have the same semisurface structure.

1. Introduction

A surface S in $\mathbf{R}^3$ is called *parallel curved* if there exists a plane P such that one of the principal directions of S at each point is parallel to P. If S is parallel curved, then such a plane as P is called a *base plane* of S. A parallel curved surface S is called *canonical* if there exist smooth functions ϕ_1, ϕ_2 on an open interval I and smooth functions ϕ_3, ϕ_4 on an open interval J satisfying $\alpha:=\sqrt{(\phi'_1)^2+(\phi'_2)^2}\neq 0$ on I, $\beta:=\sqrt{(\phi'_3)^2+(\phi'_4)^2}\neq 0$ on J, and the condition that S is the image of $I\times J$ by a map Φ defined by

$$\begin{aligned}\Phi(u, v):=&(\phi_1(u)+\phi_3(v)\phi'_2(u)/\alpha(u))\mathbf{e}_1\\&+(\phi_2(u)-\phi_3(v)\phi'_1(u)/\alpha(u))\mathbf{e}_2+\phi_4(v)\mathbf{e}_3+\mathbf{c},\end{aligned} \tag{1}$$

where $\mathbf{c}$ is a vecter of $\mathbf{R}^3$ and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis of $\mathbf{R}^3$ such that $\mathbf{e}_1$ and $\mathbf{e}_2$ form a basis of a base plane of S. Parallel curved surfaces were studied in [1], [2] and [3].

Let C_b and C_g be simple curves in $\mathbf{R}^3$ with a unique intersection $p(C_b, C_g)$ and contained in planes P_b and P_g, respectively. A pair (C_b, C_g) is called *generating* if we may choose as P_g the plane normal to C_b at $p(C_b, C_g)$; if (C_b, C_g) is generating, then C_b and C_g are called the *base curve* and the *generating curve* of (C_b, C_g), respectively. For a canonical parallel curved surface S with a base plane P_0 and each point q of S, there exists a generating pair (C_b, C_g) satisfying the following:

1. C_b and C_g are lines of curvature in S through q,

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2. P_b is parallel to P_0 (therefore P_b is a base plane of S)

(see [2]). In addition, for a generating pair (C_b, C_g), there exists a canonical parallel curved surface S satisfying the following:

1. S contains a neighborhood O_b (respectively, O_g) of $p(C_b,C_g)$ in C_b (respectively, C_g) so that O_b and O_g are lines of curvature in S;

2. P_b is a base plane of S

(see [2]). We say that S is *generated by* (C_b, C_g).

In the sequel, suppose that for a generating pair (C_b, C_g), there exists a canonical parallel curved surface S satisfying the above 1 and 2 with $O_b=C_b$ and $O_g=C_g$. The minimum of such canonical parallel curved surfaces as S is denoted by $S_{(C_b,C_g)}$. For example, if C_b and C_g are isometric to $\mathbf{R}$, then $S_{(C_b,C_g)}$ is homeomorphic to a plane; if one of C_b and C_g is isometric to $\mathbf{R}$ and the other isometric to a closed curve, then $S_{(C_b,C_g)}$ is homeomorphic to a cylinder; if C_b and C_g are closed curves, then$S_{(C_b,C_g)}$ is homeomorphic to a torus. If S is a connected, complete, real-analytic, embedded, parallel curved surface, then S is homeomorphic to a sphere, a plane, a cylinder, or a torus, and in addition,

1. if S is homeomorphic to a sphere, then S is a surface of revolution which crosses its axis of rotation at just two points;

2. if S is homeomorphic to a plane, then one of the following holds:

   (a) S is a surface of revolution which crosses its axis of rotation at just one point,
   (b) $S=S_{(C_b,C_g)}$ holds, where (C_b, C_g) is a generating pair each element of which is isometric to $\mathbf{R}$;

3. if S is homeomorphic to a cylinder, then $S=S_{(C_b,C_g)}$ holds, where (C_b,C_g) is a generating pair such that one of C_b and C_g is isometric to $\mathbf{R}$ and the other a closed curve;

4. if S is homeomorphic to a torus, then $S=S_{(C_b,C_g)}$ holds, where (C_b, C_g) is a generating pair each element of which is a closed curve

(see [1]).

The first purpose of the present paper is to represent the Gaussian curvature of a canonical parallel curved surface in terms of a generating pair. For a canonical parallel curved surface $S_{(C_b,C_g)}$, we suppose

$$C_b=\{\Phi(u, v_0);\ u\in I\},\quad C_g=\{\Phi(u_0, v);\ v\in J\}$$

for some $u_0 \in I$ and some $v_0 \in J$ satisfying $\phi_3(v_0)=0$. In addition, we suppose $\mathbf{e}_1:=(1, 0, 0)$, $\mathbf{e}_2:=(0, 1, 0)$, $\mathbf{e}_3:=(0, 0, 1)$, $c:=(0, 0, 0)$ in (1). Then the map Φ is represented as follows:

$$\Phi(u, v):=\begin{pmatrix}\phi_1(u)\\ \phi_2(u)\\ 0\end{pmatrix}+\frac{\phi_3(v)}{\alpha(u)}\begin{pmatrix}\phi'_2(u)\\ -\phi'_1(u)\\ 0\end{pmatrix}+\phi_4(v)\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}. \tag{2}$$

Let k_b and k_g be the curvatures of C_b and C_g, respectively. We shall prove

Theorem 1.1. *The Gaussian curvature K of a canonical parallel curved surface $S_{(C_b,C_g)}$ is represented as follows:*

$$K=\frac{\phi'_4(v)k_b(u)k_g(v)}{(1+\phi_3(v)k_b(u))\beta(v)}. \tag{3}$$

Remark. From (3), we see that if C_b and C_g are closed and convex, then the sign of the Gaussian curvature of a canonical parallel curved surface $S_{(C_b,C_g)}$ depends only on ϕ'_4, and that the set of the zero points of the Gaussian curvature is represented as $C_{0,1}\cup C_{0,2}$, where $C_{0,1}$ and $C_{0,2}$ are simple closed curves in $S_{(C_b,C_g)}$ such that each of $C_{0,1}$ and $C_{0,2}$ is contained in a base plane of $S_{(C_b,C_g)}$.

The second purpose of the present paper is to study a relation between generating pairs such that the corresponding canonical parallel curved surfaces have the same semisurface structure. Let S be a surface in $\mathbf{R}^3$ without any umbilical point and p a point of S. Then there exist local coordinates (u, v) on a neighborhood U_p of p such that two principal distributions are given by $\partial/\partial u$ and $\partial/\partial v$ on U_p. Such coordinates are said to be *compatible with* principal distributions on S. The first fundamental form of S is represented as $A^2du^2+B^2dv^2$ on U_p, where A and B are smooth, positive-valued functions on U_p. Let k_1 and k_2 be two principal curvature functions corresponding to $\partial/\partial u$ and $\partial/\partial v$, respectively. If the Gaussian curvature K of S is nowhere zero on U_p, then on U_p, the following holds ([3]):

$$c_{20}k_1^2+c_{11}k_1k_2+c_{02}k_2^2=0,$$

where

$$\begin{aligned} c_{20} &:= (\log|K|A^2)_v(\log B)_u - (\log B)_{uv}. \\ c_{11} &:= (\log|K|AB)_{uv} - 4(\log A)_v(\log B)_u. \\ c_{02} &:= (\log|K|B^2)_u(\log A)_v - (\log A)_{uv}. \end{aligned} \tag{4}$$

Noticing the equation of Gauss:

$$K = -\frac{1}{AB}\left\{\left(\frac{A_v}{B}\right)_v + \left(\frac{B_u}{A}\right)_u\right\}, \tag{5}$$

we see that c_{20}, c_{11}, c_{02} in (4) depend only on A, B and their partial derivatives. In addition, if (u', v') are local coordinates on U_p compatible with principal distributions such that at each point of U_p, $\partial/\partial u'$ (respectively, $\partial/\partial v'$) is represented as $\partial/\partial u$ (respectively, $\partial/\partial v$) up to a nonzero constant, then the coefficients c'_{20}, c'_{11}, c'_{02} defined as in (4) for the coordinates (u', v') satisfy $(c'_{20}, c'_{11}, c'_{02}) = \lambda(c_{20}, c_{11}, c_{02})$ for some $\lambda \in \mathbf{R}\backslash\{0\}$ at each point of U_p. For a surface S in $\mathbf{R}^3$ without any umbilical point and with nowhere zero Gaussian curvature, a neighborhood of each point of S is a canonical parallel curved surface if and only if S satisfies the following:

1. The integral curves of just one of the principal distributions on S are geodesics;

2. c_{20}, c_{11} and c_{02} in (4) vanish on S

(see [3]).

Let M be a two-dimensional Riemannian manifold and g its metric. Let D_1, D_2 be two smooth one-dimensional distributions on M. A Riemannian manifold (M, g) equipped with (D_1, D_2) is called a *semisurface* if D_1 and D_2 are orthogonal to each other at any point of M with respect to g. If (M, g, D_1, D_2) is a semisurface, then a triplet (g, D_1, D_2) is a *semisurface structure* of M. Let (M, g, D_1, D_2) be a semisurface. Then for each point $p \in M$, there exist local coordinates (u,v) on a neighborhood U_p of p satisfying $\partial/\partial u \in D_1$ and $\partial/\partial v \in D_2$ on U_p. Such coordinates are said to be *compatible with* (D_1, D_2). The Riemannian metric g may be represented as $g = A^2du^2 + B^2dv^2$ on U_p. Suppose that the curvature K of the Riemannian manifold (M, g) is nowhere zero on U_p. Then for each point p of U_p, a homogeneous polynomial

$$P_{CM,q}(X_1, X_2) := c_{20}(q)X_1^2 + c_{11}(q)X_1X_2 + c_{02}(q)X_2^2,$$

where c_{20}, c_{11}, c_{02} are defined as in (4), is determined by a given semisurface structure of M up to a nonzero constant. This polynomial $P_{CM,q}$ is called a *Codazzi-Mainardi polynomial* of a semisurface (M, g, D_1, D_2) at q. Let (M, g, D_1, D_2) be a semisurface with nowhere zero curvature. Suppose that the integral curves of just one of D_1 and D_2 are geodesics and that a

Codazzi-Mainardi polynomial vanishes at any point of M. Then there exist plural isometric immersions of a neighborhood U_p of each point p of M into $\mathbf{R}^3$ satisfying the following ([3]):

1. the images are canonical parallel curved surfaces such that arbitrary two in them are not congruent with each other in $\mathbf{R}^3$;

2. (D_1, D_2) gives a pair of two principal distributions.

Let $\Phi^{(1)}$ and $\Phi^{(2)}$ be isometric immersions of U_p into $\mathbf{R}^3$ as above. For $j \in \{1, 2\}$, let $(C_b^{(j)}, C_g^{(j)})$ be a generating pair satisfying $\Phi^{(j)}(U_p)=S_{(C_b^{(j)}, C_g^{(j)})}$. In the present paper, we shall prove

Theorem 1.2. *Let (u, v) be compatible with (D_1, D_2). Then there exist nonzero numbers λ, μ $\in \mathbf{R}\backslash\{0\}$ satisfying the following:*

$$\frac{\alpha^{(2)}(u)k_b^{(2)}(u)}{\alpha^{(1)}(u)k_b^{(1)}(u)}=\frac{(\phi_4^{(1)})'(v)k_g^{(1)}(v)}{(\phi_4^{(2)})'(v)k_g^{(2)}(v)}=\lambda, \tag{6}$$

$$\frac{\alpha^{(2)}(u)-\alpha^{(1)}(u)}{\alpha^{(1)}(u)k_b^{(1)}(u)}=\phi_3^{(1)}(v)-\lambda\phi_3^{(2)}(v)=\mu. \tag{7}$$

Noticing that (6) and (7) give a relation between two generating pairs $(C_b^{(1)}, C_g^{(1)})$ and $(C_b^{(2)}, C_g^{(2)})$, we shall also prove

Theorem 1.3. *Let $(C_b^{(1)}, C_g^{(1)})$ and $(C_b^{(2)}, C_g^{(2)})$ be generating pairs satisfying* (6) *and* (7) *for some $\lambda, \mu \in \mathbf{R}\backslash\{0\}$. Then the corresponding canonical parallel curved surfaces $S_{(C_b^{(1)}, C_g^{(1)})}$ and $S_{(C_b^{(2)}, C_g^{(2)})}$ have the same semisurface structure, i.e., there exist a semisurface (M, g, D_1, D_2) and isometric immersions $\Phi^{(1)}, \Phi^{(2)}: M\rightarrow \mathbf{R}^3$ such that for $j \in \{1, 2\}$, the following hold:*

1. $\Phi^{(j)}(M)=S_{(C_b^{(j)}, C_g^{(j)})}$;

2. *$d\Phi^{(j)}(D_i)$ is a principal distribution on $\Phi^{(j)}(M)$.*

2. Proof of Theorem 1.1

We shall prove

Proposition 2.1. *The coefficients of the first fundamental form $Edu^2+2Fdudv+Gdv^2$ of $S_{(C_b, C_g)}$ are represented as follows:*

$$E=(1+\phi_3(v)k_b(u))^2\alpha(u)^2, \quad F=0, \quad G=\beta(v)^2.$$

Proof. The following hold:

$$\Phi_u=\begin{pmatrix}\phi_1'(u)\\ \phi_2'(u)\\ 0\end{pmatrix}+\left(\frac{\phi_3(v)}{\alpha(u)}\right)_u\begin{pmatrix}\phi_2'(u)\\ -\phi_1'(u)\\ 0\end{pmatrix}+\frac{\phi_3(v)}{\alpha(u)}\begin{pmatrix}\phi_2''(u)\\ -\phi_1''(u)\\ 0\end{pmatrix}$$

$$=\begin{pmatrix}\phi_1'(u)\\ \phi_2'(u)\\ 0\end{pmatrix}-\frac{\phi_3(v)}{\alpha(u)^3}(\phi_1'(u)\phi_1''(u)+\phi_2'(u)\phi_2''(u))\begin{pmatrix}\phi_2'(u)\\ -\phi_1'(u)\\ 0\end{pmatrix}+\frac{\phi_3(v)}{\alpha(u)}\begin{pmatrix}\phi_2''(u)\\ -\phi_1''(u)\\ 0\end{pmatrix}$$

$$=\begin{pmatrix}\phi_1'(u)\\ \phi_2'(u)\\ 0\end{pmatrix}-\frac{\phi_3(v)}{\alpha(u)^3}\left\{\begin{pmatrix}(\phi_1'(u)\phi_1''(u)+\phi_2'(u)\phi_2''(u))\phi_2'(u)\\ -(\phi_1'(u)\phi_1''(u)+\phi_2'(u)\phi_2''(u))\phi_1'(u)\\ 0\end{pmatrix}-\alpha(u)^2\begin{pmatrix}\phi_2''(u)\\ -\phi_1''(u)\\ 0\end{pmatrix}\right\}$$

$$=\begin{pmatrix}\phi_1'(u)\\ \phi_2'(u)\\ 0\end{pmatrix}-\frac{\phi_3(v)}{\alpha(u)^3}\begin{pmatrix}(\phi_1''(u)\phi_2'(u)-\phi_1'(u)\phi_2''(u))\phi_1'(u)\\ (\phi_1''(u)\phi_2'(u)-\phi_1'(u)\phi_2''(u))\phi_2'(u)\\ 0\end{pmatrix}$$

$$=\begin{pmatrix}\phi_1'(u)\\ \phi_2'(u)\\ 0\end{pmatrix}+\frac{\phi_3(v)}{\alpha(u)^3}(\phi_1'(u)\phi_2''(u)-\phi_1''(u)\phi_2'(u))\begin{pmatrix}\phi_1'(u)\\ \phi_2'(u)\\ 0\end{pmatrix}$$

$$=\left\{1+\frac{\phi_3(v)}{\alpha(u)^3}(\phi_1'(u)\phi_2''(u)-\phi_1''(u)\phi_2'(u))\right\}\begin{pmatrix}\phi_1'(u)\\ \phi_2'(u)\\ 0\end{pmatrix}.$$

Since the curvature $k_b(u)$ of the plane curve given by(ϕ_1,ϕ_2) is represented as

$$k_b(u)=\frac{\phi_1'(u)\phi_2''(u)-\phi_1''(u)\phi_2'(u)}{\alpha(u)^3},$$

we obtain

$$\Phi_u=(1+\phi_3(v)k_b(u))\begin{pmatrix}\phi_1'(u)\\ \phi_2'(u)\\ 0\end{pmatrix}.$$

The following holds:

$$\Phi_v=\frac{\phi'_3(v)}{\alpha(u)}\begin{pmatrix}\phi'_2(u)\\-\phi'_1(u)\\0\end{pmatrix}+\phi'_4(v)\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Therefore we obtain

$$E=\Phi_u\cdot\Phi_u=(1+\phi_3(v)k_b(u))^2\alpha(u)^2,\quad F=\Phi_u\cdot\Phi_v=0,$$

$$G=\Phi_v\cdot\Phi_v=\frac{\phi'_3(v)^2}{\alpha^2}\alpha(u)^2+\phi'_4(v)^2=\beta(v)^2.$$

□

Proposition 2.2. *A unit normal vecter field of $S_{(C_b,C_g)}$ is represented as follows:*

$$\mathbf{e}=\frac{\phi'_4(v)}{\alpha(u)\beta(v)}\begin{pmatrix}\phi'_2(u)\\-\phi'_1(u)\\0\end{pmatrix}-\frac{\phi'_3(v)}{\beta(v)}\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Proof. The following hold:

$$\Phi_u\times\Phi_v=(1+\phi_3(v)k_b(u))\begin{pmatrix}\phi'_1(u)\\\phi'_2(u)\\0\end{pmatrix}\times\left\{\frac{\phi'_3(v)}{\alpha(u)}\begin{pmatrix}\phi'_2(u)\\-\phi'_1(u)\\0\end{pmatrix}+\phi'_4(v)\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

$$=-(1+\phi_3(v)k_b(u))\alpha(u)\phi'_3(v)\begin{pmatrix}0\\0\\1\end{pmatrix}+(1+\phi_3(v)k_b(u))\phi'_4(v)\begin{pmatrix}\phi'_2(u)\\-\phi'_1(u)\\0\end{pmatrix}.$$

Then we obtain

$$|\Phi_u\times\Phi_v|^2=\alpha(u)^2(1+\phi_3(v)k_b(u))^2(\phi'_3(v)^2+\phi'_4(v)^2)$$

and

$$|\Phi_u\times\Phi_v|=\alpha(u)(1+\phi_3(v)k_b(u))\sqrt{\phi'_3(v)^2+\phi'_4(v)^2}.$$

Therefore we obtain

$$\mathbf{e}=\frac{\Phi_u\times\Phi_v}{|\Phi_u\times\Phi_v|}=\frac{\phi'_4(v)}{\alpha(u)\beta(v)}\begin{pmatrix}\phi'_2(u)\\-\phi'_1(u)\\0\end{pmatrix}-\frac{\phi'_3(v)}{\beta(v)}\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

□

Proposition 2.3. *The coefficients of the second fundamental form* $Ldu^2+2Mdudv+Ndv^2$ *of* $S_{(C_b,C_g)}$ *are represented as follows:*

$$L=-(1+\phi_3(v)k_b(u))\frac{\phi'_4(v)k_b(u)\alpha(u)^2}{\beta(v)},\quad M=0,$$

$$N=-k_g(v)\beta(v)^2.$$

Proof. The following hold:

$$\Phi_{uu}=(1+\phi_3(v)k_b(u))_u\begin{pmatrix}\phi'_1(u)\\ \phi'_2(u)\\ 0\end{pmatrix}+(1+\phi_3(v)k_b(u))\begin{pmatrix}\phi''_1(u)\\ \phi''_2(u)\\ 0\end{pmatrix},$$

$$\Phi_{uv}=\phi'_3(v)k_b(u)\begin{pmatrix}\phi'_1(u)\\ \phi'_2(u)\\ 0\end{pmatrix},\quad \Phi_{vv}=\frac{\phi''_3(v)}{\alpha(u)}\begin{pmatrix}\phi'_2(u)\\ -\phi'_1(u)\\ 0\end{pmatrix}+\phi''_4(v)\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}.$$

Therefore we obtain

$$L=\Phi_{uu}\cdot\mathbf{e}=(1+\phi_3(v)k_b(u))\frac{\phi'_4(v)}{\alpha(u)\beta(v)}(\phi''_1(u)\phi'_2(u)-\phi'_1(u)\phi''_2(u))$$

$$=-(1+\phi_3(v)k_b(u))\frac{\phi'_4(v)k_b(u)\alpha(u)^2}{\beta(v)},$$

$$M=\Phi_{uv}\cdot\mathbf{e}=0,$$

$$N=\Phi_{vv}\cdot\mathbf{e}=\frac{\phi'_4(v)\phi''_3(v)}{\alpha(u)^2\beta(v)}\alpha(u)^2-\frac{\phi'_3(v)\phi''_4(v)}{\beta(v)}$$

$$=-\frac{\phi'_3(v)\phi''_4(v)-\phi''_3(v)\phi'_4(v)}{\beta(v)}.$$

Since the curvature $k_g(v)$ of the plane curve given by (ϕ_3, ϕ_4) is represented as

$$k_g(v)=\frac{\phi'_3(v)\phi''_4(v)-\phi''_3(v)\phi'_4(v)}{\beta(v)^3},$$

we obtain

$$N=-k_g(v)\beta(v)^2.$$

□

By Propositions 2.1, 2.2 and 2.3, we obtain

$$K=\frac{LN}{EG}=\frac{(1+\phi_3(v)k_b(u))\phi_4'(v)k_b(u)k_g(v)\alpha(u)^2\beta(v)}{(1+\phi_3(v)k_b(u))^2\alpha(u)^2\beta(v)^2}$$
$$=\frac{\phi_4'(v)k_b(u)k_g(v)}{(1+\phi_3(v)k_b(u))\beta(v)}.$$

Hence we have proved Theorem 1.1.

3. Proofs of Theorem 1.2 and Theorem 1.3

Proof of Theorem 1.2. Let $g^{(j)}$ be the first fundamental form of a canonical parallel curved surface $S^{(j)}:=\Phi^{(j)}(U_p)$ and $K^{(j)}$ the Gaussian curvature of $S^{(j)}$. Since $S^{(1)}$ and $S^{(2)}$ have the same semisurface structure, the following hold:

$$K^{(1)}=K^{(2)}, \tag{8}$$
$$E^{(1)}=E^{(2)}, \quad G^{(1)}=G^{(2)}. \tag{9}$$

Therefore by (3) together with (8), we obtain

$$\frac{(\phi_4^{(1)})'(v)k_b^{(1)}(u)k_g^{(1)}(v)}{(1+\phi_3^{(1)}(v)k_b^{(1)}(u))\beta^{(1)}(v)}=\frac{(\phi_4^{(2)})'(v)k_b^{(2)}(u)k_g^{(2)}(v)}{(1+\phi_3^{(2)}(v)k_b^{(2)}(u))\beta^{(2)}(v)}. \tag{10}$$

From Proposition 2.1 and (9), we obtain

$$(1+\phi_3^{(1)}(v)k_b^{(1)}(u))\alpha^{(1)}(u)=(1+\phi_3^{(2)}(v)k_b^{(2)}(u))\alpha^{(2)}(u), \tag{11}$$
$$\beta(v)^{(1)}=\beta(v)^{(2)}. \tag{12}$$

We rewrite (11) into

$$\frac{(1+\phi_3^{(1)}(v)k_b^{(1)}(u))}{(1+\phi_3^{(2)}(v)k_b^{(2)}(u))}=\frac{\alpha^{(2)}(u)}{\alpha^{(1)}(u)}. \tag{13}$$

Therefore from (10), (12) and (13), we obtain

$$\frac{\alpha^{(2)}(u)k_b^{(2)}(u)}{\alpha^{(1)}(u)k_b^{(1)}(u)}=\frac{(\phi_4^{(1)})'(v)k_g^{(1)}(v)}{(\phi_4^{(2)})'(v)k_g^{(2)}(v)}. \tag{14}$$

Since the left hand side (respectively, the right hand side) of (14) depends only on u (respectively, v), there exists a nonzero number $\lambda \in \mathbf{R}\backslash\{0\}$ satisfying

$$\frac{\alpha^{(2)}(u)k_b^{(2)}(u)}{\alpha^{(1)}(u)k_b^{(1)}(u)}=\frac{(\phi_4^{(1)})'(v)k_g^{(1)}(v)}{(\phi_4^{(2)})'(v)k_g^{(2)}(v)}=\lambda.$$

Therefore we obtain (6). From (6), we obtain

$$\alpha^{(2)}(u)k_b^{(2)}(u)=\lambda\alpha^{(1)}(u)k_b^{(1)}(u). \tag{15}$$

Therefore by (11) together with (15), we obtain

$$\frac{\alpha^{(2)}(u)-\alpha^{(1)}(u)}{\alpha^{(1)}(u)k_b^{(1)}(u)}=\phi_3^{(1)}(v)-\lambda\phi_3^{(2)}(v). \tag{16}$$

Since the left hand side (respectively, the right hand side) of (16) depends only on u (respectively, v), there exists a nonzero number $\mu\in\mathbf{R}\backslash\{0\}$ satisfying

$$\frac{\alpha^{(2)}(u)-\alpha^{(1)}(u)}{\alpha^{(1)}(u)k_b^{(1)}(u)}=\phi_3^{(1)}(v)-\lambda\phi_3^{(2)}(v)=\mu.$$

Therefore we obtain (7). Hence we have proved Theorem 1.2. □

Proof of Theorem 1.3. We may suppose

$$\beta^{(1)}=\beta^{(2)}=1. \tag{17}$$

By (7), we obtain

$$\alpha^{(1)}(u)+\phi_3^{(1)}(v)\alpha^{(1)}(u)k_b^{(1)}(u)=\alpha^{(2)}(u)+\lambda\phi_3^{(2)}(v)\alpha^{(1)}(u)k_b^{(1)}(u). \tag{18}$$

By (6), we obtain

$$\alpha^{(2)}(u)k_b^{(2)}(u)=\lambda\alpha^{(1)}(u)k_b^{(1)}(u). \tag{19}$$

Therefore applying (19) into (18), we obtain

$$(1+\phi_3^{(1)}(v)k_b^{(1)}(u))\alpha^{(1)}(u)=(1+\phi_3^{(2)}(v)k_b^{(2)}(u))\alpha^{(2)}(u). \tag{20}$$

From Proposition 2.1, (17) and (20), we obtain

$$E^{(1)}=E^{(2)},\quad G^{(1)}=G^{(2)}.$$

Therefore canonical parallel curved surfaces $S_{(C_b^{(1)},C_g^{(1)})}$ and $S_{(C_b^{(2)},C_g^{(2)})}$, respectively, have the same semisurface structure. Hence we have proved Theorem 1.3. □

4. Examples of canonical parallel curved surfaces

Example. Let C_b and C_g be ellipses. Then we shall compute the Gaussian curvature of the corresponding canonical parallel curved surface $S_{(C_b,C_g)}$. We set

$$\phi_1(u):=a\cos u,\quad \phi_2(u):=b\sin u,$$
$$\phi_3(v):=c+c\cos v,\quad \phi_4(v):=d\sin v,$$

where a, b, c, $d>0$. Then since

$$\phi_1'(u)=-a\sin u,\quad \phi_2'(u)=b\cos u,$$
$$\phi_3'(v)=-c\sin v,\quad \phi_4'(v)=d\cos v,$$

the map Φ is represented as follows:

$$\Phi(u,\ v):=\begin{pmatrix} a\cos u \\ b\sin u \\ 0 \end{pmatrix}+\frac{c+c\cos v}{\sqrt{b^2\cos^2 u+a^2\sin^2 u}}\begin{pmatrix} b\cos u \\ a\sin u \\ 0 \end{pmatrix}+d\sin v\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The following hold:

$$\phi_1''(u)=-a\cos u,\quad \phi_2''(u)=-b\sin u,$$
$$\phi_3''(v)=-c\cos v,\quad \phi_4''(v)=-d\sin v.$$

Therefore if we set

$$\alpha(u):=\sqrt{b^2\cos^2 u+a^2\sin^2 u},\quad \beta(v):=\sqrt{d^2\cos^2 v+c^2\sin^2 v},$$

then we obtain

$$k_b(u)=\frac{\phi_1'(u)\phi_2''(u)-\phi_1''(u)\phi_2'(u)}{\alpha(u)^3}=\frac{ab}{\alpha(u)^3},$$
$$k_g(v)=\frac{\phi_3'(v)\phi_4''(v)-\phi_3''(v)\phi_4'(v)}{\beta(v)^3}=\frac{cd}{\beta(v)^3}.$$

Therefore we obtain

$$\begin{aligned} K&=d\cos v\cdot\frac{ab}{\alpha(u)^3}\cdot\frac{cd}{\beta(v)^3}\cdot\frac{1}{(1+\phi_3(v)ab/\alpha(u)^3)\beta(v)} \\ &=\frac{abcd^2\cos v}{(\alpha(u)^3+abc(1+\cos v))\beta(v)^4}. \end{aligned}$$

Suppose that C_b and C_g are circles, that is, set $a=b=R$, $c=d=r$. Then the map Φ is represented as follows:

$$\Phi(u,\ v):=\begin{pmatrix} R\cos u \\ R\sin u \\ 0 \end{pmatrix}+r(1+\cos v)\begin{pmatrix} \cos u \\ \sin u \\ 0 \end{pmatrix}+r\sin v\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix};$$

the Gaussian curvature K is represented as follows:

$$K=r\cos v\cdot\frac{1}{R}\cdot\frac{1}{r}\cdot\frac{1}{(1+r(1+\cos v)\cdot 1/R)r}=\frac{\cos v}{r(R+r(1+\cos v))}.$$

Example. Let C_b be an ellipse, and C_g a parabola. Then we shall compute the Gaussian curvature of the corresponding canonical parallel curved surface $S_{(C_b,C_g)}$. We set

$$\phi_1(u):=a\cos u,\quad \phi_2(u):=b\sin u,\quad \phi_3(v):=v^2/2,\quad \phi_4(v):=v.$$

Then since

$$\phi_1'(u)=-a\sin u,\quad \phi_2'(u)=b\cos u,\quad \phi_3'(v)=v,\quad \phi_4'(v)=1,$$

the map Φ is represented as follows:

$$\Phi(u,\ v):=\begin{pmatrix} a\cos u \\ b\sin u \\ 0 \end{pmatrix}+\frac{v^2}{2\sqrt{b^2\cos^2 u+a^2\sin^2 u}}\begin{pmatrix} b\cos u \\ a\sin u \\ 0 \end{pmatrix}+v\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The following hold:

$$\phi_1''(u)=-a\cos u,\quad \phi_2''(u)=-b\sin u,\quad \phi_3''(v)=1,\quad \phi_4''(v)=0.$$

Therefore if we set

$$\alpha(u):=\sqrt{b^2\cos^2 u+a^2\sin^2 u},\quad \beta(v):=\sqrt{1+v^2},$$

then we obtain

$$k_b(u)=\frac{\phi_1'(u)\phi_2''(u)-\phi_1''(u)\phi_2'(u)}{\alpha(u)^3}=\frac{ab}{\alpha(u)^3},$$

$$k_g(v)=\frac{\phi_3'(v)\phi_4''(v)-\phi_3''(v)\phi_4'(v)}{\beta(v)^3}=-\frac{1}{\beta(v)^3}.$$

Therefore we obtain

$$K=\frac{ab}{\alpha(u)^3}\cdot\frac{-1}{\beta(v)^3}\cdot\frac{1}{(1+abv^2/2\alpha(u)^3)\beta(v)}=-\frac{ab}{(\alpha(u)^3+abv^2/2)\beta(v)^4}.$$

Example. Let C_b be a parabola and C_g an ellipse. Then we shall compute the Gaussian curvature of the corresponding canonical parallel curved surface $S_{(C_b,C_g)}$. We set

$$\phi_1(u):=u^2/2,\quad \phi_2(u):=u,\quad \phi_3(v):=-c+c\cos v,\quad \phi_4(v):=d\sin v.$$

Then since

$$\phi'_1(u)=u,\quad \phi'_2(u)=1,\quad \phi'_3(v)=-c\sin v,\quad \phi'_4(v)=d\cos v,$$

the map Φ is represented as follows:

$$\Phi(u,\ v)=\begin{pmatrix} u^2/2 \\ u \\ 0 \end{pmatrix}-\frac{(c-c\cos v)}{\sqrt{1+u^2}}\begin{pmatrix} 1 \\ -u \\ 0 \end{pmatrix}+d\sin v\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The following hold:

$$\phi''_1(u)=1,\quad \phi''_2(u)=0,\quad \phi''_3(v)=-c\cos v,\quad \phi''_4(v)=-d\sin v.$$

Therefore if we set

$$\alpha(u):=\sqrt{1+u^2},\quad \beta(v):=\sqrt{d^2\cos^2 v+c^2\sin^2 v},$$

then we obtain

$$k_b(u)=\frac{\phi'_1(u)\phi''_2(u)-\phi''_1(u)\phi'_2(u)}{\alpha(u)^3}=-\frac{1}{\alpha(u)^3},$$

$$k_g(v)=\frac{\phi'_3(v)\phi''_4(v)-\phi''_3(v)\phi'_4(v)}{\beta(v)^3}=\frac{cd}{\beta(v)^3}$$

Therefore we obtain

$$K=d\cos v\cdot\frac{-1}{\alpha(u)^3}\cdot\frac{cd}{\beta(v)^3}\cdot\frac{1}{(1+c(1-\cos v)/\alpha(u)^3)\beta(v)}$$

$$=-\frac{cd^2\cos v}{(\alpha(u)^3+c(1-\cos v))\beta(v)^4}.$$

Example. Let C_b and C_g be parabolas. Then we shall compute the Gaussian curvature of the corresponding canonical parallel curved surface $S_{(C_b,C_g)}$. We set

$$\phi_1(u):=u^2/2,\quad \phi_2(u):=u,\quad \phi_3(v):=-v^2/2,\quad \phi_4(v):=v.$$

Then since

$$\phi'_1(u)=u,\quad \phi'_2(u)=1,\quad \phi'_3(v)=-v,\quad \phi'_4(v)=1,$$

the map Φ is represented as follows:

$$\Phi(u, v)=\begin{pmatrix} u^2/2 \\ u \\ 0 \end{pmatrix}-\frac{v^2}{2\sqrt{1+u^2}}\begin{pmatrix} 1 \\ -u \\ 0 \end{pmatrix}+v\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The following hold:

$$\phi_1''(u)=1, \quad \phi_2''(u)=0, \quad \phi_3''(v)=-1, \quad \phi_4''(v)=0.$$

Therefore if we set

$$\alpha(u):=\sqrt{1+u^2}, \quad \beta(v):=\sqrt{1+v^2},$$

then we obtain

$$k_b(u)=\frac{\phi_1'(u)\phi_2''(u)-\phi_1''(u)\phi_2'(u)}{\alpha(u)^3}=-\frac{1}{\alpha(u)^3},$$

$$k_g(v)=\frac{\phi_3'(v)\phi_4''(v)-\phi_3''(v)\phi_4'(v)}{\beta(v)^3}=\frac{1}{\beta(v)^3}.$$

Therefore we obtain

$$K=\frac{-1}{\alpha(u)^3}\cdot\frac{1}{\beta(v)^3}\cdot\frac{1}{(1+v^2/2\alpha(u)^3)\beta(v)}=-\frac{1}{(\alpha(u)^3+v^2/2)\beta(v)^4}.$$

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