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On topological pressure of Cantor minimal systems and a strong orbit equivalence

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Abstract

This is an announcement of a result on topological pressure of Cantor minimal systems and a strong orbit equivalence. Later we will show the detail of this result in the paper [S4]: F. Sugisaki, Topological pressure of cantor minimal systems within a strong orbit equivalence class.

The concept of orbit equivalence for measurable dynamical system was introduced by H. Dye in 1959 ([D1], [D2]). Dye showed that all finite ergodic measure preserving dynamical systems are orbit equivalent. However in those days not many specialists in studying ergodic theory took much notice of this result because orbit equivalence is a much weaker than measure theoretic isomorphism, which they usually considered, as an equivalence relation on dynamical systems. Later, W. Krieger showed that two Von Neumann algebras associated with non-singular ergodic dynamical systems are $*$-isomorphic if and only if the dynamical systems are orbit equivalent ([K1], [K2]). Let us call this connection between non-singular measurable dynamical system and von Neumann algebra a measure theoretic version. Then, we can easily raise a question whether there is a topological version. That is, is there a connection of the same sort between C^* algebras and topological dynamics? The answer to this question was given by Giordano, Putnam and Skau in 1993 ([GPS]), which is stated as follows: Two C^* algebras associated with minimal homeomorphisms on Cantor sets are $*$-isomorphic if and only if the topological dynamical systems are strongly orbit equivalent. We will refer to a minimal homeomorphism on a Cantor set as a *Cantor minimal system*.

On the relationship between strong orbit equivalence of Cantor minimal systems and topological entropy, it was proven in [S1] and [S2] that for any strong orbit equivalence class of a Cantor minimal system it contains Cantor minimal systems of all topological entropies. This is an analogy to Dye's results in topological version, that is, Dye showed that for any (mesure theoretic) orbit equivalence class of a measure preserving erogdic dynamical system it contains ergodic systems of all (measure theoretic) entropies. This result tells us two things. The first is that two equivalence classes among Cantor minimal systems, one for strong orbit equivalence and the other for having the same entropy, are independent of each other. The second is that there are many kinds of dynamical systems within any strong orbit equivalence class.

In this paper we announce a result on topological pressure of Cantor minimal systems and a strong orbit equivalence, which is a generalized result of [S1], [S2] and [S3] by the following. For a topological dynamical system (X, T), denote $M(X)$ ty the set of Borel probability measures on X and $M(X, T)$ by the set of T-invariant Borel probability measures on X.

Theorem 1. *Suppose that* (X, φ) *is a Cantor minimal system and* f *is a potensial function on* X. *Choose any* α *with*

$$\exp\left(\sup\left\{\int f d\mu \middle| \mu \in M(X,\varphi)\right\}\right) \leq \alpha \leq \infty \tag{1}$$

and fix it. Then there exists a Cantor minimal system (Y, ψ) *strongly orbit equivalent to* (X, φ) *such that*

$$P(\psi, f\circ\theta^{-1}) = \log\alpha,$$

where $P(\psi, \cdot)$ *is the topological pressure of* ψ *and* $\theta: X \to Y$ *is strong orbit equivalence map. If* α *is finite, we can take* ψ *as an expansive homeomorphism.*

Remark 2. (i) For a topological dynamical system (X, T) and potential function $f \in C(X, \mathbf{R})$, the variational principle of topological pressure (see Theorem 9. 10 in [W1])

$$P(T,f) = \sup\left\{h_\mu(T) + \int f d\mu \middle| \mu \in M(X,T)\right\}$$

implies that

- $P(T,f) \geq \sup\{\int f d\mu | \mu \in M(X,T)\}$,
- $P(T,f) = \sup\{\int f d\mu | \mu \in M(X,T)\}$ iff $h(T)=0$.

So(1)is the best possible inequality which α can take.
(ii)If $f=0$, then P(ψ, 0) is equal to the topological entropy of ψ.

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