On perfect isometries for blocks with abelian defect groups and cyclic hyperfocal subgroups

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Let $p$ be a prime number and $(K, O, F)$ be a $p$-modular system such that $K$ is algebraically closed. Let $G$ be a finite group and $B$ be a $(p)$ block of $G$ with defect group $P$. Also let $B_0$ be the Brauer correspondent of $B$, that is, $B_0$ is the block of $N_G(P)$ associated with B. M. Broué conjectured in [1] that there exists a perfect isometry between $B$ and $B_0$ when $P$ is abelian. This is verified for several blocks as in [1], §5. When $p=2$, Broué’s conjecture on perfect isometries is true for principal 2-blocks ([4]). Let $b$ be a block of $C_G(P)$ associated with $B$. When $N_G(P)/C_G(P)$ is an elementary abelian 2-group or when $|N_G(P)/C_G(P)|$ is small, Broué’s conjecture on perfect isometries is true for $B$ ([9], [11], [10], [12], [13] and [14]). See [5] also. In this article we show Broué’s conjecture on perfect isometry is true when $[(N_G(P), b), P]$ is cyclic (see Corollary below). Note that the commutator subgroup $[(N_G(P), b), P]$ of $N_G(P)$ and $P$ is a hyperfocal subgroup of a pointed group of a pointed group $G_{un}$ on $OG$. (For the definition of hyperfocal subgroups, see [8])

1. Perfect isometry

Let $CF_k(G)$ be the vector space of $K$-valued class functions of $G$ and $BCF_k(G)$ be the vector space of $K$-valued class functions on the set $G_{pr}$ of $p'$-elements of $G$. Then the set of irreducible characters of $G$ is a $K$-basis of $CF_k(G)$, and the set of Brauer irreducible characters of $G$ is a $K$-basis of $BCF_k(G)$. Similarly we define $CF_0(G)$ and $BCF_0(G)$. $BCF_k(G)$ can be regarded as a subspace of $CF_k(G)$ canonically, and also $BCF_0(G)$ can be regarded as a subspace of $CF_0(G)$. Further we consider $CF_k(G)$ endowed with the usual inner product. For $\chi, \chi' \in CF_k(G)$, we denote by $\langle \chi, \chi' \rangle_G$ or for short $\langle \chi, \chi' \rangle$ the inner product of $\chi$ and $\chi'$.

Let $u$ be a $p$-element of $G$. Let $d_u^\chi : CF_k(G) \to BCF_k(C_G(u))$ be a surjective $K$-linear map defined by $d_u^\chi(s) = \chi(us)$ for any $\chi \in CF_k(G)$ and $s \in C_G(u)$. Let $e_u^\chi : BCF_k(C_G(u)) \to CF_k(G)$ be a section of $d_u^\chi$ which satisfies that for $\varphi \in BCF_k(C_G(u))$, $e_u^\chi(\varphi)(g) = 0$ if the $p$-part of $g$ is not conjugate to $u$ in $G$.

Let $B$ be a block of $G$ and let $\text{Irr}(B)$ (resp. $\text{Br}(B)$) denote the set of ordinary (resp. Brauer) irreducible characters in $B$. Further let $CF_k(G, B) = \sum_{\chi \in \text{Irr}(B)} K\chi$ and $L_k(G, B)$ denote the group of generalized characters in $B$. Set $k(B) = |\text{Irr}(B)|$ and $l(B) = |\text{Br}(B)|$. Also let
\( \mathcal{CF}_\kappa(G, B) = (\text{Ker } d') \cap \mathcal{CF}_\kappa(G, B) \) and \( L^\kappa(G, B) = (\text{Ker } d') \cap \mathcal{L}^\kappa(G, B) \). Moreover we set \( \mathcal{CF}(G, B) = \mathcal{CF}_\kappa(G, B) \cap \mathcal{CF}(G, B) \), \( \mathcal{BCF}(G, B) = \mathcal{BCF}_\kappa(G, B) \cap \mathcal{CF}(G, B) \) and \( \mathcal{BCF}(G, B) = \mathcal{BCF}_\kappa(G, B) \cap \mathcal{CF}(G, B) \). For the further notations and terminologies, we follow [15] and [7]. For the rest of this article suppose that \( B \) has an abelian defect group \( P \) and let \((P, b)\) be a maximal \((G, B)\)-Brauer pair. Further we set
\[
N = N_G(P, b), \quad E = N(C_G(P), L = E \ltimes P
\]
and
\[
P_1 = C_P(N) \text{ and } P_2 = [N, P].
\]
By [6], Theorem 5.2.3, we have \( P = P_1 \times P_2 \) and hence \( L = P_1 \times (E \ltimes P_2) \). For an \( N \)-stable element \( \lambda \in \mathcal{CF}_\kappa(P) \) and \( x \in \mathcal{CF}_\kappa(G, B) \), we denote by \( \lambda \star x \) the Broué-Puig's \( G \)-central function belonging to \( \mathcal{CF}_\kappa(G, B) \) (see [2], and [9], 2.12 also). For a character \( \lambda \) of \( P_1 \) and for \( \xi \in L^\kappa(G, B) \), \( \lambda \star \xi \) is a generalized character regarding \( \lambda \) as a character of \( P \). Moreover if \( \lambda \) and \( \chi \) are irreducible, then \( \lambda \star \chi \) is irreducible. We prove the following by using [9], §3.

**Theorem** With the above notations, suppose that \( E \) is cyclic and that \( C_2(x) = 1 \) for any \( x \in P \) \(-1\). If \( l(B) = |E| \), then there exists a perfect isometry \( I: L^\kappa(N, b^w) \to L^\kappa(G, B) \) such that \( I(\lambda \star \xi) = \lambda \star I(\xi) \) for \( \lambda \in \text{Irr}(P_1) \) and \( \xi \in L^\kappa(N, b^w) \).

Let \( \tilde{N}_G(P, b) = N_G(P, b)/P \) and \( \tilde{C}_G(P) = C_G(P)/P \), and denote by \( \tilde{b} \) the image of \( b \in F\tilde{C}_G(P) \).

Since \( F\tilde{C}_G(P) \tilde{b} \) is simple, the action of \( \tilde{N}_G(P, b) \) on \( F\tilde{C}_G(P) \tilde{b} \) determines an \( F^* \)-central extension \( 1 \to F^* \to \tilde{N}_G(P, b) \xrightarrow{\tilde{g}} \tilde{N}_G(P, b) \to 1 \) such that \( g^{-1}(\tilde{C}_G(P)) \) is isomorphic to \( F^* \times \tilde{C}_G(P) \). Here for an \( F \)-algebra \( A \), \( A^* \) denotes the set of invertible elements of \( A \). So \( \tilde{N}_G(P, b)/\tilde{C}_G(P) \) is an \( F^* \)-central extension of \( E \), where \( \tilde{C}_G(P) \) is embedded in \( \tilde{N}_G(P, b) \) by \( \tilde{g} \). Let \( \tilde{E} \) be the opposite group and \( \tilde{L} = \tilde{E} \times P \), and we denote by \( O * \tilde{L} \) the twisted group algebra. Note that \( O * \tilde{L} \) has a unique block. Moreover if \( E \) is cyclic, then \( O * \tilde{L} \) is a group algebra \( OL \). In [9], 3.2, Puig and Usami defined a notion \((G, B)\)-local system related to \( O * \tilde{L} \). Since the situation we treat here is the case where \( E \) is cyclic, we state the definition of \((G, B)\)-local system under the assumption that \( E \) is cyclic.

**Definition** (Puig-Usami [9], 3.2) With the above notations, assume that \( E \) is cyclic. Let \( X \) be an \( E \)-stable non-empty set of subgroups of \( P \) and assume that \( X \) contains any subgroup of \( P \) containing an element of \( X \). Let \( \Gamma \) be a map over \( X \) sending \( Q \in X \) to a bijective isometry
\[
\Gamma_e : \mathcal{BCF}_\kappa(C_G(Q)) \cong \mathcal{BCF}_\kappa(C_G(Q), b^{G_{C_Q}}).
\]
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If $\Gamma$ satisfies the following conditions, then $\Gamma$ is called a $(G, B)$-local system over $X$.

(i) For any $Q \in X$, any $\eta \in BCF_X(C_\ell(Q))$ and any $s \in E$, we have $\Gamma_\ell(\eta)^s = \Gamma_\ell(\eta^s)$.

(ii) For any $Q \in X$ and any $\eta \in L_\kappa(C_\ell(Q))$, the sum

$$\sum_{\eta} e_{\ell, \eta}(\Gamma_{\ell, \eta}^{-\varepsilon}(d_{\ell, \eta}(\eta)))$$

where $\varepsilon$ runs over a set of representatives $U_0$ for the orbits of $C_\ell(Q)$ in $P$, is a generalized character of $C_\ell(Q)$.

For any $Q \in X$, let $\Delta_\phi : CF_X(C_\ell(Q)) \to CF_X(C_\ell(Q), \delta^{C_\ell(Q)})$ be defined by

$$\Delta_\phi(\eta) = \sum_{\eta \in U_0} e_{\ell, \eta}(\Gamma_{\ell, \eta}^{-\varepsilon}(d_{\ell, \eta}(\eta))).$$

([9], (3.3.1)) By [9], 3.3 and 3.4, $\Delta_\phi$ gives a perfect isometry between the principal block of $C_\ell(Q)$ and $\delta^{C_\ell(Q)}$ and satisfies the following

$$\Delta_\phi(\lambda \ast \eta) = \lambda \ast \Delta_\phi(\eta)$$

for any $\lambda \in CF_X(P)^{C_\ell(Q)}$ and $\eta \in CF_X(C_\ell(Q))$. Here $CF_X(P)^{C_\ell(Q)}$ is the set of $C_\ell(Q)$-invariant elements of $CF_X(P)$. Therefore if $X$ contains the identity group, then $\Delta_{\ell, \eta}$ is a perfect isometry between the principal block of $L$ and $B$ ([9], (3.3.8), (3.4.1) and (3.3.5)). In fact, by [9], 1.6, this is an isotypy in the sense of [1].

2. Proof of Theorem

When $E = 1$, the theorem is well known ([1], 5B). So we may assume $E \neq 1$. In order to get a perfect isometry $I$ it suffices to show that there exists a perfect isometry $\Delta$ between the principal block of $L$ and $B$ such that $\Delta(\lambda \ast \eta) = \lambda \ast \Delta(\eta)$ for $\lambda \in \text{Irr}(P)$ and $\eta \in L_\kappa(L)$. Since $E$ is cyclic, therefore it suffices to show that there is a $(G, B)$-local system over the set of all subgroups of $P$. Let $X$ be an $E$-stable non-empty set of subgroups of $P$ and assume that $X$ contains any subgroup of $P$ containing an element of $X$ and $\Gamma$ be a $(G, B)$-local system over $X$. (We use the notations in Definition above) Such a set $X$ exists by [9], 3.4.2. Suppose that $(1) \notin X$ and let $Q$ be a subgroup of $P$ maximal such that $Q \notin X$. We will show that there is a $(G, B)$-local system $\Gamma'$ extending $\Gamma$ over the union $X'$ of $X$ and the $E$-orbit of $Q$. Let

$$f = \delta^{C_\ell(Q)}, \quad C_\ell(Q) = C_\ell(Q)/Q \text{ and } \overline{C}_\ell(Q) = C_\ell(Q)/Q$$

and let $\overline{f}$ be the block of $\overline{C}_\ell(Q)$ corresponding to $f$. Let

$$\Delta_\phi(\lambda \ast \eta) = \sum_{\eta \in U_0} e_{\ell, \eta}(\Gamma_{\ell, \eta}^{-\varepsilon}(d_{\ell, \eta}(\eta))).$$
By [9], Proposition 3.7 and Remark 3.8, $\Delta^\delta$ induces a bijective isometry

$$\Delta^\delta : \mathbb{CF}_k(C_\ell(Q)) \cong \mathbb{CF}_k(C_c(Q), \bar{f})$$

such that

$$\Delta^\delta : L^k(C_\ell(Q)) \cong L^k(C_c(Q), \bar{f}). \tag{2.1}$$

Let $\bar{P} = P/Q$, $\bar{P}_1 = P_1Q/Q$ and $\bar{P}_2 = P_2Q/Q$. By [9], Proposition 3.11 and 4.3, in order to get $\Gamma^\prime$, it suffices to show that $\Delta^\delta_{L^k(C_\ell(Q))}$ can be extended to an $N_\varepsilon(Q)$-stable bijective isometry

$$\Delta^\delta : L^\varepsilon(C_\ell(Q)) \cong L^\varepsilon(C_c(Q), \bar{f}). \tag{2.2}$$

**Case 1** Assume $|C_\varepsilon(Q)| = 1$.

Then we see $\bar{f}$ is a nilpotent block of $C_c(Q)$ with defect group $\bar{P}$ because $\bar{f}$ has inertial index 1. Also $C_\ell(Q) = \bar{P}$, and hence we have $|\text{Irr}(C_\ell(Q))| = |\text{Irr}(\bar{P})| = |\text{Irr}(\bar{f})|$. Moreover $L^\varepsilon(C_\ell(Q)) = \sum_{\xi \in \text{Irr}(\bar{P})} \mathbb{Z}(\xi - \bar{\xi})$ where $\bar{\xi}$ is the trivial character of $\bar{P}$ and $L^\varepsilon(C_c(Q), \bar{f}) = \sum_{\xi \in \text{Irr}(\bar{P})} \mathbb{Z}(\xi - \bar{\xi})$ by [3]. Since $(\Delta^\delta(\xi - \bar{\xi}), \Delta^\delta(\xi - \bar{\xi})) = 2$ for $\xi, \bar{\xi} \in \text{Irr}(\bar{P})$ and $\Delta^\delta(\xi - \bar{\xi})(1) = 0$ where $\xi \neq \bar{\xi}$, $\Delta^\delta$ can be extended to an $N_\varepsilon(Q)$-stable isometry in (2.2) by the same argument as in [9], 4.4. Note that if $|\text{Irr}(C_\ell(Q))| = 2$, then any character in $\text{Irr}(C_\ell(Q))$ and in $\text{Irr}(\bar{f})$ is $N_\varepsilon(Q)$-invariant, hence by the assumption we may assume $C_\varepsilon(Q) = E$, that is, $Q \subseteq P$.

**Case 2** Assume $C_\varepsilon(Q) = E$.

Then $N_\varepsilon(Q) = C_\varepsilon(Q)$. Hence it suffices to show that $\Delta^\delta$ can be extended to an isometry in (2.2).

Set $e = |E|$ and $u = (|\bar{P}_2| - 1)/e$. By the assumption $u$ is an integer and there are exactly $u$ $E$-conjugacy classes of non-trivial linear characters of $\bar{P}_2$. Because $\bar{P}_2$ is isomorphic to $P_2$. Let $\nu_1, \nu_2, \ldots, \nu_u$ be a set of representatives for the $E$-conjugacy classes of non-trivial linear characters of $\bar{P}_2$. Put $\mu_i = \nu_i^E (i = 1, 2, \ldots, u)$ where we set $\bar{L} = C_\ell(Q) = L/Q$ where $\nu_i$ is regarded as a character of $\bar{P}$. On the other hand let $\xi_1, \xi_2, \ldots, \xi_e$ be the linear characters of $E$. Since $C_\ell(Q) = \bar{P}_1 \times (EP_2)$, we have

$$\text{Irr}(\bar{C}_\ell(Q)) = \{ \lambda\mu_i | \lambda \in \text{Irr}(\bar{P}_1), 1 \leq i \leq u \} \cup \{ \lambda\xi_j | \lambda \in \text{Irr}(\bar{P}_2), 1 \leq j \leq e \}$$

where $E$ is embedded in $\bar{C}_\ell(Q)$. Note $\lambda\mu_i = \lambda^*\mu_i$ and $\lambda\xi_j = \lambda^*\xi_j$. Moreover $\xi_{i\alpha_i}, i = 1, 2, \ldots, e$, are the Brauer irreducible characters of $\bar{L}$. Let $\eta \in L^\varepsilon(\bar{L})$ and set $\eta = \sum_{i=1}^u a_i \lambda^* \mu_i + \sum_{j=1}^e b_j \lambda^* \xi_j (a_{ii}, b_{jj} \in \mathbb{Z})$. Then we have $\sum_{i=1}^{ul_1} a_{ii} = 0$ for any $j$ and hence we have $\eta = \sum_{i=1}^u a_{ii} \lambda^* (\mu_i - \sum_{k=1}^u \xi_{ik} (\lambda - 1)) + \sum_{j=1}^e (b_j + \sum_{i=1}^e a_{ii})(\lambda - 1) \xi_j$. Thus the following set is a $\mathbb{Z}$-basis of $L^\varepsilon(\bar{L})$. 

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\[
\{ (\lambda \sum_{i=1}^{u} \epsilon_{i} \xi_{i} - \mu_{i}) \mid \lambda \in \text{Irr}(\overline{P}_{j}), 1 \leq i \leq u \} \cup \{(1_{\overline{P}_{j}} - \lambda)\xi_{j} \mid \lambda \in \text{Irr}(\overline{P}_{j}) \setminus \{1_{\overline{P}_{j}}\}, 1 \leq j \leq e \}.
\]

Put \( \rho = \sum_{i=1}^{u} \epsilon_{i} \xi_{i} = (1_{\overline{P}_{j}})^{e} \), \( M = \{\mu_{j} \mid 1 \leq j \leq u\} \) and \( \Delta^{0} = \Delta^{0}(\text{Irr}(\overline{P}_{j})) \). From (2.1), any irreducible character in \( \text{Irr}(\overline{f}) \) appears in either \( \Delta^{0}(\lambda_{i} - \mu_{i}) \) for some \( \lambda_{i} \in \text{Irr}(\overline{P}_{j}) \) and \( i (1 \leq i \leq u) \) or \( \Delta^{0}((1_{\overline{P}_{j}} - \lambda)\xi_{j}) \) for some \( \lambda \in \text{Irr}(\overline{P}_{j}) \) and some \( j (1 \leq j \leq e) \). On the other hand by [16], Theorem 1 we have \( l(\overline{f}) = l(b^{\text{det}(\mu)}(B)) = l(B) \) because \( C_{C}(Q) \equiv C_{C}(P_{j}) \). So by the assumption \( l(B) = e, l(\overline{f}) = e \). Therefore \( l(\overline{f}) = e \). This and (2.1) imply \( k(\overline{f}) = |\text{Irr}(\overline{L})| = |\overline{P}_{j}|(e + u) \).

**Case 2.1** Assume \( \overline{P}_{j} = 1 \).

At first suppose that \( u = 1 \). Then \( |\text{Irr}(\overline{f})| = e + 1 \) and any irreducible character \( \chi \) in \( \overline{f} \) appears in \( \Delta^{0}(\rho - \mu_{j}) \). Since \( \Delta^{0}(\rho - \mu_{j}) = e^{+} \), \( \chi \) appears in \( \Delta^{0}(\rho - \mu_{j}) \) with multiplicity \( \pm 1 \).

Let \( \text{Irr}(\overline{f}) = \{\chi_{0}, \chi_{1}, \ldots, \chi_{e}\} \). We have \( \Delta^{0}(\rho - \mu_{j}) = \sum_{i=1}^{e} \epsilon_{i} \chi_{i}, \epsilon_{i} = \pm 1 \) \( (i = 0, 1, \ldots, e) \). So let \( \overline{\Delta} \) be the isometry from \( L_{\kappa}(\overline{L}) \) onto \( L_{\kappa}(C_{C}(Q), \overline{f}) \) defined by \( \overline{\Delta}(\xi_{i}) = \epsilon_{i} \chi_{i} \) \( (i = 1, 2, \ldots, e) \) and \( \overline{\Delta}(\mu_{j}) = -\epsilon_{0} \chi_{0} \). Then \( \overline{\Delta} \) is a required isometry.

Next suppose that \( u = 2 \). Since \( \Delta^{0}(\mu_{1} - \mu_{2}) = 2 \) and \( \Delta^{0}(\mu_{1} - \mu_{2})(1) = 0 \), there exists \( \chi_{m}, \chi_{n} \in \text{Irr}(\overline{f}) \) and a sign \( \epsilon \) such that \( \Delta^{0}(\mu_{1} - \mu_{2}) = \epsilon (\chi_{m} - \chi_{n}) = -\epsilon (\chi_{n} - \chi_{m}) \). Now since \( k(\overline{f}) = e + 2 \), let \( \chi_{1}, \chi_{2}, \ldots, \chi_{e} \) be the irreducible characters in \( \overline{f} \) other than \( \chi_{m} \) and \( \chi_{n} \). Since \( \Delta^{0}(\rho - \mu_{j}) = \Delta^{0}(\rho - \mu_{1}) + \Delta^{0}(\mu_{1} - \mu_{2}) \), each \( \chi_{i} \) appears in \( \Delta^{0}(\rho - \mu_{i}) \). As the inner product of \( \Delta^{0}(\rho - \mu_{i}) \) is \( e + 1 \), we have

\[
\Delta^{0}(\rho - \mu_{i}) = \sum_{i=1}^{e} \epsilon_{i} \chi_{i} + \epsilon' \chi, \epsilon_{i} = \pm 1 \quad (i = 1, 2, \ldots, e), \quad \epsilon' = \pm 1,
\]

where \( \chi \in \{\chi_{m}, \chi_{n}\} \). From (2.3) we have

\[
\Delta^{0}(\rho - \mu_{i}) = \sum_{i=1}^{e} \epsilon_{i} \chi_{i} + \epsilon' \chi + \epsilon (\chi_{m} - \chi_{n}).
\]

Considering the inner product of \( \Delta^{0}(\rho - \mu_{i}) \) and changing \( \chi_{m} \) and \( \chi_{n} \), and \( \epsilon \) and \( -\epsilon \) if necessary, we have \( \epsilon' \chi = -\epsilon \chi_{m} \). So (2.3) and (2.4) imply

\[
\Delta^{0}(\rho - \mu_{i}) = \sum_{i=1}^{e} \epsilon_{i} \chi_{i} - \epsilon \chi_{m} \quad \text{and} \quad \Delta^{0}(\rho - \mu_{i}) = \sum_{i=1}^{e} \epsilon_{i} \chi_{i} - \epsilon \chi_{n}.
\]

From (2.5), it is not difficult to extend \( \Delta^{0} \) to an isometry from \( L_{\kappa}(\overline{L}) \) onto \( L_{\kappa}(\overline{C_{C}(Q)}, \overline{f}) \).

Finally suppose that \( u \geq 3 \). Since \( \Delta^{0}(\mu_{i} - \mu_{j}) = \delta_{ij} - \delta_{ij} - \delta_{ij} + \delta_{ij} \) and \( \Delta^{0}(\mu_{i} - \mu_{j})(1) = 0 \) \( (1 \leq i, j, i', j' \leq u) \), we can see \( \Delta^{0}(\mu_{i} - \mu_{j}) = \epsilon (\chi_{m} - \chi_{n}) \) \( (1 \leq i, j, i' \leq u) \), \( \epsilon = \pm 1 \) where \( \chi_{m} \),
\( \chi_{n}, \ldots, \chi_{m} \in \text{Irr}(\bar{f}) \) are pairwise orthogonal. Let \( \chi_{1}, \chi_{2}, \ldots, \chi_{e} \) be the irreducible characters in \( \bar{f} \) other than \( \chi_{n}, \chi_{n}, \ldots, \chi_{m} \). Let \( \mu \in M \setminus \{\mu_{1}\} \). Since

\[
\Delta^{\varphi}(\rho - \mu) = \Delta^{\varphi}(\rho - \mu) + \Delta^{\varphi}(\mu_{1} - \mu) = \Delta^{\varphi}(\rho - \mu) + \varepsilon(\chi_{n_{1}} - \chi_{n}),
\]

\( \chi_{i} \) appears in \( \Delta^{\varphi}(\rho - \mu) \) with multiplicity \( \pm 1 \):

\[
\Delta^{\varphi}(\rho - \mu) = \sum_{i=1}^{e} \varepsilon_{\chi_{i}} + \varepsilon' \chi_{i}, \varepsilon_{\chi_{i}} = \pm 1 \text{ and } \varepsilon' = \pm 1,
\]

where \( \chi \in \{\chi_{n}, \ldots, \chi_{m}\} \). On the other hand since \( (\Delta^{\varphi}(\rho - \mu), \Delta^{\varphi}(\mu_{1} - \mu)) = -1 \) for \( \mu \in M \setminus \{\mu_{1}\} \) and \( u - 1 \geq 2 \), we have \( \varepsilon' \chi_{i} = -\varepsilon_{\chi_{n}} \). So we have \( \Delta^{\varphi}(\rho - \mu) = \sum_{i=1}^{e} \varepsilon_{\chi_{i}} - \varepsilon_{\chi_{n}} \), and hence \( \Delta^{\varphi}(\rho - \mu) = \sum_{i=1}^{e} \varepsilon_{\chi_{i}} - \varepsilon_{\chi_{n}} (\mu \in M) \). From this \( \Delta^{\varphi} \) can be extended to an isometry \( \Delta \) on \( L_{\kappa}(L \bar{c}(Q), \bar{f}) \). In fact \( \Delta \) is defined as follows \( \Delta(\xi_{i}) = \varepsilon_{\chi_{i}} (1 \leq i \leq e) \) and \( \Delta(\mu) = \varepsilon_{\chi_{n}} (\mu \in M) \).

Case 2.2 Assume \( \bar{F}_{i} \neq 1 \).

At first we note that \( \Delta^{\varphi}(\lambda \ast \eta) = \lambda \ast \Delta^{\varphi}(\eta) \) for \( \lambda \in \text{Irr}(\bar{F}_{i}) \) and \( \eta \in L^{\varphi}_{\kappa}(\bar{C}_{i}(Q)) \) by the definition of \( \Delta^{\varphi} \). Let \( \lambda \in \text{Irr}(\bar{F}_{i}) \setminus \{1_{\bar{F}_{i}}\} \) be fixed. We have \( \Delta^{\varphi}(1_{\bar{F}_{i}} - \lambda) \xi_{i} = 2\delta_{u} \) and \( \Delta^{\varphi}(1_{\bar{F}_{i}} - \lambda) \xi_{j} = 0 \) for all \( i, j (1 \leq i, j \leq e) \). Hence we can see

\[
\Delta^{\varphi}(1_{\bar{F}_{i}} - \lambda) \xi_{i} = \varepsilon_{\chi_{i} - \chi_{n}}, \varepsilon_{\chi_{i}} = \pm 1 (1 \leq i \leq e)
\]

where \( \chi_{i} \in \text{Irr}(\bar{f}) \) (1 \( \leq i \leq e \)) and \( \chi_{n_{1}} \in \text{Irr}(\bar{f}) \) (1 \( \leq j \leq e \)) are pairwise orthogonal. Suppose that \( \text{Irr}(\bar{F}_{i}) \setminus \{1_{\bar{F}_{i}}\} \) is not empty and let \( \lambda' \in \text{Irr}(\bar{F}_{i}) \setminus \{1_{\bar{F}_{i}}, \lambda\} \) be fixed. Since we have \( \Delta^{\varphi}(1_{\bar{F}_{i}} - \lambda) \xi_{i} = \delta_{u} \) (1 \( \leq i, j \leq e \)), by changing \( \chi_{i} \) and \( \chi_{n_{1}} \) and \( \varepsilon_{\chi_{i}} \) and \( -\varepsilon_{\chi_{i}} \) if necessary, we can see

\[
\Delta^{\varphi}(1_{\bar{F}_{i}} - \lambda) \xi_{i} = \varepsilon_{\chi_{i} - \chi_{n_{1}}}, (1 \leq i, j \leq e)
\]

where \( \chi_{n_{1}} \in \text{Irr}(\bar{f}) \) (1 \( \leq i \leq e \)). Moreover (2.6) and (2.7) imply that \( \chi_{i} \in \text{Irr}(\bar{f}) \) (1 \( \leq i \leq e \)) and \( \chi_{n_{1}} \in \text{Irr}(\bar{f}) \) (1 \( \leq i \leq e \)) are pairwise orthogonal.

If \( |\bar{F}_{i}| \geq 4 \), then \( \chi_{i} \) must be a common irreducible constituent of \( \Delta^{\varphi}(1_{\bar{F}_{i}} - \lambda) \xi_{i} \) (\( \lambda \in \text{Irr}(\bar{F}_{i}) \setminus \{1_{\bar{F}_{i}}\} \)) since \( \Delta^{\varphi}(1_{\bar{F}_{i}} - \lambda) \xi_{i} = \delta_{u} \) (1 \( \leq i, j \leq e \)) when \( \lambda \neq \lambda' \). Further we have

\[
\Delta^{\varphi}(1_{\bar{F}_{i}} - \lambda) \xi_{i} = \varepsilon_{\chi_{i} - \chi_{n_{1}}}, (1 \leq i \leq e, \lambda \in \text{Irr}(\bar{F}_{i}) \setminus \{1_{\bar{F}_{i}}\})
\]

where \( \chi_{i} \) (1 \( \leq i \leq e \)) \( \in \text{Irr}(\bar{f}) \) and \( \chi_{n_{1}} \in \text{Irr}(\bar{f}) \) (1 \( \leq j \leq e \), \( \lambda \in \text{Irr}(\bar{F}_{i}) \setminus \{1_{\bar{F}_{i}}\} \)) are pairwise orthogonal.

Now let \( \lambda \in \text{Irr}(\bar{F}_{i}) \setminus \{1_{\bar{F}_{i}}\} \) be fixed again. By the definition of \( \Delta^{\varphi} \), we have \( \Delta^{\varphi}(\lambda^{-1}(1_{\bar{F}_{i}} - \lambda) \xi_{i}) = \lambda^{-1} \ast (\Delta^{\varphi}(1_{\bar{F}_{i}} - \lambda) \xi_{i}) = \varepsilon_{\chi_{i} - \chi_{n_{1}} - 1} \ast \chi_{i} \) and we have also \( \Delta^{\varphi}(\lambda^{-1}(1_{\bar{F}_{i}} - \lambda) \xi_{i}) = -\Delta^{\varphi}(1_{\bar{F}_{i}} - \lambda^{-1} \xi_{i}) = -\varepsilon_{\chi_{i} - \chi_{n_{1}} - 1} \). Therefore \( \lambda^{-1} \ast \chi_{n_{1}} = \chi_{i} \) and hence we have \( \chi_{n_{1}} = \lambda^{-1} \ast \chi_{i} \) for any \( i \) (1 \( \leq i \leq e \)).
Let $\mu \in M$. Since $(\Delta^q(\rho - \mu), \Delta^q(1_{\overline{F}}, -\lambda) \zeta_i) = 1$ for each $i$, $\lambda \in \text{Irr}(F)$, there exists exactly one character $\lambda \in \text{Irr}(F)$ that appears in $\Delta^q(\rho - \mu)$ for each $i$. On the other hand, we have $\Delta^q(\rho - \mu), \Delta^q(\rho - \mu) = e + 1$. Therefore, for each $i$, exactly one of the characters $\lambda \in \text{Irr}(F)$ appears in $\Delta^q(\rho - \mu)$ with multiplicity $\pm 1$. In fact, for each $i$, $\chi_i$ appears in $\Delta^q(\rho - \mu)$ with multiplicity $e$, because $\Delta^q(\rho - \mu), \Delta^q(1_{\overline{F}}, -\lambda) = 1$ for any $\lambda \in \text{Irr}(F)$. Here we replace $\chi_i$ by $\chi_i \cdot (\lambda + 1_{\overline{F}})$ if necessary. Noting $\Delta^q(\lambda(\rho - \mu)) = \lambda \cdot (\Delta^q(\rho - \mu))$, hence the sum of numbers of irreducible characters which appear in $\Delta^q((1_{\overline{F}}, -\lambda) \zeta_i)$ for $1 \leq i \leq e$, $\lambda \in \text{Irr}(F)$, and $\Delta^q(\lambda(\rho - \mu))$ for $\mu \in M$, $\lambda \in \text{Irr}(F)$, is at most $|F|(e + u)$. Recalling $|\text{Irr}(\overline{F})| = |F|(e + u)$, these imply

\begin{equation}
\Delta^q(\rho - \mu) = \sum_{\lambda \in \text{Irr}(F)} \epsilon_i \chi_i \zeta_i - \epsilon_i \chi_i, \epsilon_i = \pm 1,
\end{equation}

\begin{equation}
\Delta^q(\lambda(\rho - \mu)) = \sum_{\lambda \in \text{Irr}(F)} \epsilon_i (\lambda \chi_i - \epsilon_i \lambda \chi_i),
\end{equation}

where $\chi_i \in \text{Irr}(\overline{F})$. Moreover $\lambda \chi_i (1 \leq i \leq e, \lambda \in \text{Irr}(F)), \lambda \chi_i (\mu \in M, \lambda \in \text{Irr}(F))$ are pairwise orthogonal, and these are the ordinary irreducible characters in $\overline{F}$. (As $\Delta^q(\mu(\rho - \mu)) - \Delta^q(\rho - \mu) = \epsilon_i \chi_i - \epsilon_i \chi_i$, we have $\epsilon_i = \epsilon_i$.) Thus we can define a bijective isometry $\Delta$ from $L_K(L)$ onto $L_K(C_{\text{c}}(Q), \overline{F})$ such that $\Delta(\lambda \xi_i) = \epsilon_i (\lambda \chi_i)$ for $1 \leq i \leq e, \lambda \in \text{Irr}(F)$ and $\Delta(\lambda \mu) = \epsilon_i \lambda \chi_i$, $\mu \in M, \lambda \in \text{Irr}(F)$. Then $\Delta$ is a required extension of $\Delta^q$ from (2.8) and (2.9). This completes the proof.

By the above theorem and [16], Corollary 2, we have the following.

**Corollary** Let $B$ be a block of $G$ with abelian defect group $P$ and root $b$ in $C_{\text{c}}(P)$. If $[N_{\text{c}}(P, b) \cap P]$ is cyclic, then $B$ and $\Delta^q(B, b)$ are isotypic.

**References**


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