The Gaussian curvature of a parallel curved surface

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Abstract

We shall represent the Gaussian curvature of a canonical parallel curved surface in terms of a generating pair. In addition, we shall study a relation between generating pairs such that the corresponding canonical parallel curved surfaces have the same semisurface structure.

1. Introduction

A surface $S$ in $\mathbb{R}^3$ is called parallel curved if there exists a plane $P$ such that one of the principal directions of $S$ at each point is parallel to $P$. If $S$ is parallel curved, then such a plane as $P$ is called a base plane of $S$. A parallel curved surface $S$ is called canonical if there exist smooth functions $\phi_1, \phi_2$ on an open interval $I$ and smooth functions $\phi_3, \phi_4$ on an open interval $J$ satisfying $a := \sqrt{(\phi'_1)^2 + (\phi'_2)^2} \neq 0$ on $I$, $b := \sqrt{(\phi'_3)^2 + (\phi'_4)^2} \neq 0$ on $J$, and the condition that $S$ is the image of $I \times J$ by a map $\Phi$ defined by

$$\Phi(u, v) := (\phi_1(u) + \phi_2(v)\phi'_2(u)/a(u))e_1 + (\phi_3(u) - \phi_4(v)\phi'_4(u)/a(u))e_2 + \phi_2(v)e_3 + c,$$

(1)

where $c$ is a vector of $\mathbb{R}^3$ and $(e_1, e_2, e_3)$ is an orthonormal basis of $\mathbb{R}^3$ such that $e_1$ and $e_2$ form a basis of a base plane of $S$. Parallel curved surfaces were studied in [1], [2] and [3].

Let $C_0$ and $C_\theta$ be simple curves in $\mathbb{R}^3$ with a unique intersection $p(C_0, C_\theta)$ and contained in planes $P_0$ and $P_\theta$, respectively. A pair $(C_0, C_\theta)$ is called generating if we may choose as $P_0$ the plane normal to $C_0$ at $p(C_0, C_\theta)$; if $(C_0, C_\theta)$ is generating, then $C_0$ and $C_\theta$ are called the base curve and the generating curve of $(C_0, C_\theta)$, respectively. For a canonical parallel curved surface $S$ with a base plane $P_0$ and each point $q$ of $S$, there exists a generating pair $(C_0, C_\theta)$ satisfying the following:

1. $C_0$ and $C_\theta$ are lines of curvature in $S$ through $q$,

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2. $P_b$ is parallel to $P_b$ (therefore $P_b$ is a base plane of $S$) (see [2]). In addition, for a generating pair $(C_b, C_p)$, there exists a canonical parallel curved surface $S$ satisfying the following:

1. $S$ contains a neighborhood $O_b$ (respectively, $O_p$) of $\rho(C_b, C_p)$ in $C_b$ (respectively, $C_p$) so that $O_b$ and $O_p$ are lines of curvature in $S$;

2. $P_b$ is a base plane of $S$

(see [2]). We say that $S$ is generated by $(C_b, C_p)$.

In the sequel, suppose that for a generating pair $(C_b, C_p)$, there exists a canonical parallel curved surface $S$ satisfying the above 1 and 2 with $O_b = C_b$ and $O_p = C_p$. The minimum of such canonical parallel curved surfaces as $S$ is denoted by $S_{(C_b, C_p)}$. For example, if $C_b$ and $C_p$ are isometric to $\mathbb{R}$, then $S_{(C_b, C_p)}$ is homeomorphic to a plane; if one of $C_b$ and $C_p$ is isometric to $\mathbb{R}$ and the other isometric to a closed curve, then $S_{(C_b, C_p)}$ is homeomorphic to a cylinder; if $C_b$ and $C_p$ are closed curves, then $S_{(C_b, C_p)}$ is homeomorphic to a torus. If $S$ is a connected, complete, real-analytic, embedded, parallel curved surface, then $S$ is homeomorphic to a sphere, a plane, a cylinder, or a torus, and in addition,

1. if $S$ is homeomorphic to a sphere, then $S$ is a surface of revolution which crosses its axis of rotation at just two points;

2. if $S$ is homeomorphic to a plane, then one of the following holds:

   (a) $S$ is a surface of revolution which crosses its axis of rotation at just one point,
   (b) $S = S_{(C_b, C_p)}$ holds, where $(C_b, C_p)$ is a generating pair each element of which is isometric to $\mathbb{R}$;

3. if $S$ is homeomorphic to a cylinder, then $S = S_{(C_b, C_p)}$ holds, where $(C_b, C_p)$ is a generating pair such that one of $C_b$ and $C_p$ is isometric to $\mathbb{R}$ and the other a closed curve;

4. if $S$ is homeomorphic to a torus, then $S = S_{(C_b, C_p)}$ holds, where $(C_b, C_p)$ is a generating pair each element of which is a closed curve

(see [1]).

The first purpose of the present paper is to represent the Gaussian curvature of a canonical parallel curved surface in terms of a generating pair. For a canonical parallel curved surface $S_{(C_b, C_p)}$, we suppose
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\[ C_b = \{ \Phi(u, v_0); u \in I \}, \quad C_v = \{ \Phi(u_0, v); v \in J \} \]

for some \( u_0 \in I \) and some \( v_0 \in J \) satisfying \( \phi_0(v_0) = 0 \). In addition, we suppose \( e_1 := (1, 0, 0), e_2 := (0, 1, 0), e_3 := (0, 0, 1), c := (0, 0, 0) \) in (1). Then the map \( \Phi \) is represented as follows:

\[
\Phi(u, v) := \begin{pmatrix}
\phi_1(u) \\
\phi_2(u) \\
\phi_3(u)
\end{pmatrix} + \begin{pmatrix}
\phi_1'(u) \\
\phi_2'(u) \\
\phi_3'(u)
\end{pmatrix} + \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}.
\]

(2)

Let \( k_b \) and \( k_v \) be the curvatures of \( C_b \) and \( C_v \), respectively. We shall prove

**Theorem 1.1.** The Gaussian curvature \( K \) of a canonical parallel curved surface \( S(c_0, c_v) \) is represented as follows:

\[
K = \frac{\phi_2'(v)k_b(u)k_v(v)}{(1 + \phi_3(v)k_0(u))\beta(v)}.
\]

(3)

**Remark.** From (3), we see that if \( C_b \) and \( C_v \) are closed and convex, then the sign of the Gaussian curvature of a canonical parallel curved surface \( S(c_0, c_v) \) depends only on \( \phi_3 \), and that the set of the zero points of the Gaussian curvature is represented as \( C_{0,1} \cup C_{0,2} \), where \( C_{0,1} \) and \( C_{0,2} \) are simple closed curves in \( S(c_0, c_v) \) such that each of \( C_{0,1} \) and \( C_{0,2} \) is contained in a base plane of \( S(c_0, c_v) \).

The second purpose of the present paper is to study a relation between generating pairs such that the corresponding canonical parallel curved surfaces have the same semi-surface structure. Let \( S \) be a surface in \( \mathbb{R}^3 \) without any umbilical point and \( p \) a point of \( S \). Then there exist local coordinates \( (u, v) \) on a neighborhood \( U_p \) of \( p \) such that two principal distributions are given by \( \partial/\partial u \) and \( \partial/\partial v \) on \( U_p \). Such coordinates are said to be compatible with principal distributions on \( S \). The first fundamental form of \( S \) is represented as \( A^2 du^2 + B^2 dv^2 \) on \( U_p \), where \( A \) and \( B \) are smooth, positive-valued functions on \( U_p \). Let \( k_1 \) and \( k_2 \) be two principal curvature functions corresponding to \( \partial/\partial u \) and \( \partial/\partial v \), respectively. If the Gaussian curvature \( K \) of \( S \) is nowhere zero on \( U_p \), then on \( U_p \), the following holds ([3]):

\[
c_2k_1^2 + c_1k_1k_2 + c_0k_2^2 = 0,
\]

where
\[\begin{align*}
c_{20} := & (\log|K|A^2)_{uv} - (\log B)_{uu} \\
c_{11} := & (\log|K|AB)_{uu} - 4(\log A)_{uv}(\log B)_{u} \\
c_{22} := & (\log|K|B^2)_{uv}(\log A)_{u} - (\log A)_{uu}
\end{align*}\]

Noticing the equation of Gauss:

\[K = -\frac{1}{AB} \left( \frac{A_{uv}}{B} \right)_u + \left( \frac{B_{uv}}{A} \right)_v\]

we see that \(c_{20}, c_{11}, c_{22}\) in (4) depend only on \(A, B\) and their partial derivatives. In addition, if \((u', v')\) are local coordinates on \(U_p\) compatible with principal distributions such that at each point of \(U_p\), \(\partial/\partial u'\) (respectively, \(\partial/\partial v'\)) is represented as \(\partial/\partial u\) (respectively, \(\partial/\partial v\)) up to a nonzero constant, then the coefficients \(c_{20}', c_{11}', c_{22}'\) defined as in (4) for the coordinates \((u', v')\) satisfy \((c_{20}', c_{11}', c_{22}') = \lambda(c_{20}, c_{11}, c_{22})\) for some \(\lambda \in \mathbb{R} \setminus \{0\}\) at each point of \(U_p\). For a surface \(S\) in \(\mathbb{R}^3\) without any umbilical point and with nowhere zero Gaussian curvature, a neighborhood of each point of \(S\) is a canonical parallel curved surface if and only if \(S\) satisfies the following:

1. The integral curves of just one of the principal distributions on \(S\) are geodesics;
2. \(c_{20}, c_{11}\) and \(c_{22}\) in (4) vanish on \(S\)

(see [3]).

Let \(M\) be a two-dimensional Riemannian manifold and \(g\) its metric. Let \(D_1, D_2\) be two smooth one-dimensional distributions on \(M\). A Riemannian manifold \((M, g)\) equipped with \((D_1, D_2)\) is called a semisurface if \(D_1\) and \(D_2\) are orthogonal to each other at any point of \(M\) with respect to \(g\). If \((M, g, D_1, D_2)\) is a semisurface, then a triplet \((g, D_1, D_2)\) is a semisurface structure of \(M\). Let \((M, g, D_1, D_2)\) be a semisurface. Then for each point \(p \in M\), there exist local coordinates \((u, v)\) on a neighborhood \(U_p\) of \(p\) satisfying \(\partial/\partial u \in D_1\) and \(\partial/\partial v \in D_2\) on \(U_p\). Such coordinates are said to be compatible with \((D_1, D_2)\). The Riemannian metric \(g\) may be represented as \(g = A^2 du^2 + B^2 dv^2\) on \(U_p\). Suppose that the curvature \(K\) of the Riemannian manifold \((M, g)\) is nowhere zero on \(U_p\). Then for each point \(p\) of \(U_p\), a homogeneous polynomial

\[P_{M,s}(X_1, X_2) := c_{20}(q)X_1^2 + c_{11}(q)X_1X_2 + c_{22}(q)X_2^2\]

where \(c_{20}, c_{11}, c_{22}\) are defined as in (4), is determined by a given semisurface structure of \(M\) up to a nonzero constant. This polynomial \(P_{M,s}\) is called a Codazzi-Mainardi polynomial of a semisurface \((M, g, D_1, D_2)\) at \(q\). Let \((M, g, D_1, D_2)\) be a semisurface with nowhere zero curvature. Suppose that the integral curves of just one of \(D_1\) and \(D_2\) are geodesics and that a
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Codazzi-Mainardi polynomial vanishes at any point of $M$. Then there exist plural isometric immersions of a neighborhood $U_p$ of each point $p$ of $M$ into $\mathbb{R}^3$ satisfying the following ([3]):

1. the images are canonical parallel curved surfaces such that arbitrary two in them are not congruent with each other in $\mathbb{R}^3$;

2. $(D_1, D_2)$ gives a pair of two principal distributions.

Let $\Phi^{(1)}$ and $\Phi^{(2)}$ be isometric immersions of $U_p$ into $\mathbb{R}^3$ as above. For $j \in \{1, 2\}$, let $(C^j_x, C^j_y)$ be a generating pair satisfying $\Phi^{(j)}(U_p) = S(c_x^j, c_y^j)$. In the present paper, we shall prove

**Theorem 1.2.** Let $(u, v)$ be compatible with $(D_1, D_2)$. Then there exist nonzero numbers $\lambda, \mu \in \mathbb{R}\setminus\{0\}$ satisfying the following:

$$
\frac{\partial \gamma^{(j)}(u)}{\partial u^{(j)}(u)} \frac{k^{(j)}(u)}{k_{z}^{(j)}(u)} = \frac{\partial \gamma^{(j)}(u)}{\partial u^{(j)}(u)} \frac{k^{(j)}(v)}{k_{z}^{(j)}(v)} = \lambda, \tag{6}
$$

$$
\frac{\partial \gamma^{(j)}(u) - \partial \gamma^{(j)}(u)}{\partial u^{(j)}(u)} \eta_{z}^{(j)}(u) = \phi^{(j)}(v) - \lambda \phi^{(j)}(v) = \mu. \tag{7}
$$

Noticing that (6) and (7) give a relation between two generating pairs $(C^{(1)}_x, C^{(1)}_y)$ and $(C^{(2)}_x, C^{(2)}_y)$, we shall also prove

**Theorem 1.3.** Let $(C^{(1)}_x, C^{(1)}_y)$ and $(C^{(2)}_x, C^{(2)}_y)$ be generating pairs satisfying (6) and (7) for some $\lambda, \mu \in \mathbb{R}\setminus\{0\}$. Then the corresponding canonical parallel curved surfaces $S(c_x^{(1)}, c_y^{(1)})$ and $S(c_x^{(2)}, c_y^{(2)})$ have the same semisurface structure, i.e., there exist a semisurface $(M, g, D_1, D_2)$ and isometric immersions $\Phi^{(1)}, \Phi^{(2)} : M \rightarrow \mathbb{R}^3$ such that for $j \in \{1, 2\}$, the following hold:

1. $\Phi^{(j)}(M) = S(c_x^j, c_y^j)$;

2. $d\Phi^{(j)}(D_i)$ is a principal distribution on $\Phi^{(j)}(M)$.

**2. Proof of Theorem 1.1**

We shall prove

**Proposition 2.1.** The coefficients of the first fundamental form $Edu^2 + 2Fdudv + Gdv^2$ of $S(c_x, c_y)$ are represented as follows:

$$
E = (1 + \phi(v)k_z(u))^2\alpha(u)^2, \quad F = 0, \quad G = \beta(v)^2.
$$
Proof. The following hold:

\[
\Phi_v = \begin{pmatrix}
\phi'(u) \\
\phi''(u) \\
0 \\
\end{pmatrix} + \frac{\phi_u(v)}{\alpha(u)} \begin{pmatrix}
\phi''(v) \\
-\phi'(v) \\
0 \\
\end{pmatrix} + \frac{\phi_v(v)}{\alpha(u)} \begin{pmatrix}
\phi''(v) \\
-\phi'(v) \\
0 \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\phi'(u) \\
\phi''(u) \\
0 \\
\end{pmatrix} - \frac{\phi_u(v)}{\alpha(u)} \left(\phi'(u)\phi''(u) + \phi''(u)\phi'(u)\right) \begin{pmatrix}
\phi''(v) \\
-\phi'(v) \\
0 \\
\end{pmatrix} + \frac{\phi_v(v)}{\alpha(u)} \begin{pmatrix}
\phi''(v) \\
-\phi'(v) \\
0 \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\phi'(u) \\
\phi''(u) \\
0 \\
\end{pmatrix} - \frac{\phi_u(v)}{\alpha(u)} \left(\phi'(u)\phi''(u) + \phi''(u)\phi'(u)\right) \begin{pmatrix}
\phi''(v) \\
-\phi'(v) \\
0 \\
\end{pmatrix} + \alpha(u)^2 \begin{pmatrix}
0 \\
0 \\
\phi''(u) \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\phi'(u) \\
\phi''(u) \\
0 \\
\end{pmatrix} - \frac{\phi_u(v)}{\alpha(u)} \left(\phi'(u)\phi''(u) - \phi''(u)\phi'(u)\right) \begin{pmatrix}
\phi'(v) \\
\phi''(v) \\
0 \\
\end{pmatrix}
\]

\[
= \left[1 + \frac{\phi_u(v)}{\alpha(u)} \left(\phi'(u)\phi''(u) - \phi''(u)\phi'(u)\right)\right] \begin{pmatrix}
\phi'(u) \\
\phi''(u) \\
0 \\
\end{pmatrix}
\]

Since the curvature \(k_o(u)\) of the plane curve given by \((\phi_1, \phi_2)\) is represented as

\[
k_o(u) = \frac{\phi'(u)\phi''(u) - \phi''(u)\phi'(u)}{\alpha(u)^2},
\]

we obtain

\[
\Phi_v = (1 + \phi_u(v)k_o(u)) \begin{pmatrix}
\phi'(u) \\
\phi''(u) \\
0 \\
\end{pmatrix}
\]

The following holds:
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\[
\Phi_v = \frac{\phi_2^2(v)}{\alpha(u)} \begin{pmatrix} \phi_2(u) \\ -\phi_1(u) \\ 0 \end{pmatrix} + \phi_4(v) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

Therefore we obtain

\[
E = \Phi_u \cdot \Phi_v = (1 + \phi_3(v)k_3(u))^2\alpha(u)^2, \quad F = \Phi_u \cdot \Phi_v = 0,
\]

\[
G = \Phi_u \cdot \Phi_v = \frac{\phi_2^2(v)}{\alpha(u)} - \alpha(u)^2 + \phi_4^2(v) = \beta(v)^2.
\]

**Proposition 2.2.** A unit normal vector field of \(S(\alpha, \beta)\) is represented as follows:

\[
e = \frac{\phi_4(v)}{\alpha(u)\beta(v)} \begin{pmatrix} \phi_2(u) \\ -\phi_1(u) \\ 0 \end{pmatrix} - \frac{\phi_2^2(v)}{\alpha(u)\beta(v)} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

**Proof.** The following hold:

\[
\Phi_u \times \Phi_v = (1 + \phi_3(v)k_3(u)) \begin{pmatrix} \phi_1(u) \\ \phi_2(u) \\ 0 \end{pmatrix} \times \left[ \begin{pmatrix} \phi_1(u) \\ \phi_2(u) \\ 0 \end{pmatrix} \right] = -(1 + \phi_3(v)k_3(u))\alpha(u)\phi_3(v) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (1 + \phi_3(v)k_3(u))\phi_4(v) \begin{pmatrix} \phi_2(u) \\ 0 \\ 0 \end{pmatrix}.
\]

Then we obtain

\[
|\Phi_u \times \Phi_v|^2 = \alpha(u)^2(1 + \phi_3(v)k_3(u))^2(\phi_3(v)^2 + \phi_4(v)^2)
\]

and

\[
|\Phi_u \times \Phi_v| = \alpha(u)(1 + \phi_3(v)k_3(u))\sqrt{\phi_3(v)^2 + \phi_4(v)^2}.
\]

Therefore we obtain

\[
e = \frac{\Phi_u \times \Phi_v}{|\Phi_u \times \Phi_v|} = \frac{\phi_4(v)}{\alpha(u)\beta(v)} \begin{pmatrix} \phi_2(u) \\ -\phi_1(u) \\ 0 \end{pmatrix} - \frac{\phi_2^2(v)}{\alpha(u)\beta(v)} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

\[
\square
\]
Proposition 2.3. The coefficients of the second fundamental form \( Ldu^2 + 2Mdudv + Ndv^2 \) of \( S_{(c_0, c_d)} \) are represented as follows:

\[
L = -(1 + \phi_3(v)k_3(u))\frac{\phi_1'(u)k_3(u)\sigma(u)^2}{\beta(v)}, \quad M = 0,
\]
\[
N = -k_3(v)\beta(v)^2.
\]

Proof. The following hold:

\[
\Phi_{uu} = (1 + \phi_3(v)k_3(u))\begin{pmatrix}
\phi_1'(u) \\
\phi_2'(u) \\
0
\end{pmatrix} + (1 + \phi_3(v)k_3(u))\begin{pmatrix}
\phi_3'(u) \\
\phi_4'(u) \\
0
\end{pmatrix},
\]
\[
\Phi_{uv} = \phi_3'(v)k_3(u)\begin{pmatrix}
\phi_1'(u) \\
\phi_2'(u) \\
0
\end{pmatrix}, \quad \Phi_{vv} = \frac{\phi_3'(v)}{\alpha(u)}\begin{pmatrix}
\phi_3'(u) \\
-\phi_1'(u) + \phi_3'(v) \\
0
\end{pmatrix}.
\]

Therefore we obtain

\[
L = \Phi_{uu} \cdot e = (1 + \phi_3(v)k_3(u))\frac{\phi_1'(u)k_3(u)\sigma(u)^2}{\beta(v)},
\]
\[
M = \Phi_{uu} \cdot e = 0,
\]
\[
N = \Phi_{uv} \cdot e = \frac{\phi_3'(v)\phi_2'(v)}{\alpha(u)\beta(v)}\phi_3'(u) - \frac{\phi_3'(v)\phi_3'(u)}{\beta(v)}.
\]

Since the curvature \( k_3(v) \) of the plane curve given by \( (\phi_3, \phi_4) \) is represented as

\[
k_3(v) = \frac{\phi_3'(v)\phi_4'(v) - \phi_3'(v)\phi_3'(v)}{\beta(v)^2},
\]

we obtain

\[
N = -k_3(v)\beta(v)^2.
\]

By Propositions 2.1, 2.2 and 2.3, we obtain
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\[
K = \frac{LN}{EG} = \frac{(1 + \phi_1(v)k_1(u))\phi_2(v)k_2(u)k_3(v)\alpha(u)\beta(v)}{(1 + \phi_2(v)k_2(u))^2\alpha(u)^2\beta(v)^2}
\]

\[= \frac{\phi_2(v)k_2(u)k_3(v)}{(1 + \phi_2(v)k_2(u))\beta(v)}.\]

Hence we have proved Theorem 1.1.

3. Proofs of Theorem 1.2 and Theorem 1.3

Proof of Theorem 1.2. Let \( g^{(1)} \) be the first fundamental form of a canonical parallel curved surface \( S^{(1)} := \Phi^{(1)}(U) \) and \( K^{(1)} \) the Gaussian curvature of \( S^{(1)} \). Since \( S^{(1)} \) and \( S^{(2)} \) have the same semisurface structure, the following hold:

\[
K^{(1)} = K^{(2)},
\]

\[
E^{(1)} = E^{(2)}, \quad G^{(1)} = G^{(2)}.
\]

Therefore by (3) together with (8), we obtain

\[
\frac{(\phi_1^{(1)}(v)k_1^{(1)}(u))k_2^{(1)}(v)}{(1 + \phi_1^{(1)}(v)k_1^{(1)}(u))\beta^{(1)}(v)} = \frac{(\phi_1^{(2)}(v)k_1^{(2)}(u))k_2^{(2)}(v)}{(1 + \phi_1^{(2)}(v)k_1^{(2)}(u))\beta^{(2)}(v)}.
\]

From Proposition 2.1 and (9), we obtain

\[
(1 + \phi_1^{(1)}(v)k_1^{(1)}(u))\alpha^{(1)}(u) = (1 + \phi_1^{(2)}(v)k_1^{(2)}(u))\alpha^{(2)}(u),
\]

\[
\beta^{(1)}(v) = \beta^{(2)}(v).
\]

We rewrite (11) into

\[
\frac{(1 + \phi_1^{(1)}(v)k_1^{(1)}(u))}{(1 + \phi_1^{(2)}(v)k_1^{(2)}(u))} = \frac{\alpha^{(1)}(u)}{\alpha^{(2)}(u)}.
\]

Therefore from (10), (12) and (13), we obtain

\[
\frac{\alpha^{(1)}(u)k_1^{(1)}(u)}{\alpha^{(2)}(u)k_1^{(2)}(u)} = \frac{(\phi_1^{(1)}(v)k_1^{(1)}(v))}{(\phi_1^{(2)}(v)k_1^{(2)}(v))}
\]

Since the left hand side (respectively, the right hand side) of (14) depends only on \( u \) (respectively, \( v \)), there exists a nonzero number \( \lambda \in \mathbb{R}\setminus\{0\} \) satisfying

\[
\frac{\alpha^{(1)}(u)k_1^{(1)}(u)}{\alpha^{(2)}(u)k_1^{(2)}(u)} = \frac{(\phi_1^{(1)})(v)k_1^{(1)}(v)}{(\phi_1^{(2)})(v)k_1^{(2)}(v)} = \lambda.
\]

Therefore we obtain (6). From (6), we obtain

\[
\alpha^{(1)}(u)k_1^{(1)}(u) = \lambda\alpha^{(2)}(u)k_1^{(2)}(u).
\]
Therefore by (11) together with (15), we obtain

\[ \frac{a^{(3)}(u) - a^{(1)}(u)}{a^{(1)}(u) k^{(1)}_b(u)} = \phi^{(1)}_b(v) - \lambda \phi^{(1)}(v). \]  

(16)

Since the left hand side (respectively, the right hand side) of (16) depends only on \( u \) (respectively, \( v \)), there exists a nonzero number \( \mu \in \mathbb{R} \setminus \{0\} \) satisfying

\[ \frac{a^{(3)}(u) - a^{(1)}(u)}{a^{(1)}(u) k^{(1)}_b(u)} = \phi^{(1)}_b(v) - \lambda \phi^{(1)}(v) = \mu. \]

Therefore we obtain (7). Hence we have proved Theorem 1.2.

\[ \square \]

**Proof of Theorem 1.3.** We may suppose

\[ \beta^{(1)} = \beta^{(3)} = 1. \]  

(17)

By (7), we obtain

\[ a^{(1)}(u) + \phi^{(1)}_b(v) a^{(1)}(u) k^{(1)}_b(u) = a^{(3)}(u) + \lambda \phi^{(1)}(v) a^{(1)}(u) k^{(1)}_b(u). \]  

(18)

By (6), we obtain

\[ a^{(3)}(u) k^{(3)}_b(u) = \lambda a^{(1)}(u) k^{(1)}_b(u). \]  

(19)

Therefore applying (19) into (18), we obtain

\[ (1 + \phi^{(1)}(v) k^{(1)}_b(u)) a^{(1)}(u) = (1 + \phi^{(3)}_b(v) k^{(3)}_b(v)) a^{(3)}(u). \]  

(20)

From Proposition 2.1, (17) and (20), we obtain

\[ E^{(1)} = E^{(3)}, \quad G^{(1)} = G^{(3)}. \]

Therefore canonical parallel curved surfaces \( S_{cc}^{(1)}, c^{(1)}_b \) and \( S_{cc}^{(3)}, c^{(3)}_b \), respectively, have the same semisurface structure. Hence we have proved Theorem 1.3.

\[ \square \]

4. **Examples of canonical parallel curved surfaces**

**Example.** Let \( C_b \) and \( C_\sigma \) be ellipses. Then we shall compute the Gaussian curvature of the corresponding canonical parallel curved surface \( S_{(c_b, c_\sigma)} \). We set

\[ \phi_1(u) := a \cos u, \quad \phi_2(u) := b \sin u, \]

\[ \phi_3(v) := c + c \cos v, \quad \phi_4(v) := d \sin v, \]

where \( a, b, c, d > 0 \). Then since
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\[ \phi_i(u) = -a \sin u, \quad \phi_j(u) = b \cos u, \]
\[ \phi_i'(v) = -c \sin v, \quad \phi_j'(v) = d \cos v, \]

the map \( \Phi \) is represented as follows:

\[
\Phi(u, v) := \begin{pmatrix}
    a \cos u \\
    b \sin u \\
    0
\end{pmatrix} + \frac{c + c \cos v}{\sqrt{b^2 \cos^2 u + a^2 \sin^2 u}} \begin{pmatrix}
    b \cos u \\
    a \sin u \\
    0
\end{pmatrix} + d \sin v \begin{pmatrix}
    0 \\
    0 \\
    1
\end{pmatrix}.
\]

The following hold:

\[ \phi_i''(u) = -a \cos u, \quad \phi_j''(u) = -b \sin u, \]
\[ \phi_i''(v) = -c \cos v, \quad \phi_j''(v) = -d \sin v. \]

Therefore if we set

\[ a(u) := \sqrt{b^2 \cos^2 u + a^2 \sin^2 u}, \quad \beta(v) := \sqrt{d^2 \cos^2 v + c^2 \sin^2 v}, \]

then we obtain

\[ k_v(u) = \frac{\phi_i'(u) \phi_j''(u) - \phi_i''(u) \phi_j'(u)}{a(u)^3}, \]
\[ k_v(v) = \frac{\phi_i'(v) \phi_j''(v) - \phi_i''(v) \phi_j'(v)}{\beta(v)^3}. \]

Therefore we obtain

\[
K = dcov \left( \frac{ab}{a(u)^3} \frac{cd}{\beta(v)^3} \frac{1}{(1 + \phi_i'(u)ab/a(u)^3)\beta(v)} \right)
\]
\[ = \frac{abcd^3 \cos v}{(a(u)^3 + abc(1 + \cos v))\beta(v)^3}. \]

Suppose that \( C_\phi \) and \( C_\theta \) are circles, that is, set \( a = b = R, \ c = d = r. \) Then the map \( \Phi \) is represented as follows:

\[
\Phi(u, v) := \begin{pmatrix}
    R \cos u \\
    R \sin u \\
    0
\end{pmatrix} + r(1 + \cos v) \begin{pmatrix}
    \cos u \\
    \sin u \\
    0
\end{pmatrix} + r \sin v \begin{pmatrix}
    0 \\
    0 \\
    1
\end{pmatrix};
\]

the Gaussian curvature \( K \) is represented as follows:
\[ K = r \cos v \frac{\cos v}{R} \frac{1}{r} \frac{1}{(1 + \rho(1 + \cos v) \cdot 1/R)} = \frac{\cos v}{r(R + r(1 + \cos v))}. \]

**Example.** Let \( C_0 \) be an ellipse, and \( C_0 \) a parabola. Then we shall compute the Gaussian curvature of the corresponding canonical parallel curved surface \( S_{C_0,C_0} \). We set

\[ \phi_1(u) := \cos u, \quad \phi_2(u) := b \sin u, \quad \phi_3(v) := v^2/2, \quad \phi_4(v) := v. \]

Then since

\[ \phi_1'(u) = -a \sin u, \quad \phi_2'(u) = b \cos u, \quad \phi_3'(v) = v, \quad \phi_4'(v) = 1, \]

the map \( \Phi \) is represented as follows:

\[ \Phi(u,v) := \begin{pmatrix} \cos u \\ b \sin u \\ 0 \end{pmatrix} + \frac{v}{\beta(u)^2} \begin{pmatrix} b \cos u \\ a \sin u \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \]

The following hold:

\[ \phi_1''(u) = -a \cos u, \quad \phi_2''(u) = -b \sin u, \quad \phi_3''(v) = 1, \quad \phi_4''(v) = 0. \]

Therefore if we set

\[ a(u) := \sqrt{b^2 \cos^2 u + a^2 \sin^2 u}, \quad \beta(v) := \sqrt{1 + v^2}, \]

then we obtain

\[ k_0(u) = \frac{\phi_1''(u) \phi_2''(u) - \phi_1'(u) \phi_2''(u)}{a(u)^3} = \frac{ab}{a(u)^3}, \]

\[ k_0(v) = \frac{\phi_3''(v) \phi_4''(v) - \phi_3'(v) \phi_4''(v)}{\beta(v)^3} = -\frac{1}{\beta(v)^3}. \]

Therefore we obtain

\[ K = \frac{ab}{a(u)^3} \frac{-1}{\beta(v)^3} \frac{1}{(1 + abu^2/2a(u)^2)\beta(v)} = -\frac{ab}{(a(u)^3 + abu^2/2)\beta(v)^3}. \]

**Example.** Let \( C_0 \) be a parabola and \( C_0 \) an ellipse. Then we shall compute the Gaussian curvature of the corresponding canonical parallel curved surface \( S_{C_0,C_0} \). We set

\[ \phi_1(u) := u^2/2, \quad \phi_2(u) := u, \quad \phi_3(v) := -c + ccos v, \quad \phi_4(v) := d \sin v. \]

Then since
The Gaussian curvature of a parallel curved surface

\[ \phi'_i(u) = u, \quad \phi'_2(u) = 1, \quad \phi'_3(v) = -c\sin v, \quad \phi'_4(v) = d\cos v, \]

the map \( \Phi \) is represented as follows:

\[
\Phi(u, v) = \begin{pmatrix} u^2/2 \\ u \\ 0 \end{pmatrix} - \frac{(c - \cos v)}{\sqrt{1 + u^2}} \begin{pmatrix} 1 \\ -u \\ 0 \end{pmatrix} + d\sin v \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

The following hold:

\[ \phi_i(u) = 1, \quad \phi_2(u) = 0, \quad \phi_3(v) = -c\cos v, \quad \phi_4(v) = -d\sin v. \]

Therefore if we set

\[ \alpha(u) := \sqrt{1 + u^2}, \quad \beta(v) := \sqrt{d^2 \cos^2 v + c^2 \sin^2 v}, \]

then we obtain

\[
k_\phi(u) = \phi'_i(u)\phi''_i(u) - \phi'_i(u)\phi'_i(u) = -\frac{1}{\alpha(u)^3},
\]

\[
k_\psi(v) = \phi'_3(v)\phi''_3(v) - \phi'_3(v)\phi'_3(v) = \frac{cd}{\beta(v)^3}.
\]

Therefore we obtain

\[
K = d\cos v \cdot \frac{-1}{\alpha(u)^3} \cdot \frac{cd}{\beta(v)^3} \cdot \frac{1}{(1 + c(1 - \cos v)/\alpha(u)^3)\beta(v)}
\]

\[
= \frac{cd^2 \cos v}{(\alpha(u)^3 + c(1 - \cos v))\beta(v)^3}.
\]

**Example.** Let \( C_\phi \) and \( C_\sigma \) be parabolas. Then we shall compute the Gaussian curvature of the corresponding canonical parallel curved surface \( S_{(C_\phi, C_\sigma)} \). We set

\[ \phi_1(u) := u^2/2, \quad \phi_2(u) := u, \quad \phi_3(v) := -v^2/2, \quad \phi_4(v) := v. \]

Then since

\[ \phi'_1(u) = u, \quad \phi'_2(u) = 1, \quad \phi'_3(v) = -v, \quad \phi'_4(v) = 1, \]

the map \( \Phi \) is represented as follows:
\[ \Phi(u, v) = \begin{pmatrix} \frac{u^2}{2} \\ u \\ 0 \end{pmatrix} - \frac{v^2}{2\sqrt{1+u^2}} \begin{pmatrix} 1 \\ -u \\ 0 \end{pmatrix} + v \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \]

The following hold:

\[ \phi_1(u) = 1, \quad \phi_2(u) = 0, \quad \phi_3(u) = -1, \quad \phi_4(u) = 0. \]

Therefore if we set

\[ \alpha(u) := \sqrt{1+u^2}, \quad \beta(v) := \sqrt{1+v^2}, \]

then we obtain

\[ k_\alpha(u) = \frac{\phi_1(u)\phi_2(u) - \phi_3(u)\phi_4(u)}{\alpha(u)^2} = -\frac{1}{\alpha(u)^3}, \]

\[ k_\beta(v) = \frac{\phi_3(v)\phi_4(v) - \phi_1(v)\phi_2(v)}{\beta(v)^3} = \frac{1}{\beta(v)^3}. \]

Therefore we obtain

\[ K = -\frac{1}{\alpha(u)^2} \cdot \frac{1}{\beta(v)^3} \cdot \frac{1}{(1+v^2/2\alpha(u)^2)\beta(v)} = -\frac{1}{(\alpha(u)^3 + v^2/2)\beta(v)^3}. \]

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References


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