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On perfect isometries for blocks with abelian defect groups and cyclic hyperfocal subgroups

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Let p be a prime number and $(K, \mathcal{O}, F)$ be a p-modular system such that K is algebraically closed. Let G be a finite group and B be a (p-)block of G with defect group P. Also let B_0 be the Brauer correspondent of B, that is, B_0 is the block of $N_G(P)$ associated with B. M. Broué conjectured in [1] that there exists a perfect isometry between B and B_0 when P is abelian. This is verified for several blocks as in [1], §5. When $p=2$, Broué's conjecture on perfect isometries is true for principal 2-blocks ([4]). Let b be a block of $C_G(P)$ associated with B. When $N_G(P,b)/C_G(P)$ is an elementary abelian 2-group or when $|N_G(P,b)/C_G(P)|$ is small, Broué's conjecture on perfect isometries is true for B ([9], [11], [10], [12], [13] and [14]). See [5] also. In this article we show Broué's conjecture on perfect isometry is true when $[(N_G(P,b), P]$ is cyclic (see Corollary below). Note that the commutator subgroup $[(N_G(P,b), P]$ of $N_G(P,b)$ and P is a hyperfocal subgroup of a defect pointed group of a pointed group $G_{\{B\}}$ on $\mathcal{O}G$. (For the definition of hyperfocal subgroups, see [8])

1. Perfect isometry

Let $\mathcal{CF}_K(G)$ be the vector space of K-valued class functions of G and $\mathcal{BCF}_K(G)$ be the vector space of K-valued class functions on the set $G_{p'}$ of p'-elements of G. Then the set of irreducible characters of G is a K-basis of $\mathcal{CF}_K(G)$, and the set of Brauer irreducible characters of G is a K-basis of $\mathcal{BCF}_K(G)$. Similarly we define $\mathcal{CF}_\mathcal{O}(G)$ and $\mathcal{BCF}_\mathcal{O}(G)$. $\mathcal{BCF}_K(G)$ can be regarded as a subspace of $\mathcal{CF}_K(G)$ canonically, and also $\mathcal{BCF}_\mathcal{O}(G)$ can be regarded as a subspace of $\mathcal{CF}_\mathcal{O}(G)$. Further we consider $\mathcal{CF}_K(G)$ endowed with the usual inner product. For χ, $\chi' \in \mathcal{CF}_K(G)$, we denote by $(\chi, \chi')_G$ or for short (χ, χ') the inner product of χ and χ'.

Let u be a p-element of G. Let d_G^u: $\mathcal{CF}_K(G) \to \mathcal{BCF}_K(C_G(u))$ be a surjective K-linear map defined by $d_G^u(\chi)(s)=\chi(us)$ for any $\chi \in \mathcal{CF}_K(G)$ and $s \in C_G(u)_{p'}$. Let e_G^u: $\mathcal{BCF}_K(C_G(u)) \to \mathcal{CF}_K(G)$ be a section of d_G^u which satisfies that for $\varphi \in \mathcal{BCF}_K(C_G(u))$, $e_G^u(\varphi)(g)=0$ if the p-part of g is not conjugate to u in G.

Let B be a block of G and let $\mathrm{Irr}(B)$ (resp. $\mathrm{IBr}(B)$) denote the set of ordinary (resp. Brauer) irreducible characters in B. Further let $\mathcal{CF}_K(G, B)=\sum_{\chi \in \mathrm{Irr}(B)} K\chi$ and $\mathcal{L}_K(G, B)$ denote the group of generalized characters in B. Set $k(B)=|\mathrm{Irr}(B)|$ and $l(B)=|\mathrm{IBr}(B)|$. Also let

$\mathcal{CF}^0_K(G, B)=(\mathrm{Ker}\ d^1_G)\cap \mathcal{CF}_K(G, B)$ and $\mathcal{L}^0_K(G, B)=(\mathrm{Ker}\ d^1_G)\cap \mathcal{L}_K(G, B)$. Moreover we set $\mathcal{CF}_{\mathcal{O}}(G, B)=\mathcal{CF}_K(G, B)\cap \mathcal{CF}_{\mathcal{O}}(G)$, $\mathcal{BCF}_K(G, B)=\mathcal{BCF}_K(G)\cap \mathcal{CF}_K(G, B)$ and $\mathcal{BCF}_{\mathcal{O}}(G, B)=\mathcal{BCF}_{\mathcal{O}}(G)\cap \mathcal{CF}_{\mathcal{O}}(G, B)$. For the further notations and terminologies, we follow [15] and [7].

For the rest of this ariticle suppose that B has an abelian defect group P and let (P, b) be a maximal (G, B)-Brauer pair. Further we set

$$N=N_G(P,b),\ E=N/C_G(P),\ L=E\ltimes P$$

and

$$P_1=C_P(N) \text{ and } P_2=[N, P].$$

By [6], Theorem 5.2.3, we have $P=P_1\times P_2$ and hence $L=P_1\times(E\ltimes P_2)$. For an N-stable element $\lambda\in \mathcal{CF}_K(P)$ and $\chi\in \mathcal{CF}_K(G, B)$, we denote by $\lambda * \chi$ the Broué-Puig's G-central function belonging to $\mathcal{CF}_K(G, B)$ (see [2], and [9], 2.12 also). For a character λ of P_1 and for $\zeta\in \mathcal{L}_K(G, B)$, $\lambda * \zeta$ is a generalized character regarding λ as a character of P. Moreover if λ and χ are irreducible, then $\lambda * \chi$ is irreducible. We prove the following by using [9], §3.

Theorem *With the above notations, suppose that E is cyclic and that $C_E(x)=1$ for any $x\in P_2-\{1\}$. If $l(B)=|E|$, then there exists a perfect isometry $I:\mathcal{L}_K(N, b^N)\to \mathcal{L}_K(G, B)$ such that $I(\lambda * \zeta)=\lambda * I(\zeta)$ for $\lambda\in \mathrm{Irr}(P_1)$ and $\zeta\in \mathcal{L}_K(N, b^N)$.*

Let $\bar{N}_G(P,b)=N_G(P,b)/P$ and $\bar{C}_G(P)=C_G(P)/P$, and denote by $\bar{b}$ the image of $b\in F\bar{C}_G(P)$. Since $F\bar{C}_G(P)\bar{b}$ is simple, the action of $\bar{N}_G(P,b)$ on $F\bar{C}_G(P)\bar{b}$ determines an $F^\times$-central extension $1\to F^\times\to \hat{\bar{N}}_G(P,b)\overset{g}{\to} \bar{N}_G(P,b)\to 1$ such that $g^{-1}(\bar{C}_G(P))$ is isomorphic to $F^\times\times \bar{C}_G(P)$. Here for an F-algebra A, $A^\times$ denotes the set of invertible elements of A. So $\hat{\bar{N}}_G(P,b)/\bar{C}_G(P)$ is an $F^\times$-central extension of E, where $\bar{C}_G(P)$ is embedded in $\hat{\bar{N}}_G(P,b)$ by g. Let $\hat{E}$ be the opposite group and $\hat{L}=\hat{E}\ltimes P$, and we denote by $\mathcal{O}_*\hat{L}$ the twisted group algebra. Note that $\mathcal{O}_*\hat{L}$ has a unique block. Moreover if E is cyclic, then $\mathcal{O}_*\hat{L}$ is a group algebra $\mathcal{O}L$. In [9], 3.2, Puig and Usami defined a notion (G, B)-local system related to $\mathcal{O}_*\hat{L}$. Since the situation we treat here is the case where E is cyclic, we state the definition of (G, B)-local system under the assumption that E is cyclic.

Definition (Puig-Usami [9], 3.2) With the above notations, assume that E is *cyclic*. Let X be an E-stable non-empty set of subgroups of P and assume that X contains any subgroup of P containing an element of X. Let Γ be a map over X sending $Q\in X$ to a bijective isometry

$$\Gamma_Q:\ \mathcal{BCF}_K(C_L(Q))\cong \mathcal{BCF}_K(C_G(Q),\ b^{C_G(Q)}).$$

If Γ satisfies the following conditions, then Γ is called a *(G, B)-local system over X*.
(i) For any $Q \in X$, any $\eta \in \mathcal{B}\mathcal{CF}_K(C_L(Q))$ and any $s \in E$, we have $\Gamma_Q(\eta)^s = \Gamma_{Q^s}(\eta^s)$.
(ii) For any $Q \in X$ and any $\eta \in \mathcal{L}_K(C_L(Q))$, the sum

$$\sum_u e^u_{C_G(Q)}(\Gamma_{Q \cdot <u>}(d^u_{C_L(Q)}(\eta)))$$

where u runs over a set of representatives U_Q for the orbits of $C_E(Q)$ in P, is a generalized character of $C_G(Q)$.

For any $Q \in X$, let Δ_Q: $\mathcal{CF}_K(C_L(Q)) \to \mathcal{CF}_K(C_G(Q), b^{C_G(Q)})$ be defined by

$$\Delta_Q(\eta) = \sum_{u \in U_Q} e^u_{C_G(Q)}(\Gamma_{Q \cdot <u>}(d^u_{C_L(Q)}(\eta))).$$

([9], (3.3.1)) By [9], 3.3 and 3.4, Δ_Q gives a perfect isometry between the principal block of $C_L(Q)$ and $b^{C_G(Q)}$ and satisfies the following

$$\Delta_Q(\lambda * \eta) = \lambda * \Delta_Q(\eta)$$

for any $\lambda \in \mathcal{CF}_K(P)^{C_E(Q)}$ and $\eta \in \mathcal{CF}_K(C_L(Q))$. Here $\mathcal{CF}_K(P)^{C_E(Q)}$ is the set of $C_E(Q)$-invariant elements of $\mathcal{CF}_K(P)$. Therefore if X contains the identity group, then $\Delta_{<1>}$ is a perfect isometry between the principal block of L and B ([9], (3.3.8), (3.4.1) and (3.3.5)). In fact, by [9], 1.6, this is an isotypy in the sense of [1].

2. Proof of Theorem

When $E=1$, the theorem is well known ([1], 5B). So we may assume $E \neq 1$. In order to get a perfect isometry I it suffices to show that there exists a perfect isometry Δ between the principal block of L and B such that $\Delta(\lambda * \eta) = \lambda * \Delta(\eta)$ for $\lambda \in \mathrm{Irr}(P_1)$ and $\eta \in \mathcal{L}_K(L)$. Since E is cyclic, therefore it suffices to show that there is a (G, B)-local system over the set of all subgroups of P. Let X be an E-stable non-empty set of subgroups of P and assume that X contains any subgroup of P containing an element of X and Γ be a (G, B)-local system over X. (We use the notations in Difinition above) Such a set X exists by [9], 3.4.2. Suppose that $\{1\} \notin X$ and let Q be a subgroup of P maximal such that $Q \notin X$. We will show that there is a (G, B)-local system Γ' extending Γ over the union X' of X and the E-orbit of Q. Let

$$f = b^{C_G(Q)},\ \bar{C}_L(Q) = C_L(Q)/Q \text{ and } \bar{C}_G(Q) = C_G(Q)/Q$$

and let $\bar{f}$ be the block of $\bar{C}_G(Q)$ corresponding to f. Let

$$\Delta^0_Q = \sum_{u \in U_Q - Q} e^u_{C_G(Q)} \circ \Gamma_{Q \cdot <u>} \circ d^u_{C_L(Q)}.$$

([9], (3.6.2)) By [9], Proposition 3.7 and Remark 3.8, Δ_Q^0 induces a bijective isometry

$$\bar{\Delta}_Q^0:\ \mathcal{CF}_K^0(\bar{C}_L(Q))\cong\mathcal{CF}_K^0(\bar{C}_G(Q),\bar{f})$$

such that

$$\bar{\Delta}_Q^0(\mathcal{L}_K^0(\bar{C}_L(Q))=\mathcal{L}_K^0(\bar{C}_G(Q),\bar{f}). \tag{2.1}$$

Let $\bar{P}=P/Q$, $\bar{P}_1=P_1Q/Q$ and $\bar{P}_2=P_2Q/Q$. By [9], Proposition 3.11 and 4.3, in order to get Γ', it suffices to show that $\bar{\Delta}_Q^0|_{\mathcal{L}_K^0(\bar{C}_L(Q))}$ can be extended to an $N_E(Q)$-stable bijective isometry

$$\bar{\Delta}_Q:\ \mathcal{L}_K(\bar{C}_L(Q))\cong\mathcal{L}_K(\bar{C}_G(Q),\bar{f}). \tag{2.2}$$

Case 1 *Assume* $|C_E(Q)|=1$.

Then we see $\bar{f}$ is a nilpotent block of $\bar{C}_G(Q)$ with defect group $\bar{P}$ because $\bar{f}$ has inertial index 1. Also $\bar{C}_L(Q)=\bar{P}$, and hence we have $|\mathrm{Irr}(\bar{C}_L(Q))|=|\mathrm{Irr}(\bar{P})|=|\mathrm{Irr}(\bar{f})|$. Moreover $\mathcal{L}_K^0(\bar{C}_L(Q))=\sum_{\zeta\in\mathrm{Irr}(\bar{P})}\mathbf{Z}(\zeta-\xi)$ where ξ is the trivial character of $\bar{P}$ and $\mathcal{L}_K^0(\bar{C}_G(Q),\bar{f})=\sum_{\zeta,\zeta'\in\mathrm{Irr}(\bar{f})}\mathbf{Z}(\zeta-\zeta')$ by [3]. Since $(\bar{\Delta}^0(\zeta-\zeta'),\bar{\Delta}^0(\zeta-\zeta'))=2$ for $\zeta,\zeta'\in\mathrm{Irr}(\bar{P})$ and $\bar{\Delta}^0(\zeta-\zeta')(1)=0$ where $\zeta\neq\zeta'$, $\bar{\Delta}_Q^0$ can be extended to an $N_E(Q)$-stable isometry in (2.2) by the same argument as in [9], 4.4. Note that if $|\mathrm{Irr}(\bar{C}_L(Q))|=2$, then any character in $\mathrm{Irr}(\bar{C}_L(Q))$ and in $\mathrm{Irr}(\bar{f})$ is $N_E(Q)$-invariant because $2\not|\,|N_E(Q):C_E(Q)|$. Hence by the assumption we may assume $C_E(Q)=E$, that is, $Q\subseteq P_1$.

Case 2 *Assume* $C_E(Q)=E$.

Then $N_E(Q)=C_E(Q)$. Hence it suffices to show that $\bar{\Delta}_Q^0$ can be extended to an isometry in (2.2). Set $e=|E|$ and $u=(|\bar{P}_2|-1)/e$. By the assumption u is an integer and there are exactly u E-conjugacy classes of non-trivial linear characters of $\bar{P}_2$ because $\bar{P}_2$ is isomrophic to P_2. Let $\nu_1,\nu_2,\cdots,\nu_u$ be a set of representatives for the E-conjugacy classes of non-trivial linear characters of $\bar{P}_2$. Put $\mu_i=\nu_i^{\bar{L}}$ $(i=1,2,\cdots,u)$ where we set $\bar{L}=\bar{C}_L(Q)=L/Q$ where ν_i is regarded as a character of $\bar{P}$. On the other hand let $\zeta_1,\zeta_2,\cdots,\zeta_e$ be the linear characters of E. Since $\bar{C}_L(Q)=\bar{P}_1\times(EP_2)$, we have

$$\mathrm{Irr}(\bar{C}_L(Q))=\{\lambda\mu_i\mid\lambda\in\mathrm{Irr}(\bar{P}_1),\ 1\le i\le u\}\cup\{\lambda\zeta_j\mid\lambda\in\mathrm{Irr}(\bar{P}_1),\ 1\le j\le e\}$$

where E is embedded in $\bar{C}_L(Q)$. Note $\lambda\mu_i=\lambda*\mu_i$ and $\lambda\zeta_j=\lambda*\zeta_j$. Moreover $\zeta_i|_{G_{p'}}$, $i=1,2,\cdots,e$, are the Brauer irreducible characters of $\bar{L}$. Let $\eta\in\mathcal{L}_K^0(\bar{L})$ and set $\eta=\sum_{\lambda,i}a_{\lambda i}\lambda\mu_i+\sum_{\lambda,j}b_{\lambda j}\lambda\zeta_j$ $(a_{\lambda i},b_{\lambda,j}\in\mathbf{Z})$. Then we have $\sum_{\lambda,i}a_{\lambda i}+\sum_{\lambda}b_{\lambda j}=0$ for any j and hence we have $\eta=\sum_{\lambda,i}a_{\lambda i}\lambda(\mu_i-\sum_{k=1}^{e}\zeta_k)+\sum_{\lambda,j}(b_{\lambda j}+\sum_i a_{\lambda i})(\lambda-1_{\bar{P}_1})\zeta_j$. Thus the following set is a $\mathbf{Z}$-basis of $\mathcal{L}_K^0(\bar{L})$.

$$\{\lambda(\sum_{k=1}^{e}\zeta_k-\mu_i)\mid \lambda\in \mathrm{Irr}(\bar{P}_1),\ 1\leq i\leq u\}\ \cup\ \{(1_{\bar{P}_1}-\lambda)\zeta_j\mid \lambda\in \mathrm{Irr}(\bar{P}_1)\backslash\{1_{\bar{P}_1}\},\ 1\leq j\leq e\}.$$

Put $\rho=\sum_{k=1}^{e}\zeta_k=(1_{\bar{P}})^{\bar{L}}$, $M=\{\mu_j|1\leq j\leq u\}$ and $\bar{\Delta}^0=\bar{\Delta}^0_Q|_{L^0_K(\bar{C}_L(Q))}$. From (2.1), any irreducible character in $\mathrm{Irr}(\bar{f})$ appears in either $\bar{\Delta}^0(\lambda(\rho-\mu_i))$ for some $\lambda\in \mathrm{Irr}(\bar{P}_1)$ and i $(\leq i\leq u)$ or $\bar{\Delta}^0((1_{\bar{P}_1}-\lambda')\zeta_j))$ for some $\lambda'\in \mathrm{Irr}(\bar{P}_1)$ and some j $(1\leq j\leq e)$. On the other hand by [16], Theorem 1 we have $l(f)=l(b^{C_G(P_1)})=l(B)$ because $C_G(Q)\supseteq C_G(P_1)$. So by the assumption $l(B)=e$, $l(f)=e$. Therefore $l(\bar{f})=e$. This and (2.1) imply $k(\bar{f})=|\mathrm{Irr}(\bar{L})|=|\bar{P}_1|(e+u)$.

Case 2.1 *Assume* $\bar{P}_1=1$.

At first suppose that $u=1$. Then $|\mathrm{Irr}(\bar{f})|=e+1$ and any irreducible character χ in $\bar{f}$ appears in $\bar{\Delta}^0(\rho-\mu_1)$. Since $(\bar{\Delta}^0(\rho-\mu_1), \bar{\Delta}^0(\rho-\mu_1))=e+1$, χ appears in $\bar{\Delta}^0(\rho-\mu_1)$ with multiplicity ± 1. Let $\mathrm{Irr}(\bar{f})=\{\chi_0, \chi_1,\cdots, \chi_e\}$. We have $\bar{\Delta}^0(\rho-\mu_1)=\sum_{i=0}^{e}\varepsilon_i\chi_i$, $\varepsilon_i=\pm 1$ $(i=0, 1,\cdots, e)$. So let $\bar{\Delta}$ be the isometry from $L_K(\bar{L})$ onto $L_K(\bar{C}_G(Q), \bar{f})$ defined by $\bar{\Delta}(\zeta_i)=\varepsilon_i\chi_i$ $(i=1, 2,\cdots, e)$ and $\bar{\Delta}(\mu_1)=-\varepsilon_0\chi_0$. Then $\bar{\Delta}$ is a required isometry.

Next suppose that $u=2$. Since $(\bar{\Delta}^0(\mu_1-\mu_2), \bar{\Delta}^0(\mu_1-\mu_2))=2$ and $(\bar{\Delta}^0(\mu_1-\mu_2))(1)=0$, there exists $\chi_{\mu_1}, \chi_{\mu_2}\in \mathrm{Irr}(\bar{f})$ and a sign ε such that $\bar{\Delta}^0(\mu_1-\mu_2)=\varepsilon(\chi_{\mu_1}-\chi_{\mu_2})=-\varepsilon(\chi_{\mu_2}-\chi_{\mu_1})$. Now since $k(\bar{f})=e+2$, let $\chi_1, \chi_2,\cdots, \chi_e$ be the irreducible characters in $\bar{f}$ other than χ_{μ_1} and χ_{μ_2}. Since $\bar{\Delta}^0(\rho-\mu_2)=\bar{\Delta}^0(\rho-\mu_1)+\bar{\Delta}^0(\mu_1-\mu_2)$, each χ_i appears in $\bar{\Delta}^0(\rho-\mu_1)$. As the inner product of $\bar{\Delta}(\rho-\mu_1)$ is $e+1$, we have

$$\bar{\Delta}^0(\rho-\mu_1)=\sum_{i=1}^{e}\varepsilon_i\chi_i+\varepsilon'\chi,\ \varepsilon_i=\pm 1\ (i=1, 2,\cdots, e),\ \varepsilon'=\pm 1, \tag{2.3}$$

where $\chi\in\{\chi_{\mu_1}, \chi_{\mu_2}\}$. From (2.3) we have

$$\bar{\Delta}^0(\rho-\mu_2)=\sum_{i=1}^{e}\varepsilon_i\chi_i+\varepsilon'\chi+\varepsilon(\chi_{\mu_1}-\chi_{\mu_2}). \tag{2.4}$$

Considering the inner product of $\bar{\Delta}^0(\rho-\mu_2)$ and changing χ_{μ_1} and χ_{μ_2}, and ε and $-\varepsilon$ if necessary, we have $\varepsilon'\chi=-\varepsilon\chi_{\mu_1}$. So (2.3) and (2.4) imply

$$\bar{\Delta}_0(\rho-\mu_1)=\sum_{i=1}^{e}\varepsilon_i\chi_i-\varepsilon\chi_{\mu_1}\ \text{and}\ \bar{\Delta}^0(\rho-\mu_2)=\sum_{i=1}^{e}\varepsilon_i\chi_i-\varepsilon\chi_{\mu_2}. \tag{2.5}$$

From (2.5), it is not difficult to extend $\bar{\Delta}^0$ to an isometry from $L_K(\bar{L})$ onto $L_K(\bar{C}_G(Q), \bar{f})$.

Finally suppose that $u\geq 3$. Since $(\bar{\Delta}^0(\mu_i-\mu_j), \bar{\Delta}^0(\mu_{i'}-\mu_{j'}))=\delta_{ii'}-\delta_{ij'}-\delta_{ji'}+\delta_{jj'}$ and $\bar{\Delta}^0(\mu_i-\mu_j)(1)=0$ $(1\leq i, j, i', j'\leq u)$, we can see $\bar{\Delta}^0(\mu_i-\mu_j)=\varepsilon(\chi_{\mu_i}-\chi_{\mu_j})$ $(1\leq i, j\leq u)$, $\varepsilon=\pm 1$ where χ_{μ_1},

$\chi_{\mu_2}, \cdots, \chi_{\mu_u} \in \mathrm{Irr}(\bar{f})$ are pairwise orthogonal. Let $\chi_1, \chi_2, \cdots, \chi_e$ be the irreducible characters in $\bar{f}$ other than $\chi_{\mu_1}, \chi_{\mu_2}, \cdots, \chi_{\mu_u}$. Let $\mu \in M\backslash\{\mu_1\}$. Since

$$\bar{\Delta}^0(\rho-\mu)=\bar{\Delta}^0(\rho-\mu_1)+\bar{\Delta}^0(\mu_1-\mu)=\bar{\Delta}^0(\rho-\mu_1)+\varepsilon(\chi_{\mu_1}-\chi_\mu),$$

χ_i appears in $\bar{\Delta}^0(\rho-\mu_1)$ with multiplicity ± 1:

$$\bar{\Delta}^0(\rho-\mu_1)=\sum_{i=1}^{e} \varepsilon_i\chi_i+\varepsilon'\chi,\ \varepsilon_i=\pm 1 \text{ and } \varepsilon'=\pm 1,$$

where $\chi \in \{\chi_{\mu_1}, \cdots, \chi_{\mu_u}\}$. On the other hand since $(\bar{\Delta}^0\rho-\mu_1), \bar{\Delta}^0(\mu_1-\mu))=-1$ for $\mu \in M\backslash\{\mu_1\}$ and $u-1\geq 2$, we have $\varepsilon'\chi=-\varepsilon\chi_{\mu_1}$. So we have $\bar{\Delta}^0(\rho-\mu_1)=\sum_{i=1}^{e}\varepsilon_i\chi_i-\varepsilon\chi_{\mu_1}$ and hence $\bar{\Delta}^0(\rho-\mu)=\sum_{i=1}^{e}\varepsilon_i\chi_i-\varepsilon\chi_\mu$ $(\mu \in M)$. From this $\bar{\Delta}^0$ can be extended to an isometry $\bar{\Delta}$ from $L_K(\bar{L})$ onto $L_K(\bar{C}_G(Q), \bar{f})$. In fact $\bar{\Delta}$ is defined as follows $\bar{\Delta}(\zeta_i)=\varepsilon_i\chi_i$ $(1\leq i\leq e)$ and $\bar{\Delta}(\mu)=\varepsilon\chi_\mu$ $(\mu \in M)$.

Case 2.2 *Assume* $\bar{P}_1\neq 1$.

At first we note that $\bar{\Delta}^0_Q(\lambda * \eta)=\lambda * \bar{\Delta}^0_Q(\eta)$ for $\lambda \in \mathrm{Irr}(\bar{P}_1)$ and $\eta \in L^0_K(\bar{C}_L(Q))$ by the definition of $\bar{\Delta}^0_Q$. Let $\lambda \in \mathrm{Irr}(\bar{P}_1)\backslash\{1_{\bar{P}_1}\}$ be fixed. We have $(\bar{\Delta}^0((1_{\bar{P}_1}-\lambda)\zeta_i), \bar{\Delta}^0((1_{\bar{P}_1}-\lambda)\zeta_j))=2\delta_{ij}$ and $\bar{\Delta}^0((1_{\bar{P}_1}-\lambda)\zeta_i)(1)=0$ for all i, j $(1\leq i, j\leq e)$. Hence we can see

(2.6) $$\bar{\Delta}^0((1_{\bar{P}_1}-\lambda)\zeta_i)=\varepsilon_i(\chi_i-\chi_{\lambda,i}),\ \varepsilon_i=\pm 1\ (1\leq i\leq e)$$

where $\chi_i \in \mathrm{Irr}(\bar{f})$ $(1\leq i\leq e)$ and $\chi_{\lambda,j} \in \mathrm{Irr}(\bar{f})$ $(1\leq j\leq e)$ are pairwise orthogonal. Suppose that $\mathrm{Irr}(\bar{P}_1)\backslash\{1_{\bar{P}_1}, \lambda\}$ is not empty and let $\lambda' \in \mathrm{Irr}(\bar{P}_1)\backslash\{1_{\bar{P}_1}, \lambda\}$ be fixed. Since we have $(\bar{\Delta}^0((1_{\bar{P}_1}-\lambda)\zeta_i), \bar{\Delta}^0((1_{\bar{P}_1}-\lambda')\zeta_j))=\delta_{ij}$ $(1\leq i, j\leq e)$, by changing χ_i and $\chi_{\lambda,i}$ and ε_i and $-\varepsilon_i$ if necessary, we can see

(2.7) $$\bar{\Delta}^0((1_{\bar{P}_1}-\lambda')\zeta_i)=\varepsilon_i(\chi_i-\chi_{\lambda',i})\ (1\leq i, j\leq e)$$

where $\chi_{\lambda',i} \in \mathrm{Irr}(\bar{f})$ $(1\leq i\leq e)$. Moreover (2.6) and (2.7) imply that $\chi_i \in \mathrm{Irr}(\bar{f})$ $(1\leq i\leq e)$ and $\chi_{\lambda,i} \in \mathrm{Irr}(\bar{f})$ $(1\leq i\leq e)$ and $\chi_{\lambda',i} \in \mathrm{Irr}(\bar{f})$ $(1\leq i\leq e)$ are pairwise orthogonal.

If $|\bar{P}_1|\geq 4$, then χ_i must be a common irreducible constituent of $\bar{\Delta}^0((1-\lambda)\zeta_i)$ $(\lambda \in \mathrm{Irr}(\bar{P}_1)\backslash\{1_{\bar{P}_1}\})$ since $(\bar{\Delta}^0((1_{\bar{P}_1}-\lambda)\zeta_i), \bar{\Delta}^0((1_{\bar{P}_1}-\lambda')\zeta_j))=\delta_{ij}$ $(1\leq i, j\leq e)$ when $\lambda\neq\lambda'$. Further we have

(2.8) $$\bar{\Delta}^0((1_{\bar{P}_1}-\lambda)\zeta_i)=\varepsilon_i(\chi_i-\chi_{\lambda,i}),\ \varepsilon_i=\pm 1\ (1\leq i\leq e,\ \lambda \in \mathrm{Irr}(\bar{P}_1)\backslash\{1_{\bar{P}_1}\})$$

where χ_i $(1\leq i\leq e) \in \mathrm{Irr}(\bar{f})$ and $\chi_{\lambda,j} \in \mathrm{Irr}(\bar{f})$ $(1\leq j\leq e,\ \lambda \in \mathrm{Irr}(\bar{P}_1)\backslash\{1_{\bar{P}_1}\})$ are pairwise orthogonal.

Now let $\lambda \in \mathrm{Irr}(\bar{P}_1)\backslash\{1_{\bar{P}_1}\}$ be fixed again. By the definition of $\bar{\Delta}^0$, we have $(\bar{\Delta}^0(\lambda^{-1}(1_{\bar{P}_1}-\lambda)\zeta_i))=\lambda^{-1} * (\bar{\Delta}^0((1_{\bar{P}_1}-\lambda)\zeta_i))=\varepsilon_i(\lambda^{-1} * \chi_i-\lambda^{-1} * \chi_{\lambda,i})$ and we have also $\bar{\Delta}^0((\lambda^{-1}(1_{\bar{P}_1}-\lambda)\zeta_i))=-\bar{\Delta}^0((1_{\bar{P}_1}-\lambda^{-1})\zeta_i))=-\varepsilon_i(\chi_i-\chi_{\lambda^{-1},i})$. Therefore $\lambda^{-1} * \chi_{\lambda,i}=\chi_i$ and hence we have $\chi_{\lambda,i}=\lambda * \chi_i$ for any i $(1\leq i\leq e)$.

Let $\mu \in M$. Since $(\bar{\Delta}^0(\rho-\mu), \bar{\Delta}^0((1_{\bar{P}_1}-\lambda)\zeta_i))=1$ $(1\leq i\leq e, \lambda \in \mathrm{Irr}(\bar{P}_1)\backslash\{1_{\bar{P}_1}\})$, at least one of the characters $\lambda * \chi_i$ $(\lambda \in \mathrm{Irr}(\bar{P}_1))$ appears in $\bar{\Delta}^0(\rho-\mu)$ for each i. On the other hand we have $(\bar{\Delta}^0(\rho-\mu), \bar{\Delta}^0(\rho-\mu))=e+1$. Therefore for each i, exactly one of the characters $\lambda * \chi_i$ $(\lambda \in \mathrm{Irr}(\bar{P}_1))$ appears in $\bar{\Delta}^0(\rho-\mu)$ with multiplicity ± 1. In fact for each i, χ_i appears in $\bar{\Delta}^0(\rho-\mu)$ with multiplicity ε_i because $(\bar{\Delta}^0(\rho-\mu), \bar{\Delta}^0((1_{\bar{P}_1}-\lambda)\zeta_i))=1$ for any $\lambda \in \mathrm{Irr}(\bar{P}_1)\backslash\{1_{\bar{P}_1}\}$. Here we replace χ_i by $\chi_{\lambda,i}$ $(\lambda \neq 1_{\bar{P}_1})$ if necessary when $|\bar{P}_1|=2$. Noticing $\bar{\Delta}^0(\lambda(\rho-\mu))=\lambda * (\bar{\Delta}^0(\rho-\mu))$, hence the sum of numbers of irreducible characters which appear in $\bar{\Delta}^0((1_{\bar{P}_1}-\lambda)\zeta_i)$ $(1\leq i\leq e$, $\lambda \in \mathrm{Irr}(\bar{P}_1)\backslash\{1_{\bar{P}_1}\})$ and $\bar{\Delta}^0(\lambda(\rho-\mu))$ $(\mu \in M, \lambda \in \mathrm{Irr}(\bar{P}_1))$ is at most $|\bar{P}_1|(e+u)$. Recalling $|\mathrm{Irr}(\bar{f})|=|\bar{P}|(e+u)$, these imply

(2.9)
$$\bar{\Delta}^0(\rho-\mu)=\sum_{i=1}^{e} \varepsilon_i\chi_i-\varepsilon_\mu\chi_\mu, \quad \varepsilon_\mu=\pm 1,$$
$$\bar{\Delta}^0(\lambda(\rho-\mu))=\sum_{i=1}^{e} \varepsilon_i(\lambda * \chi_i)-\varepsilon_\mu(\lambda * \chi_\mu),$$

where $\chi_\mu \in \mathrm{Irr}(\bar{f})$. Moreover $\lambda * \chi_i$ $(1\leq i\leq e, \lambda \in \mathrm{Irr}(\bar{P}_1))$, $\lambda * \chi_\mu$ $(\mu \in M, \lambda \in \mathrm{Irr}(\bar{P}_1))$ are pairwise orthogonal, and these are the ordinary irreducible characters in $\bar{f}$. (As $\bar{\Delta}^0(\mu-\mu')=\bar{\Delta}^0(\rho-\mu')-\bar{\Delta}^0(\rho-\mu)=\varepsilon_\mu\chi_\mu-\varepsilon_{\mu'}\chi_{\mu'}$, we have $\varepsilon_\mu=\varepsilon_{\mu'}$.) Thus we can define a bijective isometry $\bar{\Delta}$ from $L_K(\bar{L})$ onto $L_K(\bar{C}_G(Q), \bar{f})$ such that $\bar{\Delta}(\lambda\zeta_i)=\varepsilon_i(\lambda * \chi_i)$ $(1\leq i\leq e, \lambda \in \mathrm{Irr}(\bar{P}_1))$ and $\bar{\Delta}(\lambda\mu)=\varepsilon_\mu\lambda * \chi_\mu$ $(\mu \in M, \lambda \in \mathrm{Irr}(\bar{P}_1))$. Then $\bar{\Delta}$ is a required extension of $\bar{\Delta}^0$ from (2.8) and (2.9). This completes the proof.

By the above theorem and [16], Corollary 2, we have the following.

Corollary *Let B be a block of G with abelian defect group P and root b in $C_G(P)$. If $[N_G(P,b), P]$ is cyclic, then B and $b^{N_G(P,b)}$ are isotypic.*

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