Kumamoto J. Math.
Vol. 18, 103-106 March (2005)

On a distribution formula for solutions of linear autonomous systems with fluctuations

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(Received January 31, 2005)

Abstract

In this communication, we announce a formula on the distribution of solutions of 2-dimensional linear autonomous systems with fluctuations, whose details can be seen in [1] and will be published elsewhere.

Let a, b, c, d be real constants, and $S(t)$, $T(t)$ be scattered functions whose means are 0, and variances are positive constants σ_S^2, σ_T^2 respectively. Let ρ_{ST} be the constant correlation coefficient between S and T.

We consider the equation

$$\begin{cases} dx=(a\cdot x+b\cdot y)dt+S(t)\sqrt{dt} \\ dy=(c\cdot x+d\cdot y)dt+T(t)\sqrt{dt} \end{cases}$$

with the initial condition

$$\begin{cases} x(0)=x_0 \\ y(0)=y_0 \end{cases}$$

Let λ and μ be distinct complex eigenvalues of the matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and X be a (2, 2)-matrix such that $AX=X\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. Then the mean of the distribution of solutions is given by

$$\begin{pmatrix} m_x(t) \\ m_y(t) \end{pmatrix}=X\begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{pmatrix}X^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

The fluctuation of solutions can be calculated by the following Brownian integral.

$$\int_0^t X\begin{pmatrix} e^{\lambda(t-s)} & 0 \\ 0 & e^{\mu(t-s)} \end{pmatrix}X^{-1}\begin{pmatrix} S(s) \\ T(s) \end{pmatrix}\sqrt{ds}$$

Let

$$\begin{pmatrix} g_{11}(s,t) & g_{12}(s,t) \\ g_{21}(s,t) & g_{22}(s,t) \end{pmatrix} = X \begin{pmatrix} e^{\lambda(t-s)} & 0 \\ 0 & e^{\mu(t-s)} \end{pmatrix} X^{-1},$$

then the variances and the covariance of the solutions are given by the following formula.

$$\sigma_x^2(t) = \sigma_S^2 \int_0^t g_{11}(s,t)^2 ds + 2\rho_{ST}\sigma_S\sigma_T \int_0^t g_{11}(s,t) g_{12}(s,t) ds + \sigma_T^2 \int_0^t g_{12}(s,t)^2 ds$$

$$\sigma_y^2(t) = \sigma_S^2 \int_0^t g_{21}(s,t)^2 ds + 2\rho_{ST}\sigma_S\sigma_T \int_0^t g_{21}(s,t) g_{22}(s,t) ds + \sigma_T^2 \int_0^t g_{22}(s,t)^2 ds$$

$$\sigma_{xy}^2(t) = \sigma_S^2 \int_0^t g_{11}(s,t) g_{21}(s,t) ds$$

$$+ \rho_{ST}\sigma_S\sigma_T \int_0^t \{g_{11}(s,t) g_{22}(s,t) + g_{12}(s,t) g_{21}(s,t)\} ds + \sigma_T^2 \int_0^t g_{12}(s,t) g_{22}(s,t) ds$$

The correlation coefficient of the solutions is given by

$$\rho_{xy}(t) = \frac{\sigma_{xy}^2(t)}{\sigma_x(t)\sigma_y(t)}.$$

Theorem. The distribution of the solutions is the 2-dimensional normal distribution

$$N\Big(m_x(t), m_y(t), \sigma_x^2(t), \sigma_y^2(t), \rho_{xy}(t)\Big)$$

at each t, and the density function is given by

$$u(t,x,y) = \frac{1}{2\pi\sqrt{1-\rho_{xy}(t)^2}\,\sigma_x(t)\sigma_y(t)}$$

$$\exp\left(-\frac{\left(\frac{x-m_x(t)}{\sigma_x(t)}\right)^2 - 2\rho_{xy}(t)\left(\frac{x-m_x(t)}{\sigma_x(t)}\right)\left(\frac{y-m_y(t)}{\sigma_y(t)}\right) + \left(\frac{y-m_y(t)}{\sigma_y(t)}\right)^2}{2\left(1-\rho_{xy}(t)^2\right)} \right)$$

Example. Let $A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$, $\sigma_S = \sigma_T = 0.05, \rho_{ST} = -0.7, x_0 = 1, y_0 = -1$,

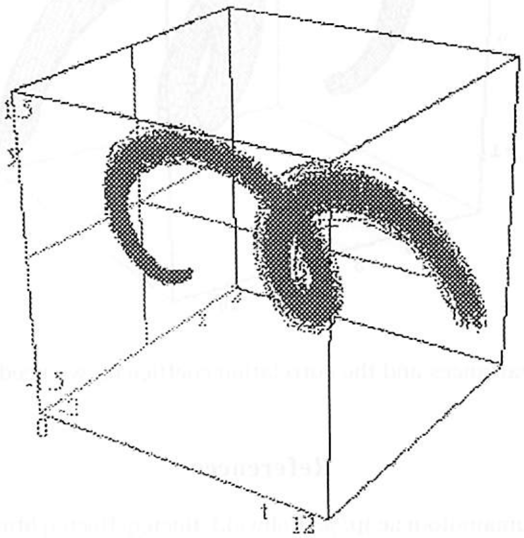
then the following equation has the required properties.

$$\begin{cases} dx = (x+2y)dt + 0.05 \cdot N_{0.3,0.5}^{0}(t)\sqrt{dt} \\ dy = (-x-y)dt + 0.05 \cdot N_{0.3,0.5}^{0.746817}(t)\sqrt{dt} \end{cases}$$

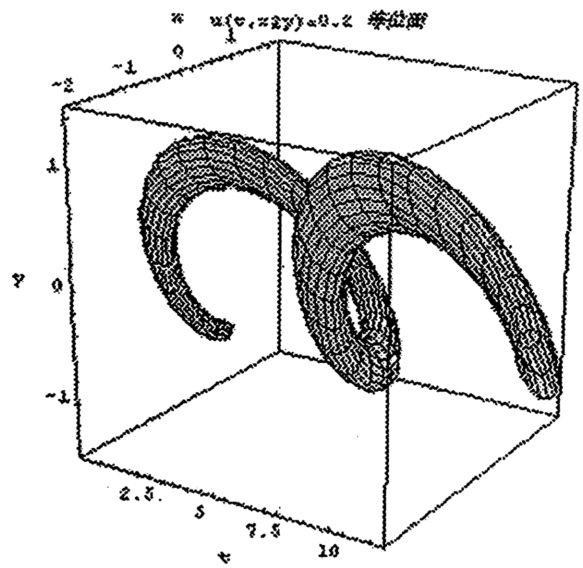
Because the relation $\rho\left(N^{0}_{0.3,0.5}, N^{\theta}_{0.3,0.5}\right)=\cos \pi\theta$ holds[2], then

$$\theta=\frac{1}{\pi}\cos^{-1}(-0.7)\cong 0.746817.$$

A perspective view of 100 solutions is the following.



The level surface $u(t, x, y)=0.2$ is the following figure.



To check means, variances and the correlation coefficient, we need much more numerical experiments[1].

References

[1] http://www.sci.kumamoto-u.ac.jp/%7Eohwaki/flucteq/flucteq.html(in Japanese)

[2] S. Ohwaki, K. Matsuda, On a formula for correlation coefficients between normally scattered functions, *Kumamoto Journal of Mathematics*, **12** (1999), 73-79.

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