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Lines on Del Pezzo surfaces with Gorenstein singularities

Yasuhiro Ohshima

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1. Introduction.

1.1. A normal Gorenstein projective algebraic surface X over $\mathbf{C}$ is called the *del Pezzo surface* if the anti-canonical divisor $-K_X$ is ample. We put $d:=K_X^2>0$, which is called the *degree* of X. Let us denote by $x:=\mathrm{Sing}X=\{x_1,x_2,\ldots,x_k\}$ the singular locus of X, where $k=0$ if X is smooth. Let $\pi: M\longrightarrow X$ be the minimal resolution with the exceptional set $C:=\mathrm{Exc}(\pi)$ $=\pi^{-1}(x)=\cup_i C_i$, where C_i is irreducible. Then we have $K_M=\pi^*K_X-\sum_i m_i C_i$ $(0\leq m_i\in\mathbf{Z})$

Then Hidaka-Watanabe (cf. Demazure) obtain the following:

Theorem 1. ([3], [5])

The minimal resolution M is rational or a $\mathbf{P}^1$-bundle $M\xrightarrow{\phi}\mathbf{T}$ over an elliptic curve $\mathbf{T}$. Moreover,

(1) *M is rational $\Longrightarrow$ $\mathrm{Sing}X$ consists of at worst rational double points, and hence $K_M=\pi^*K_X$, in particular, $-K_X$ is linearly equivalent to an elliptic curve Γ with $\Gamma\cap\mathrm{Sing}X=\emptyset$, hence $-K_M=\hat{\Gamma}$, where $\hat{\Gamma}$ is the proper transform of Γ in M.*

(2) *M is a $\mathbf{P}^1$-bundle over an elliptic curve $\mathbf{T}\Longrightarrow X$ is a cone over $\mathbf{T}$. In particular, $C=\mathrm{Exc}(\pi)$ is the negative section of M and $K_M=\pi^*K_X-C$.*

Remark 1. *In case where X is rational and $d\geq 7$, one has easily the following:*

(1) $d=9\Longrightarrow M=X=\mathbf{P}^2$

(2) $d=8\Longrightarrow M=X=\mathbf{P}^1\times\mathbf{P}^1$, $\mathbf{F}_1$ *or* $X=\mathbf{Q}_0^2$, *where $\mathbf{F}_1$ is the Hirzebruch surface of degree 1 (the blowing up of $\mathbf{P}^2$ at one point), and $\mathbf{Q}_0$ is a quadric cone in $\mathbf{P}^3$.*

(3) $d=7\Longrightarrow M$ *is obtained from $\mathbf{F}_2$ (resp. $\mathbf{P}^2$) by the blowing up of one point which is not on the negative section (resp. two points in $\mathbf{P}^2$ infinitely near points allowed).*

1.2. Now in this paper we shall consider the case where X is rational and $1\leq d\leq 6$. Since X has at worst rational double points, each component C_i is a smooth rational curve with the self-intersection number $C_i^2=-2$, which we call the (-2)-curve. The anti-canonical divisor

$-K_X$ is linearly equivalent to an elliptic curve $\Gamma(\cong T)$ with $x\cap\Gamma=\emptyset$ by Theorem 1. Moreover $-K_X$ is very ample if $d\geq 3$, Bs $|-K_X|=\emptyset$ if $d=2$ and Bs $|-K_X|=\{p\}$(one point) with $p\notin \mathrm{Sing}X$ if $d=1$. We note that a normal cubic surface in $\mathbf{P}^3$ is a del Pezzo surface with degree $d=3$.

1.3. A curve ℓ in X is called a *line* on X if

(1) ℓ is smooth rational curve on X,

(2) $(K_X\cdot\ell)_X=-1$,

Applying the adjunction formula, we have easily

Lemma 1. *A curve ℓ in X is a "line" if and only if the proper transform $\hat{\ell}$ of ℓ in the minimal resolution M is the exceptional curve of first kind, that is, the smooth rational curve with the self-intersection number $\hat{\ell}^2=-1$ (we call simply the $(-1)-$curve).*

Remark 2. if $d\geq 3$, then the line is a real line (in $\mathbf{P}^d$) because $-K_X$ is very ample if $d\geq 3$.

1.4. Now let us introduce the following invariants:

- δ: the number of lines on X.
- $k\geq 0$: the number of singularities of X.
- $\mu:=\sum_i\mu(x_i)$ the total Milnor number, where $\mu(x_i)$ is the Milnor number of the singularity x_i.

Remark 3. *In our case, the singular locus $x=\mathrm{Sing}X$ consists of at worst rational double points. Hence the total Milnor number μ is equal to the second Betti number $b_2(C)$ of C, that is, the number of irreducible components of the exceptional curves $C=\mathrm{Exc}(\pi)=\cup_i C_i$.*

In the paper [1], Bruce-Wall find the following interesting relation among these invariants δ, k and μ:

Theorem 2. *Assume that $d=3$, that is, X is a normal cubic surface in $\mathbf{P}^3$. Then we have*

$$(\spadesuit)\qquad \delta=\frac{1}{2}(8-\mu)(7-\mu)+k-1$$

Remark 4.

(1) *In Theorem 2, if X is smooth (that is, $k=\mu=0$), then we have $\delta=27$ on X,whichi is the well-known number of lines on a smooth cubic-surface* [6].

(2) *Applying the complete list of the classification of (δ, k, μ) for cubic surfaces in $\mathbf{P}^3$(see* [1]*), Bruce-Wall obtain the formula $(\spadesuit)$ for cubic surfaces.*

Problem 1.

(1) *Give a direct proof of the formula* (♠) *for cubic surfaces.*

(2) *Find a relation for the remaining cases where* $1 \leq d \leq 6$, $(d \neq 3)$, *especially, the interesting cases of* $d=1,2$.

2. Main theorems.

2.1. Now we shall give a similar relation for the cases that $d=5$ and $d=7$.

Theorem A. *Assume that* $d=5$, *that is,* X *is a del Pezzo surface of degree 5 in* $\mathbf{P}^5$. *Then*

(♡)
$$\delta=\frac{1}{2}(5-\mu)(4-\mu)+k$$

In particular, if S *is smooth, then* $\delta=10$.

Theorem B. *Assume that* $d=7$. *Then we have*

(◇)
$$\delta=\frac{1}{2}(3-\mu)(2-\mu)+k$$

In particular, if S *is smooth, then* $\delta=3$.

3. Proof of Theorem A and B

3.1. Assume that $d=5$. Let X be a normal del Pezzo surface of degree 5 in $\mathbf{P}^5$. By the classification of Coray-Tsfasman [2], one has the following:

(T1)
$$(\mathrm{Sing}X,\delta)=(\emptyset\,;\,10),\ (A_1\,;\,7),\ (2A_1\,;\,5),\ (A_2\,;\,4)$$
$$(A_1A_2\,;\,3),\ (A_3\,;\,2),\ (A_4\,;\,1).$$

This implies the following

(T2)
$$(\delta,\ \mu,\ k)=(10,\ 0,\ 0),\ (7,\ 1,\ 1),\ (5,\ 2,\ 2),\ (4,\ 2,\ 1)$$
$$(3,\ 3,\ 2),\ (2,\ 3,\ 1),\ (1,\ 4,\ 1).$$

One can easily check that these datum (T2) satisfy the relation (♡). This proves Theorem A. □

3.2. Next assume that $d=7$. Then one has easily $(\mathrm{Sing}X,\ \delta)=(\emptyset\,;\,3),\ (A_1\,;\,2)$, that is,

$$(\delta,\ \mu,\ k)=(3,\ 0,\ 0),\quad (2,\ 1,\ 1).$$

From this we have the relation (◇). This proves Theorem B. □

4. Del Pezzo surfaces with exactly one line.

4.1. Let X be a del Pezzo surface with exactly one line ℓ (that is, $\delta=1$) and $\pi: M \longrightarrow X$ the minimal resolution with exceptional set $C=\cup_i C_i=\pi^{-1}(x)$, where $x=\mathrm{Sing}X$. Let $\Sigma_r=\{p_1, p_2,\dots,p_r\}$; $(3\leq r\leq 8)$ be the set of points (infinitely near points allowed) on $\mathbf{P}^2$, which are in *almost general position* (cf. [3], [5]). Then we have $M=B_{\Sigma_r}(\mathbf{P}^2)$, that is, M is the blowing up of $\mathbf{P}^2$ with center Σ_r. We have $K_M=-\hat{\Gamma}$ by Theorem 1.

Theorem C(cf. [4]). *The singular locus* $x=\mathrm{Sing}X\neq\emptyset$.

(I): *The type of the singularity is as follows:*

(1) $d=1 \Longrightarrow \mathrm{Sing}X$ *is of* E_8*-type*

(2) $d=2 \Longrightarrow \mathrm{Sing}X$ *is of* E_7*-type*

(3) $d=3 \Longrightarrow \mathrm{Sing}X$ *is of* E_6*-type*

(4) $d=4 \Longrightarrow \mathrm{Sing}X$ *is of* D_5*-type*

(5) $d=5 \Longrightarrow \mathrm{Sing}X$ *is of* A_4*-type*

(6) $d=6 \Longrightarrow \mathrm{Sing}X$ *is of* A_2+A_1*-type*

(II): *Let* ℓ *be such a unique line on X, then we have* $x\in\ell$ *and* $X-\ell\cong\mathbf{C}^2$.

Proof. The proof is done by dividing into severl steps.

Claim (a). *The* (-2)*-curve on* M *is an irreducible component of the exceptional set* C.

In fact, let E be a (-2)-curve on M. Then, by the adjunction formula, one has $(K_M\cdot E)=0$, hence $(\pi^*K_X\cdot E)=0$. This proves the claim. □

Let ℓ be such a unique line on X and $\hat{\ell}_r$ the proper transform of ℓ in $M_r:=M$. By definition the curve $\hat{\ell}_r$ is a unique (-1)-curve in M_r. Let

$$\phi: M_r \xrightarrow{\phi_r} M_{r-1} \xrightarrow{\phi_{r-1}} M_{r-2}\cdots \longrightarrow \cdots M_1 \xrightarrow{\phi_1} \mathbf{P}^2$$

be the sequence of blowing ups of $\mathbf{P}^2$ with center Σ_r. Then $\Gamma^*:=\phi(\hat{\Gamma})$ is a smooth cubic curve on $\mathbf{P}^2$ passing through all the points of Σ_r.

Claim (b). $\mathrm{b}_2(X)=1$ *and* $\mathrm{b}_2(C)=r$.

In fact, we may assume that $\phi_r(\hat{\ell}_r)=p_r$. If there is a (-1)-curve $\hat{\ell}_{r-1}$ in M_{r-1}, then we must have $p_r\in\hat{\ell}_{r-1}$ since M_r contains a unique (-1)-curve. Similarly there is a (-1)-curve $\hat{\ell}_j$ in M_j

such that $p_{j-1}:=\phi_j(\hat{\ell}_j)\in\hat{\ell}_{j-1}(2\leq j\leq r-1)$. This shows that $\hat{\ell}_{r-1}\cup\hat{\ell}_{r-2}\cup\cdots\cup\hat{\ell}_1$ is a linear tree of (-2)-curves. Then the starting point $p_1\in\Gamma^*$ must be a point of inflexion since M_r contains a unique (-1)-curve. Let ℓ_0 be a tangent line of Γ^* at the point p_1 with $(\Gamma^*\cdot\ell_0)_{p_1}=3$, which always exists. Then the proper transform $\hat{\ell}_0$ in M_r is the (-2)-curve. Since the second Betti number $b_2(C)$ is equal to the number of exceptional curves, by Claim (a), we have $b_2(C)\geq r$. On the other hand, since $b_2(M)=b_2(C)+b_2(X)$ and $b_2(M)=r+1$, we have the claim. □

Claim (c). $H_1(\partial T\,;\,\mathbf{Z})\cong\mathbf{Z}_d$.

In fact, let ∂T_i be the boundary of a sufficiently small contractible Stein neighbourhood T_i of x_i ($x_j\notin T_i$ if $i\neq j$) in X and set $\partial T=\cup_{i=1}^k\partial T_i$. Applying the Mayer-Vietoris exact sequence, one has The Poincaré exact sequence:

$$0\longrightarrow H^2(X\,;\,\mathbf{Z})\xrightarrow{P_2}H_2(X\,;\,\mathbf{Z})\longrightarrow H_1(\partial T:\mathbf{Z})\longrightarrow 0$$

We have $H^2(X\,;\,\mathbf{Z})\cong\mathbf{Z}\cdot c_1(O_X(\Gamma))$ and $H_2(X\,;\,\mathbf{Z})\cong\mathbf{Z}\ell$. Since $P_2(c_1(O_X(\Gamma)))\sim d\ell$ in $H_2(X:\mathbf{Z})$, we have the claim. □

By Theorem 1, every singular point x_i is a rational double point, that is, x_i is one of the type E_8, E_7, E_6, $D_{n+2}(n\geq 2)$, $A_{n+1}(n\geq 1)$. An easy computation shows that

(1) $H_1(\partial T_i\,;\,\mathbf{Z})\cong 0$ if E_8-type.

(2) $H_1(\partial T_i\,;\,\mathbf{Z})\cong\mathbf{Z}_2$ if E_7-type.

(3) $H_1(\partial T_i\,;\,\mathbf{Z})\cong\mathbf{Z}_3$ if E_6-type.

(4) $H_1(\partial T_i\,;\,\mathbf{Z})\cong\mathbf{Z}_4$ (resp. $\mathbf{Z}_2\oplus\mathbf{Z}_2$) if D_{n+4}-type with odd n(resp. even n).

(5) $H_1(\partial T_i\,;\,\mathbf{Z})\cong\mathbf{Z}_{n+1}$ if x_i is of A_n-type.

Taking into account that $b_2(C)=9-d$, one can easily show the assertion (I) of Theorem C.

Show the assertion (II). Since

$$\hat{\ell}_r\cup C_i=\hat{\ell}_{r-1}\cup\hat{\ell}_{r-2}\cup\cdots\cup\hat{\ell}_1\cup\hat{\ell}_0,$$

we obtain that

$$\begin{aligned}X-\ell&\cong M-(\hat{\ell}_{r-1}\cup\hat{\ell}_{r-2}\cup\cdots\cup\hat{\ell}_1\cup\hat{\ell}_0)\\&\cong\mathbf{P}^2-\ell_0\\&\cong\mathbf{C}^2.\end{aligned}$$

This completes the proof of Theorem C.

□

Finally we shall propose the following

Problem 2. Assume that $\delta\leq 4$ and $\ell_i,\ldots,\ell_s$ the lines. Then is $X-\cup_i^s\ell_i\cong\mathbf{C}^2$, $\mathbf{C}^*\times\mathbf{C}$, or $(\mathbf{C}^*)^2$?

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