Kumamoto J. Math.
Vol. 19 (2006), 1-13

On the distance between orbits under a group of isometries on a metric space

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(Received September 26, 2005)

Abstract. Let X be a metric space and let G be a group acting on X such that for each g in G, the map from X to X given by $x \mapsto gx$ is an isometry. We investigate various properties of the distance between G-orbits on X and of the associated quotient space.

1. Introduction

It is well known that the canonical valuation on a local field K has a unique extension to a fixed algebraic closure $\overline{K}$ of K. This metric rigidity offers one the possibility to define and study various metric invariants associated to algebraic elements over K. By natural limiting procedures one may quite often also associate analogous metric invariants to elements from the topological completion of $\overline{K}$ which are transcendental over K. Moreover, the fact that any continuous automorphism of this completion over K is an isometry allows one to produce such metric objects, which are either Galois invariant, or at least behave nicely under the action of the Galois group. With this context in mind, below we proceed to develop some metric notions in a more general framework. We start with a metric space X, and denote by d_X, or simply by d, the corresponding distance function. We assume the action of a group G on X is given in such a way that for each g in G, the map from X to X given by $x \mapsto gx$ is an isometry. With X and G as above, we define an equivalence relation on X as follows. If $x, y \in X$ we say that x, y are equivalent and write $x \sim_G y$, or $x \sim y$ if G is understood, provided there exists a sequence $(g_n)_{n\in\mathbb{N}}$ of elements of G such that $\lim_{n\to\infty} d(x, g_n y) = 0$. Let $\mathscr{X}$ denote the quotient space $X/\sim_G$, and let $\psi : X \to \mathscr{X}$ be the canonical map which sends each element $x \in X$ to its equivalence class. On $\mathscr{X}$ we introduce a distance $d_{\mathscr{X}}$ by

$$d_{\mathscr{X}}(u,v) = \inf\{d_X(x,y) : x \in \psi^{-1}(u),\ y \in \psi^{-1}(v)\},$$

for any $u, v \in \mathscr{X}$, which makes $\mathscr{X}$ a metric space. As an example, let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers, K a finite field extension of $\mathbb{Q}_p$, $\overline{K}$ a fixed

Mathematics Subject Classification (2000): 11S99
Keywords: metric space, orbits, isometry

algebraic closure of K, and $\mathbb{C}_p$ the completion of $\overline{K}$ with respect to the p-adic absolute value. The group $G_K := \mathrm{Gal}_{\mathrm{cont}}(\mathbb{C}_p/K)$ of all continuous automorphisms of $\mathbb{C}_p$ over K is canonically isomorphic to the Galois group $\mathrm{Gal}(\overline{\mathrm{K}}/\mathrm{K})$, and any automorphism $g \in G_K$ is an isometry on $\mathbb{C}_p$. We may then take $X = \mathbb{C}_p$ and $G = G_K$. Note that in this case G is compact, any G-orbit Gy, where $y \in \mathbb{C}_p$, is a compact subset of $\mathbb{C}_p$, and the above condition $\lim_{n\to\infty} d(x, g_n y) = 0$ from the definition of equivalence forces x to belong to the Galois orbit Gy. Thus in this case $\mathscr{X}$ coincides with the space of Galois orbits. One has more flexibility here, in the sense that X may be replaced by any of its subsets which are invariant under the action of G_K, such as any finite or infinite normal field extension of K contained in $\mathbb{C}_p$. Also, one may choose G to be any subgroup of G_K. Returning to the general case, in what follows we investigate some basic properties of the canonical map ψ, the quotient space $\mathscr{X}$ and the distance function $d_{\mathscr{X}}$.

2. The quotient space $\mathscr{X}$

Let X be a metric space, $d : X \times X \to [0, \infty)$ the distance function, and let G be a group which acts on X in such a way that for each $g \in G$ the map from X to X given by $x \mapsto gx$ is an isometry, that is,

$$d(gx, gy) = d(x, y),$$

for any x, y in X and any g in G. If $x, y \in X$ we write $x \sim y$ provided there is a sequence $(g_n)_{n\in\mathbb{N}}$ of elements from G such that $\lim_{n\to\infty} d(x, g_n y) = 0$. Note that in this case we also have $d(y, g_n^{-1}x) = d(g_n y, x) \to 0$ as $n \to \infty$, hence if $x \sim y$ then $y \sim x$. Clearly $x \sim x$ for any $x \in X$. Let now $x, y, z \in X$ such that $x \sim y$ and $y \sim z$. There are sequences $(g_n)_{n\in\mathbb{N}}$ and $(t_m)_{m\in\mathbb{N}}$ in G such that $\lim_{n\to\infty} d(x, g_n y) = \lim_{m\to\infty} d(y, t_m z) = 0$. Let us put $\tau_n = g_n t_n \in G$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} d(x, \tau_n z) &\le d(x, g_n y) + d(g_n y, \tau_n z) \\ &= d(x, g_n y) + d(y, g_n^{-1}\tau_n z) = d(x, g_n y) + d(y, t_n z). \end{aligned}$$

Here the far right side tends to 0 as $n \to \infty$. It follows that $x \sim z$, and therefore $(\sim)$ is an equivalence relation on X. We denote the set of equivalence classes by $\mathscr{X}$. Let us denote by $\psi_{G,X}$, or simply by ψ, the canonical map $\psi : X \to \mathscr{X}$ which sends each $x \in X$ to its equivalence class. For any $u, v \in \mathscr{X}$ we set

$$d_{\mathscr{X}}(u, v) = \inf\{d(x, y) : x \in \psi^{-1}(u),\ y \in \psi^{-1}(v)\} \in [0, \infty).$$

Let us show that $d_{\mathscr{X}}$ is a distance on $\mathscr{X}$. Clearly $d_{\mathscr{X}}(u, v) = d_{\mathscr{X}}(v, u)$ for any $u, v \in \mathscr{X}$. Assume now that $u, v \in \mathscr{X}$ are such that $d_{\mathscr{X}}(u, v) = 0$. Let $(\varepsilon_r)_{r\in\mathbb{N}}$ be a strictly decreasing sequence of real numbers with $\lim_{r\to\infty} \varepsilon_r = 0$. By the definition of $d_{\mathscr{X}}(u, v)$, for each $r \in \mathbb{N}$ there are $x_r \in \psi^{-1}(u)$ and $y_r \in \psi^{-1}(v)$ such that $d(x_r, y_r) < \varepsilon_r$. Let us fix elements $x^* \in \psi^{-1}(u)$ and $y^* \in \psi^{-1}(v)$, and choose an $r \in \mathbb{N}$. Since $\psi(x^*) = u = \psi(x_r)$ we have $x^* \sim x_r$, so there exists a sequence $(g_{r,n})_{n\in\mathbb{N}}$ of elements from G such that $\lim_{n\to\infty} d(x^*, g_{r,n}x_r) = 0$. Similarly, there is a sequence $(t_{r,n})_{n\in\mathbb{N}}$ in G such that $\lim_{n\to\infty} d(y^*, t_{r,n}y_r) = 0$. Choose an n_r for which we have simultaneously $d(x^*, g_{r,n_r}x_r) < \varepsilon_r$ and $d(y^*, t_{r,n_r}y_r) < \varepsilon_r$. Then

$$\begin{aligned} d(x^*, g_{r,n_r}t_{r,n_r}^{-1}y^*) &\leq d(x^*, g_{r,n_r}x_r) + d(g_{r,n_r}x_r, g_{r,n_r}y_r) + d(g_{r,n_r}y_r, g_{r,n_r}t_{r,n_r}^{-1}y^*) \\ &= d(x^*, g_{r,n_r}x_r) + d(x_r, y_r) + d(y_r, t_{r,n_r}^{-1}y^*) \\ &< 2\varepsilon_r + d(t_{r,n_r}y_r, y^*) < 3\varepsilon_r. \end{aligned}$$

It follows that the sequence $(g_{r,n_r}t_{r,n_r}^{-1})_{r\in\mathbb{N}}$ in G satisfies $\lim_{r\to\infty} d(x^*, g_{r,n_r}t_{r,n_r}^{-1}y^*) = 0$, therefore $x^* \sim y^*$, and hence $u = \psi(x^*) = \psi(y^*) = v$, as desired. Lastly, let $u, v, w \in \mathscr{X}$ and choose a real number $\varepsilon > 0$. By the definition of $d_{\mathscr{X}}(u,v)$ and $d_{\mathscr{X}}(v,w)$ there are elements $x \in \psi^{-1}(u)$, $y, y' \in \psi^{-1}(v)$, and $z \in \psi^{-1}(w)$, such that $d(x,y) < d_{\mathscr{X}}(u,v) + \varepsilon$ and $d(y', z) < d_{\mathscr{X}}(v,w) + \varepsilon$. Since $y \sim y'$, there is a sequence $(g_n)_{n\in\mathbb{N}}$ in G such that $\lim_{n\to\infty} d(y, g_n y') = 0$. Fix an $n_\varepsilon \in \mathbb{N}$ for which $d(y, g_{n_\varepsilon}y') < \varepsilon$. The element $g_{n_\varepsilon}z$ belongs to $\psi^{-1}(w)$, and we have

$$\begin{aligned} d_{\mathscr{X}}(u,v) \leq d(x, g_n z) &\leq d(x,y) + d(y, g_n y') + d(g_n y', g_n z) \\ &< d_{\mathscr{X}}(u,v) + 2\varepsilon + d(y', z) < d_{\mathscr{X}}(u,v) + d_{\mathscr{X}}(v,w) + 3\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$, it follows that $d_{\mathscr{X}}(u,w) \leq d_{\mathscr{X}}(u,v) + d_{\mathscr{X}}(v,w)$. We conclude that $(\mathscr{X}, d_{\mathscr{X}})$ is a metric space.

Next, we discuss some properties of the canonical map $\psi : X \to \mathscr{X}$. First of all, note that ψ is continuous, in fact it is 1-Lipschitzian, that is,

$$d_{\mathscr{X}}(\psi(x), \psi(y)) \leq d_X(x,y),$$

for any $x, y \in X$. Another basic property of ψ is that it is an open map, thus for any open subset U of X, $\psi(U)$ is an open subset of $\mathscr{X}$. Indeed, let U be an open subset of X, and fix a point u in $\psi(U)$. Choose an element $x \in \psi^{-1}(u)$, and a real number $\delta > 0$ such that the open ball in X centered at x, of radius δ, is contained in U. Then the open ball of radius δ in $\mathscr{X}$ centered at u is contained in $\psi(U)$. For, let $v \in \mathscr{X}$ with $d_{\mathscr{X}}(u,v) < \delta$. By the definition of $d_{\mathscr{X}}(u,v)$, there are elements $x' \in \psi^{-1}(u)$ and $y \in \psi^{-1}(v)$ for which $d(x', y) < \delta$. Since $x \sim x'$, there is a sequence $(g_n)_{n\in\mathbb{N}}$ in G such that $\lim_{n\to\infty} d(x, g_n x') = 0$. Each element $g_n y$ belongs to $\psi^{-1}(v)$, and

$$d(x, g_n y) \leq d(x, g_n x') + d(g_n x', g_n y) = d(x, g_n x') + d(x', y).$$

Here for n large enough the far right side is strictly smaller than δ, so $g_n y$ belongs to the open ball of radius δ centered at x, and hence it belongs to U. Thus v belongs to $\psi(U)$, which completes the proof that $\psi(U)$ is open.

The notion of quasi isometry naturally appears in some questions of a metric nature involving local fields (see [5], [6], [7]). If (X, d_X), (Y, d_Y) are metric spaces, a map $\varphi : X \to Y$ is said to be a quasi isometry provided that for any $x, y \in X$

$$d_Y(\varphi(x), \varphi(y)) = \inf_{z \in \varphi^{-1}(\varphi(y))} d_X(x, z).$$

Let us say that a map $\varphi : X \to Y$ is a strong quasi isometry provided it is 1-Lipschitzian, and for any $x, y \in X$ there exists a $z \in \varphi^{-1}(\varphi(y))$ such that

$$d_Y(\varphi(x), \varphi(y)) = d_X(x, z).$$

Clearly any strong quasi isometry is a quasi isometry, and any quasi isometry is 1-Lipschitzian. Also, a quasi isometry is an isometry if and only if it is injective.

Returning to our context, let us note that the map $\psi : X \to \mathscr{X}$ is a quasi isometry. Indeed, fix $x, y \in X$ and denote $u = \psi(x)$, $v = \psi(y)$. Then

$$\begin{aligned} d_{\mathscr{X}}(u, v) &= \inf\{d(x', y') : x' \in \psi^{-1}(u),\ y' \in \psi^{-1}(v)\} \\ &\leq \inf_{z \in \psi^{-1}(V)} d(x, z). \end{aligned}$$

The last inequality is in fact an equality, because for any $x' \in \psi^{-1}(u)$, $y' \in \psi^{-1}(v)$, and any real number $\varepsilon > 0$, there exists g in G such that $d(x, gx') < \varepsilon$, thus if we put $z = gy' \in \psi^{-1}(v)$, we see that

$$d(x, z) \leq d(x, gx') + d(gx', z) < \varepsilon + d(x', y').$$

In conclusion, ψ is a quasi isometry.

We collect the results of this section in the following theorem.

Theorem 1. *Let G be a group acting on a metric space X such that each map $x \mapsto gx$, with g in G, is an isometry on X. Then*

(i) *$(\mathscr{X}, d_{\mathscr{X}})$ is a metric space.*

(ii) *The canonical map $\psi : X \to \mathscr{X}$ is an open map, and it is a quasi isometry.*

Remark 1. Let (X, d_X), (Y, d_Y) be metric spaces and let $\varphi : X \to Y$ be a quasi isometry with the property that for each element $y \in X$ the set $\{z \in X : \varphi(z) = \varphi(y)\}$ is compact. Then for any $x, y \in X$ the infimum $\inf\{d(x, z) : z \in \varphi^{-1}(\varphi(y))\}$ is attained, so $d_Y(\varphi(x), \varphi(y)) = d_X(x, z)$ for some $z \in X$ with $\varphi(z) = \varphi(y)$, and hence φ is a strong quasi isometry. In particular, if one applies Theorem 1 with $X = \mathbb{C}_p$ for some prime number p, and $G = G_K$ for some finite field extension K of $\mathbb{Q}_p$, then for each element y of $\mathbb{C}_p$ the set $\{z \in \mathbb{C}_p : \psi(z) = \psi(y)\}$ coincides with the Galois orbit $G_K y = \{\sigma y : \sigma \in G_K\}$ which is compact. Therefore in this case the map ψ is a strong quasi isometry.

3. Change of space

As a mater of terminology, if X, Y are sets and a group G acts on each of the two sets X and Y, we say that a map $F : X \to Y$ is G-equivariant provided that $F(gx) = gF(x)$, for any $x \in X$ and any $g \in G$.

As an example, in the above particular case when $X = Y = \mathbb{C}_p$ for some prime number p and $G = G_K$ for some finite field extension K of $\mathbb{Q}_p$, let F be a polynomial function in one variable with coefficients in $\mathbb{C}_p$. Then one sees that F is G-equivariant if and only if all the coefficients of F belong to K.

Let now (X, d_X) and (Y, d_Y) be metric spaces, and let G be a group which acts on X and on Y such that for any $g \in G$ the maps $X \to X$, $x \mapsto gx$, and $Y \to Y$, $y \mapsto gy$, are isometries on X and respectively on Y. Denote by $\mathscr{X}$ and $\mathscr{Y}$ the sets of equivalent classes $X/{\sim_G}$ and $Y/{\sim_G}$ and by $\psi_{G,X} : X \to \mathscr{X}$ and $\psi_{G,Y} : Y \to \mathscr{Y}$ the canonical projections respectively. We are interested to see under which circumstances one can associate to a map $F : X \to Y$, a map $f : \mathscr{X} \to \mathscr{Y}$ such that $f \circ \psi_{G,X} = \psi_{G,Y} \circ F$, and what continuity or metric properties of F are inherited by f. Some such properties are collected in the following result.

Theorem 2. *Let (X, d_X), (Y, d_Y) be metric spaces, let G be a group acting on X and on Y such that for each $g \in G$ the maps $X \to X$, $x \mapsto gx$ and $Y \to Y$, $y \mapsto gy$ are isometries, and let $F : X \to Y$ be G-equivariant.*

(i) *If F is continuous, then there is a unique map $f : \mathscr{X} \to \mathscr{Y}$ such that $f \circ \psi_{G,X} = \psi_{G,Y} \circ F$, which moreover is continuous.*

(ii) *If F is a topological homeomorphism, then f is a homeomorphism.*

(iii) *If F is λ-Lipschitzian, for some real number $\lambda > 0$, then f is λ-Lipschitzian.*

(iv) *If F is an isometry, then f is an isometry.*

Proof. (i) Let X, Y, G and F be as in the statement of the theorem. If there is an $f : \mathscr{X} \to \mathscr{Y}$ such that $f \circ \psi_{G,X} = \psi_{G,Y} \circ F$, then f is uniquely determined by this condition. Thus, given an element u of $\mathscr{X}$, we choose an element $x \in \psi_{G,X}^{-1}(u)$, send it to $F(x) \in Y$, and then project it to $\psi_{G,Y}(F(x)) \in \mathscr{Y}$. Then define $f(u) := \psi_{G,Y}(F(x))$. We need to prove that the map f is well defined. In order to do this, let $u \in \mathscr{X}$ and $x, x' \in \psi_{G,X}^{-1}(u)$. We need to show that $\psi_{G,Y}(F(x)) = \psi_{G,Y}(F(x'))$. Since $x \sim_G x'$ in X, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in G for which $\lim_{n\to\infty} d_X(x, g_n x') = 0$. The continuity of F implies then that $\lim_{n\to\infty} d_Y(F(x), F(g_n x')) = 0$. Here $F(g_n x') = g_n F(x')$, so $\lim_{n\to\infty} d_Y(F(x), g_n F(x')) = 0$. This means that $F(x) \sim_G F(x')$ in Y, so $\psi_{G,Y}(F(x)) = \psi_{G,Y}(F(x'))$ as required. Thus the map f is well defined. In order to show that f is continuous, fix an element u of $\mathscr{X}$ and choose a sequence $(u_m)_{m\in\mathbb{N}}$ in $\mathscr{X}$ which converges to u. Fix an element x in $\psi_{G,X}^{-1}(u)$. By Theorem

1 (ii) the map $\psi_{G,X}$ is a quasi isometry. As a consequence, for each $m \in \mathbb{N}$ there exists an element x_m in $\psi_{G,X}^{-1}(u_m)$ for which

$$d_X(x, x_m) \leq d_{\mathscr{X}}(u, u_m) + \frac{1}{m}.$$

Therefore the sequence $(x_m)_{m\in\mathbb{N}}$ converges in X to x, and by the continuity of F, the sequence $(F(x_m))_{m\in\mathbb{N}}$ converges in Y to $F(x)$. Since $\psi_{G,Y}$ is continuous, it follows in turn that the sequence $(\psi_{G,Y}(F(x_m)))_{m\in\mathbb{N}}$ converges in $\mathscr{Y}$ to $\psi_{G,Y}(F(x))$. In other words, this says that the sequence $(f(u_m))_{m\in\mathbb{N}}$ converges in $\mathscr{Y}$ to $f(u)$. Thus the map f is continuous, which proves part (i) of the theorem.
(ii) We need to show that the map $f : \mathscr{X} \to \mathscr{Y}$, which exists, is uniquely determined and continuous by part (i), is also bijective and its inverse is continuous. Looking at the way f was defined, it is clear that f is surjective. To see this, one simply takes an arbitrary element v of $\mathscr{Y}$, lifts v to one of its representatives z in $\psi_{G,Y}^{-1}(v)$, sends z via F^{-1} to an element $x := F^{-1}(z) \in X$, and projects x to its image $u := \psi_{G,X}(x)$ in $\mathscr{X}$. Then $f(u) = v$. In order to show that f is injective, let u, u' be two elements of $\mathscr{X}$ for which $f(u) = f(u')$, and choose representatives x and x' for u and u' in $\psi_{G,X}^{-1}(u)$ and $\psi_{G,X}^{-1}(u')$ respectively. Then $\psi_{G,Y}(F(x)) = f(u) = f(u') = \psi_{G,Y}(F(x'))$, so $F(x) \sim_G F(x')$ in Y. It follows that there is a sequence $(g_n)_{n\in\mathbb{N}}$ in G for which the sequence $(g_nF(x'))_{n\in\mathbb{N}}$ converges in Y to $F(x)$. Since F is G-equivariant, the sequence $(F(g_nx'))_{n\in\mathbb{N}}$ converges in Y to $F(x)$, and since F is a homeomorphism we derive that the sequence $(g_nx')_{n\in\mathbb{N}}$ converges in X to x. Thus $x \sim_G x'$ in X, and so $u = \psi_{G,X}(x) = \psi_{G,X}(x') = u'$. This shows that the map f is injective. It remains to show that f^{-1} is continuous. Recall that f is the unique map from $\mathscr{X}$ to $\mathscr{Y}$ satisfying the equality $f\circ\psi_{G,X} = \psi_{G,Y}\circ F$. Then f^{-1} is the unique map from $\mathscr{Y}$ to $\mathscr{X}$ satisfying $\psi_{G,X}\circ F^{-1} = f^{-1}\circ(f\circ\psi_{G,X})\circ F^{-1} = f^{-1}\circ(\psi_{G,Y}\circ F)\circ F^{-1} = f^{-1}\circ\psi_{G,Y}$, and the continuity of f^{-1} follows from part (i) applied to $F^{-1} : Y \to X$.
(iii) Assume F is λ-Lipschitzian for some real number $\lambda > 0$. Thus

$$d_Y(F(x_1), F(x_2)) \leq \lambda d_X(x_1, x_2),$$

for any $x_1, x_2 \in X$. Let u_1, u_2 be two arbitrary elements of $\mathscr{X}$. We need to show that $d_{\mathscr{Y}}(f(u_1), f(u_2)) \leq \lambda d_{\mathscr{X}}(u_1, u_2)$. Fix a real number $\varepsilon > 0$, and choose representatives x_1 and x_2 for u_1 and u_2 in $\psi_{G,X}^{-1}(u_1)$ and $\psi_{G,X}^{-1}(u_2)$ respectively, such that $d_X(x_1, x_2) \leq d_{\mathscr{X}}(u_1, u_2) + \varepsilon$. Since F is λ-Lipschitzian, $d_Y(F(x_1), F(x_2)) \leq \lambda d_X(x_1, x_2) \leq \lambda d_{\mathscr{X}}(u_1, u_2) + \lambda\varepsilon$, and using the fact that $\psi_{G,Y}$ is 1-Lipschitzian we find that

$$d_{\mathscr{Y}}(\psi_{G,Y}(F(x_1)), \psi_{G,Y}(F(x_2))) \leq d_Y(F(x_1), F(x_2)) \leq \lambda d_{\mathscr{X}}(u_1, u_2) + \lambda\varepsilon.$$

Here $\psi_{G,Y}(F(x_1)) = f(u_1)$ and $\psi_{G,Y}(F(x_2)) = f(u_2)$. Letting $\varepsilon \to 0$, we deduce that $d_{\mathscr{Y}}(f(u_1), f(u_2)) \leq \lambda d_{\mathscr{X}}(u_1, u_2)$. In conclusion, f is λ-Lipschitzian.

(iv) Assume that F is an isometry (not necessarily surjective). Let u_1, u_2 be arbitrary elements of $\mathscr{X}$. We need to show that $d_{\mathscr{Y}}(f(u_1), f(u_2)) = d_{\mathscr{X}}(u_1, u_2)$. Applying part (iii) with $\lambda = 1$ we see that f is 1-Lipschitzian, so $d_{\mathscr{Y}}(f(u_1), f(u_2)) \leq d_{\mathscr{X}}(u_1, u_2)$. In order to prove the reverse inequality, fix an $\varepsilon > 0$ and choose a representative x_1 for u_1 in $\psi_{G,X}^{-1}(u_1)$. Here $F(x_1) \in \psi_{G,Y}^{-1}(f(u_1))$, and using (ii) of Theorem 1, which shows that $\psi_{G,Y}$ is a quasi isometry, it follows that there exists an element z in $\psi_{G,Y}^{-1}(f(u_2))$ for which $d_Y(F(x_1), z) < d_{\mathscr{Y}}(f(u_1), f(u_2)) + \varepsilon$. Since we are not assuming that F is surjective, we can not conclude, and in general it is not true, that z necessarily comes from an element of X via the map F. Therefore what we do is the following. We choose a representative x_2 for u_2 in $\psi_{G,X}^{-1}(u_2)$, send it to $\mathscr{Y}$ via F, and since both $F(x_2)$ and z belong to $\psi_{G,Y}^{-1}(f(u_2))$, we obtain an equivalence $F(x_2) \sim_G z$ in Y. There is then a sequence $(g_n)_{n\in\mathbb{N}}$ in G with the property that $\lim_{n\to\infty} d_Y(F(x_2), g_n z) = 0$. Now the point is that each element $g_n^{-1}F(x_2)$ is the image of an element of $\mathscr{X}$ through F, and these elements $g_n^{-1}F(x_2)$ approach z in Y, so their distance to $F(x_1)$ will approach the distance from z to $F(x_1)$. More precisely, using our assumptions that F is G-equivariant and that it is an isometry, we deduce that for each $n \in \mathbb{N}$,

$$
\begin{aligned}
d_X(x_1, g_n^{-1}x_2) = d_Y(F(x_1), F(g_n^{-1}x_2)) &= d_Y(F(x_1), g_n^{-1}x_2) \\
&\leq d_Y(F(x_1), z) + d_Y(z, g_n^{-1}x_2) \\
&< d_{\mathscr{Y}}(f(u_1), f(u_2)) + \varepsilon + d_Y(x_2, g_n z).
\end{aligned}
$$

Here the left side is larger or equal to $d_{\mathscr{X}}(u_1, u_2)$ for each n, while the far right side tends to $d_{\mathscr{Y}}(f(u_1), f(u_2)) + \varepsilon$ as $n \to \infty$. Therefore, $d_{\mathscr{X}}(u_1, u_2) < d_{\mathscr{Y}}(f(u_1), f(u_2))\ +2\varepsilon$, and since $\varepsilon > 0$ was arbitrary, we conclude that $d_{\mathscr{Y}}(f(u_1), f(u_2)) \geq d_{\mathscr{X}}(u_1, u_2)$. Combining this with the reverse inequality obtained above, we see that f is an isometry, and this completes the proof of the theorem. □

We end this section with several remarks.

Remark 2. Let Y be a metric space and G a group which acts on Y in such a way that for each $g \in G$, the map $Y \to Y$, $y \mapsto gy$ is an isometry. Further, let X be a subset of Y which is closed under the action of G, in the sense that $gx \in X$ for any $x \in X$ and any $g \in G$. Then one may apply (iv) of Theorem 2 to X, Y, G, and the map F given by the injection of X into Y. Although, as sets, the equivalence class of an element $x \in X$ in X may differ from the equivalence class of x in Y, it follows by the above theorem that $\mathscr{X}$ injects isometrically in $\mathscr{Y}$ in a canonic way. One may then identify $\mathscr{X}$ with a metric subspace of $\mathscr{Y}$. Moreover by the equality $f \circ \psi_{G,X} = \psi_{G,Y} \circ F$ from the statement of Theorem 2 we see that in the above identification the map $\psi_{G,X}$ will be given by the restriction of $\psi_{G,Y}$ to X.

Remark 3. In the context of Remark 2 above, for any element x of X, the equivalence class of x in Y, as a subset of Y, coincides with the topological closure in Y of the equivalence class of x in X. Indeed, on the one hand any element of X which is equivalent to x in X is also equivalent to x in Y, and on the other hand if $y \in Y$ is equivalent to x in Y then y is the limit of a sequence of the form $(g_n x)_{n\in\mathbb{N}}$ with $g_n \in G$ for all n . Here each $g_n x$ belongs to X and is equivalent to x in X, hence y belongs to the topological closure in Y of the equivalence class of x in X.

Remark 4. In the same context, as a consequence of Remark 3, we see that if the equivalence class of an element $x \in X$ is a complete metric space with respect to the metric obtained by restriction from X, then this equivalence class remains unchanged under any extension of X to a larger metric space Y on which one has a G-action with the required properties as above.

Remark 5. In light of Remark 4 above, a natural question that arises would be to see under which circumstances all the equivalence classes in a metric space X with a G-action for which all the maps $x \mapsto gx$ are isometries, are complete metric spaces. A sufficient condition for this to happen is to start with a metric space (X, d_X) which is complete. Indeed, each equivalence class is the inverse image through $\psi_{G,X}$, which is continuous, of a point in $\mathscr{X}$. Therefore each equivalence class is a closed subset of X, and hence it is a complete metric space if X is complete.

Let us remark that the above condition is not also sufficient, in the sense that there are metric spaces which are not complete, on which one has G-actions with the required properties, for which all the equivalence classes are complete metric spaces. For example one may take X to be the algebraic closure $\overline{K}$ of a local field K, and let G be the Galois group $\mathrm{Gal}(\overline{K}/K)$. Then X is not a complete metric space while each equivalence class is a complete metric space, being in this case a Galois orbit, which is finite.

Remark 6. In connection with Theorem 2 it would also be interesting to study the possibility of lifting a given map $f : \mathscr{X} \to \mathscr{Y}$ to a map $F : X \to Y$ as in the statement of the theorem. Let us remark that it is not always possible, given a map $f : \mathscr{X} \to \mathscr{Y}$, to find a G-equivariant map $F : X \to Y$ satisfying the equality $f \circ \psi_{G,X} = \psi_{G,Y} \circ F$, even if we do not impose on F further continuity or metric properties.

As an example, let p be a prime number, K a finite field extension of $\mathbb{Q}_p$, $\overline{K}$ an algebraic closure of K and $\mathbb{C}_p$ the completion of $\overline{K}$ with respect to the p-adic absolute value $|\cdot|$. Then let $G = G_K$, $Y = \mathbb{C}_p$ and X a finite union of Galois orbits in $\mathbb{C}_p$, $G = \cup_{j=1}^n G x_j$ say. Denote $u_j = \psi(x_j)$, so that $\mathscr{X} = \{u_1, \dots, u_n\}$, and let $f : \mathscr{X} \to \mathscr{Y}$. Given this data, is there a G-equivariant map $F : X \to Y$ such that $f \circ \psi_{G,X} = \psi_{G,Y} \circ F$? Let us consider the simple case when all the elements $x_1, \dots, x_n$ belong to K. If such an F exists, then for any $j \in \{1, 2, \dots, n\}$ and any automorphism $\sigma \in G$ one has $\sigma(F(x_j)) = F(\sigma(x_j)) = F(x_j)$. This forces $F(x_j) \in K$ for each j. We conclude that if at least one of $f(u_1), \dots, f(u_n)$ does not lie in the image of K in $\mathscr{Y}$ then there is no map $F : X \to Y$ with the required properties.

Let us consider now a case when such an F exists. With Y and G taken as before, let $X = \cup_{j=1}^{n} Gx_j$, where $x_1, \dots, x_n$ are arbitrary elements of $\mathbb{C}_p$. With $u_1, \dots, u_n$ having the same meaning as above, let $f : \mathscr{X} \to \mathscr{Y}$, and choose representatives $y_1, \dots, y_n$ for $f(u_1), \dots, f(u_n)$ in $\psi_{G,Y}^{-1}(f(u_1)), \dots, \psi_{G,Y}^{-1}(f(u_n))$ respectively. If we try to construct an F with the required properties by assigning to $x_1, \dots, x_n$ the values $F(x_1) = y_1, \dots, F(x_n) = y_n$, F is uniquely determined. More precisely, any $x \in X$ is of the form $x = \sigma(x_j)$ for some $j \in \{1, 2, \dots, n\}$ and $\sigma \in G$, and by the condition that F be G-equivariant we must have $F(x) = \sigma(F(x_j)) = \sigma(y_j)$. The question is whether F is well defined. This is so, provided for any $j \in \{1, 2, \dots, n\}$ and any $\sigma, \tau \in G$ for which $\sigma(x_j) = \tau(x_j)$, one has $\sigma(y_j) = \tau(y_j)$. This is equivalent to the condition that the closed subgroups of G, $\mathscr{H}_{x_j} = \{\sigma \in G : \sigma x_j = x_j\}$, $\mathscr{H}_{y_j} = \{\sigma \in G : \sigma y_j = y_j\}$, $1 \leq j \leq n$, are such that $\mathscr{H}_{x_j} \subseteq \mathscr{H}_{y_j}$ for any $j \in \{1, 2, \dots, n\}$. Recall that by Galois theory in $\mathbb{C}_p$ ([4], [1], [3]), there is a canonic one-to-one correspondence between the closed subgroups of G_K and the closed subfields of $\mathbb{C}_p$ which contain K. In our case, the above condition states that, if one denotes by $E_1, \dots, E_n$ the topological closure of the fields $K(x_1), \dots, K(x_n)$ in $\mathbb{C}_p$, then $y_j \in E_j$ for any $j \in \{1, 2, \dots, n\}$. We conclude that, given $x_1, \dots, x_n$, for any elements $y_1 \in E_1, \dots, y_n \in E_n$, the map $f : \mathscr{X} \to \mathscr{Y}$ which sends the equivalence class (Galois orbit) of x_j to the equivalence class of y_j, for $1 \leq j \leq n$, can be lifted to a G-equivariant map $F : X \to Y$. Let us remark that if each x_j is a generating element of $\mathbb{C}_p$ (see [2]), so that each E_j coincides with $\mathbb{C}_p$, then any map $f : \mathscr{X} \to \mathscr{Y}$ can be lifted to a G-equivariant map $F : X \to Y$ such that $f \circ \psi_{G,X} = \psi_{G,Y} \circ F$.

4. Completion

Let X be a metric space and G a group acting on X such that for each $g \in G$, the map $x \mapsto gx$ is an isometry on X. Denote as usual $\mathscr{X} = X/\sim_G$ and let $\psi = \psi_{G,X} : X \to \mathscr{X}$ be the canonical projection. Let us assume that (X, d_X) is a complete metric space. Then we show that $(\mathscr{X}, d_{\mathscr{X}})$ is a complete metric space. In the case when X is an ultrametric space, that is, when the triangle inequality holds in the stronger form

$$d_X(x, z) \leq \max\{d_X(x, y), d_X(y, z)\},$$

for all $x, y, z \in X$, the proof follows easily from the fact that ψ is a quasi isometry. More precisely, if $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathscr{X}$, then using repeatedly the defining property of a quasi isometry one can find inductively representatives $x_1, x_2, \dots, x_n, \dots$ for $u_1, u_2, \dots, u_n, \dots$ in $\psi^{-1}(u_1), \psi^{-1}(u_2), \dots, \psi^{-1}(u_n), \dots$, such that for any $n \geq 1$ one has

$$d_X(x_n, x_{n+1}) \leq 2 d_{\mathscr{X}}(u_n, u_{n+1}).$$

In the ultrametric case it follows then that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. This sequence will then converge to an element $x \in X$. Denoting $u = \psi(x)$, from the continuity of ψ it follows that the sequence $(u_n)_{n\in\mathbb{N}}$ converges to u in $\mathscr{X}$. This completes the proof in the ultrametric case. In the general case we modify the above argument as follows. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{X}$. For each $r \in \mathbb{N}$ choose an $N_r \in \mathbb{N}$ such that $d_{\mathscr{X}}(u_n, u_m) \leq \frac{1}{2^r}$ for all $n, m \geq N_r$. Put $n_1 = 1$ and select a representative x_1 for u_1 in $\psi^{-1}(u_1)$. Let $k \geq 2$ and assume that positive integers $1 = n_1 < n_2 < \cdots < n_{k-1}$ are chosen and representatives $x_1, x_2, \ldots, x_{k-1}$ for $u_{n_1}, u_{n_2}, \ldots, u_{n_{k-1}}$ are selected in $\psi^{-1}(u_{n_1}), \psi^{-1}(u_{n_2}), \ldots, \psi^{-1}(u_{n_{k-1}})$ respectively such that $d_X(x_j, x_{j-1}) \leq 2d_{\mathscr{X}}(u_{n_j}, u_{n_{j-1}})$ for $2 \leq j \leq k-1$. Then let $n_k = \max\{n_{k-1}+1, N_k\}$ and select a representative x_k for u_{n_k} in $\psi^{-1}(u_{n_k})$ for which $d_X(x_k, x_{k-1}) \leq 2d_{\mathscr{X}}(u_{n_k}, u_{n_{k-1}})$. We construct in this way a sequence of positive integers $1 = n_1 < n_2 < \cdots < n_k < \ldots$ and a sequence $x_1, x_2, \ldots, x_k, \ldots$ of representatives for $u_{n_1}, u_{n_2}, \ldots, u_{n_k}, \ldots$ in X. Note that for any positive integers $n < m$ one has

$$
\begin{aligned}
d_X(x_n, x_m) &\leq \sum_{k=n+1}^{m} d_X(x_k, x_{k-1}) \leq \sum_{k=n+1}^{m} 2d_{\mathscr{X}}(u_{n_k}, u_{n_{k-1}}) \\
&\leq \sum_{k=n+1}^{m} \frac{1}{2^{k-2}} < \frac{1}{2^{n-2}}.
\end{aligned}
$$

This shows that the sequence $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in X and hence it converges to an element $x \in X$. Denote $u = \psi(x)$. We derive that the sequence $(u_{n_k})_{k\in\mathbb{N}}$ converges in $\mathscr{X}$ to u. Since $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{X}$, it follows that the entire sequence $(u_n)_{n\in\mathbb{N}}$ converges in $\mathscr{X}$ to u. This proves that $\mathscr{X}$ is complete.

Recall from Remark 5 above that if X is a complete metric space then each equivalence class is a complete metric space. Let us see whether, conversely, if each equivalence class is complete and if $\mathscr{X}$ is complete, then X is complete. Assume that $\mathscr{X}$ and all the equivalence classes are complete metric spaces, and let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X. Denote $u_n = \psi(x_n)$ for each n. Since ψ is 1-Lipschitzian, $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{X}$ and so it converges to an element $u \in \mathscr{X}$. Using the fact that ψ is a quasi isometry one can choose for each n an element z_n in $\psi^{-1}(u)$ such that $d(x_n, z_n) \leq 2d_{\mathscr{X}}(u_n, u)$. We claim that $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Indeed, let us fix an $\varepsilon > 0$. The sequence $(u_n)_{n\in\mathbb{N}}$ converges to u, so there is an N_ε such that $d_{\mathscr{X}}(u_n, u) < \varepsilon$ for all $n > N_\varepsilon$. Also, $(x_n)_{n\in\mathbb{N}}$ being a Cauchy sequence, there is an N'_ε such that $d_{\mathscr{X}}(x_n, x_m) < \varepsilon$ for any $n, m > N'_\varepsilon$. Then for all $n, m > \max\{N_\varepsilon, N'_\varepsilon\}$ one has $d_X(z_n, z_m) \leq d_X(z_n, x_n) + d_X(x_n, x_m) + d_X(x_m, z_m) < 2d_{\mathscr{X}}(u_n, u) + \varepsilon + 2d_{\mathscr{X}}(u_m, u) < 5\varepsilon$. This proves the claim that $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Each z_n belongs to $\psi^{-1}(u)$, which is a complete metric space, therefore $(z_n)_{n\in\mathbb{N}}$ converges to an element $z \in \psi^{-1}(u)$. Moreover, since $d_X(x_n, z_n) \leq 2d_{\mathscr{X}}(u_n, u)$ for each n and since $u_n \to u$ as n tends to infinity, it follows that the sequence $(x_n)_{n\in\mathbb{N}}$ converges in X to z. In conclusion X is a complete metric space.

We record the result in the following theorem.

Theorem 3. *Let G be a group acting on a metric space X such that each map $x \mapsto gx$, with g in G, is an isometry on X. The following are equivalent:*

(i) *X is a complete metric space.*

(ii) *$\mathscr{X}$ and all the equivalence classes are complete metric spaces.*

We now consider the problem of completing a metric space X with a given action of a group G. Assume as usual that for each g in G the map $x \mapsto gx$ is an isometry on X. Then one has a canonical way of extending this action from X to the completion of X, call it Y. More precisely, the action of G on Y is defined by taking each g in G and extending by continuity the map $X \to X$, $x \mapsto gx$ to a map $Y \to Y$, $y \mapsto gy$, which is clearly an isometry on Y. Denote $\mathscr{Y} = Y/\sim_G$ and let $\psi_{G,Y}$ be the canonical projection. By Remark 2 above, we may identify $\mathscr{X}$ with a metric subspace of $\mathscr{Y}$. Moreover, the canonical projection $\psi_{G,X} : X \to \mathscr{X}$ identifies with the restriction of $\psi_{G,Y}$ to X. Since Y is the completion of X, a natural question is whether the metric space $\mathscr{Y}$ is the completion of $\mathscr{X}$. By Theorem 3 we know that $\mathscr{Y}$ is a complete metric space. Also, each element of $\mathscr{Y}$ is the limit of a sequence of elements from $\mathscr{X}$. Indeed, if $u \in \mathscr{Y}$, choose a representative y for u in Y and a sequence $(x_n)_{n\in\mathbb{N}}$ in X which converges to y. Then each $\psi_{G,Y}(x_n) = \psi_{G,X}(x_n)$ belongs to $\mathscr{X}$, and by the continuity of $\psi_{G,Y}$ the sequence $(\psi_{G,Y}(x_n))_{n\in\mathbb{N}}$ converges in $\mathscr{Y}$ to u. It follows that $\mathscr{Y}$ is the completion of $\mathscr{X}$ as stated. We collect the results in the following theorem.

Theorem 4. *Let G be a group acting on a metric space X such that each map $x \mapsto gx$, with g in G, is an isometry on X. If Y denotes the completion of X, then one has a canonical action of G on Y, and the metric space $\mathscr{Y} = Y/\sim_G$ is the completion of $\mathscr{X} = X/\sim_G$.*

As an example, if p is a prime number, K a finite field extension of $\mathbb{Q}_p$, $X = \overline{K}$ an algebraic closure of $\overline{K}$, and $G = \mathrm{Gal}(\overline{K}/K)$, then the canonical action of G on the completion $Y = \mathbb{C}_p$ of $\overline{K}$ is obtained by taking each automorphism $\sigma \in G$, which is an isometry on $\overline{K}$, and extending by continuity σ to a continuous automorphism of $\mathbb{C}_p$ over K, which is then also an isometry. In this example each element of $\mathscr{Y}$ is a G-orbit, which as a subset of Y is compact, and the space of these compact subsets of Y is the metric completion of the space of G-orbits of elements of the form $\overline{K}$, which are finite subsets of $\overline{K}$.

In the above example the elements of $\mathscr{X}$ are finite subsets of X and the elements of $\mathscr{Y}$ are compact, although not finite, subsets of Y. This phenomenon holds in more generality: If all the equivalence classes in X are finite then all the equivalence classes in Y are compact. One has in fact the following result.

Theorem 5. *Under the assumptions from Theorem 4, if all the equivalence classes in X are compact, then all the equivalence classes in Y are compact.*

In order to prove the theorem, recall that a metric space is compact if and only if it is complete and totally bounded. By Theorem 3 all the equivalence classes in Y are complete metric spaces. As for the total boundedness, let us fix an equivalence class $\psi_{G,Y}^{-1}(u)$, where $u \in \mathscr{Y}$. We need to show that for any $\delta > 0$ one can cover $\psi_{G,Y}^{-1}(u)$ with finitely many open balls of radius δ. Fix such a δ and choose an element t in $\mathscr{X}$ for which $d_{\mathscr{Y}}(t,u) < \frac{\delta}{5}$. Select a representative x for t in X. Since the equivalence class of x in X, which by Remark 4 coincides with the equivalence class of x in Y, is compact, it can be covered by finitely many open balls of radius $\frac{\delta}{5}$, call them $B_1, B_2, \dots, B_s$. Choose elements $x_1, x_2, \dots, x_s$ in $\psi_{G,X}^{-1}(t)$ such that $x_j \in B_j$, for each $j \in \{1,2,\dots,s\}$. Using the fact that $\psi_{G,Y}$ is a quasi isometry one can choose elements $y_1, y_2, \dots, y_s$ in $\psi_{G,Y}^{-1}(u)$ such that $d_Y(x_j, y_j) \le 2d_{\mathscr{Y}}(t,u)$ for all $1 \le j \le s$. Then the union of the open balls of radius δ centered at $y_1, y_2, \dots, y_s$ covers the entire equivalence class $\psi_{G,Y}^{-1}(u)$. Indeed, let w be an arbitrary element of $\psi_{G,Y}^{-1}(u)$. Using again the fact that $\psi_{G,Y}$ is a quasi isometry choose an element z in $\psi_{G,X}^{-1}(t) = \psi_{G,Y}^{-1}(t)$ for which $d_Y(z,w) \le 2d_{\mathscr{Y}}(t,u)$. There exists an $i \in \{1,2,\dots,s\}$ such that $d_X(x_i, z) < \frac{\delta}{5}$. Then

$$d_Y(w,y_i) \le d_Y(w,z) + d_X(z,x_i) + d_Y(x_i,y_i) < 2d_{\mathscr{Y}}(t,u) + \frac{\delta}{5} + 2d_{\mathscr{Y}}(t,u) < \delta,$$

which shows that the open balls of radius δ centered at $y_1, y_2, \dots, y_s$ cover $\psi_{G,Y}^{-1}(u)$. Thus each equivalence class in Y is complete and totally bounded, and hence compact, which completes the proof of the theorem.

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