Kumamoto J. Math.
Vol. **19** (2006), 15–23

On the middle convolution and birational symmetries of the sixth Painlevé equation

Galina Filipuk*

(Received October 11, 2005)

Abstract. In this paper we apply the Dettweiler and Reiter algorithm of middle convolution for a linear Fuchsian system with four singularities the isomonodromy deformation of which leads to the sixth Painlevé equation. We obtain Okamoto's first degree birational transformation as a result of the middle convolution with a special choice of the parameter.

1. Introduction

Recently Dettweiler and Reiter gave a purely algebraic analogue of Katz' middle convolution functor in [5, 6, 7] and presented an algorithm for the construction of Fuchsian systems corresponding to irreducible rigid local systems under the Riemann-Hilbert correspondence. Earlier in [11] Katz described all irriducible and physically rigid systems on the punctured affine line and introduced a middle convolution functor on the category of perverse sheaves which preserves important properties of local systems such as a number of singularities, the index of rigidity and irreducibility but in general changes the rank and monodromy group. By Katz's algorithm one can obtain any irreducible rigid local system on the punctured affine line from rank one local system by applying a suitable sequence of middle convolutions and scalar multiplications. The main approach in [5, 6] was the generalization of normal forms of the Pochhammer equation. Dettweiler and Reiter not only reproduced Katz' main result, but also presented both the multiplicative and additive versions of their algebraic analogue, studied their main properties, gave a cohomological interpretation and applied their theory for the construction of explicit algebraic solutions of the sixth Painlevé equation in [7].

Mathematics Subject Classification (2000): 34M55
Keywords: Painlevé equations, birational transformations, isomonodromy deformations, middle convolution
*The work is supported by the Japan Society for the Promotion of Science (JSPS) postdoctoral fellowship P04303.

The sixth Painlevé equation

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right), \quad (1)$$

α, β, γ, δ being arbitrary parameters, describes monodromy preserving deformations of 2×2 linear Fuchsian system with four singular points on $\mathbb{P}^1$ [10]. The Dettweiler and Reiter algorithm can formally be applied to any Fuchsian system, not necessarily the rigid one. It guarantees that after applying the middle convolution functor a new linear system has the same number of singularities and is irreducible. However, the dimension of the matrices of the Fuchsian system may change. In case the dimension remains the same, i.e., we get 2×2 matrices after the middle convolution transformation, we automatically get, as a deformation equation, the sixth Painlevé equation possibly with different values of the parameters which gives rise to birational transformations [14]. In this paper we show how to calculate explicitly Okamoto's birational transformation via middle convolution.

Recall that birational transformations for the sixth Painlevé equation were discovered by Okamoto by studying the associated Hamiltonian system. He gave explicit forms of canonical transformations of the Hamiltonian system associated with the affine transformations of the space of parameters and obtained a nonlinear representation of the affine Weyl group of the root system of type $D_4^{(1)}$. In a series of papers [2, 3, 4] Conte and Musette studied an extension to ODEs of the singular manifold method originally introduced for PDEs. They showed that by considering truncated Laurent series of the solution of the sixth Painlevé equation it is possible to recover the first degree birational transformations. Their method is direct in a sense that it is based only on the singularity structure of the differential equation. The Schlesinger transformations of the Fuchsian linear system under which monodromy data are invariant were studied in [10, 13, 15]. These transformations conserve two monodromy exponents and shift two others by an integer. The Schlesinger transformations of the linear system generate transformations for the solutions of the Painlevé equation and their relation to the repeated applications of Okamoto's birational transformations was studied in [8]. However, to find the Schlesinger transformation explicitly it is also necessary to examine local behavior of solutions of the linear system and solve a certain Riemann-Hilbert problem. The main feature of our approach presented below is its simplicity. One needs to know only the Fuchsian system related to the sixth Painlevé equation since the algorithm [5, 6] operates on the matrices of the system in terms of linear algebra. Another treatment of Okamoto's birational transformations is given in [12] where they are related to a simple gauge transformation of the system with irregular singularity associated to the sixth Painlevé equation.

To obtain Okamoto's first degree birational transformation equivalent to w_2

[14] (see also [2]) via middle convolution we use the additive version of the middle convolution transformation mc_μ for a specific value of μ equal to one of the eigenvalues of the matrix of the Fuchsian system at infinity. In case the dimension of the matrices of the new linear system increases by one, we may apply the approach given in [1, 9] and get explicitly the Hamiltonian of the original equation.

2. The middle convolution functor mc_μ

In this section we briefly outline the Dettweiler and Reiter algorithm following [5, 6, 7] as it is necessary for our construction. The multiplicative version of middle convolution functor denoted by MC_λ is a functor of the category of finite dimensional $\mathbb{C}[F_r]$−modules of the free group F_r on r generators to itself (local systems), where $\lambda \in \mathbb{C}^\times$ is a parameter. It is a transformation sending r matrices in $GL_n(\mathbb{C})$ to another r matrices in $GL_m(\mathbb{C})$, where usually m is not equal to n. Up to a simultaneous conjugation in $GL_m(\mathbb{C})$ this transformation commutes with the Artin braid group [6]. There exists a parallel functor in the category of the Fuchsian systems, mc_μ, which is related to MC_λ via the Riemann-Hilbert correspondence by a monodromy map.

Let $\mathbf{A} = (A_1, \ldots, A_r)$, $A_k \in \mathbb{C}^{n\times n}$. For $\mu \in \mathbb{C}$ one defines the convolution matrices $\mathbf{B} = mc_\mu(\mu) = (B_1, \ldots, B_r)$ as follows:

$$B_k = \begin{pmatrix} 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ A_1 & \ldots & A_{k-1} & A_k + \mu & A_{k+1} & \ldots & A_r \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \in \mathbb{C}^{nr\times nr} \qquad (2)$$

such that B_k is zero outside the k-th block row.

There are the following invariant subspaces of the column vector space $\mathbb{C}^{nr}$:

$$\mathscr{L}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{Ker}(A_k) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (k\text{-th entry}), \quad k = 1, \ldots, r, \qquad (3)$$

and

$$\mathscr{K} = \bigcap_{k=1}^{r} \mathrm{Ker}(B_k) = \mathrm{Ker}(B_1 + \ldots + B_r). \qquad (4)$$

Let $\mathscr{L} = \oplus_{k=1}^{r} \mathscr{L}_k$ and fix an isomorphism between $\mathbb{C}^{nr}/(\mathscr{K} + \mathscr{L})$ and $\mathbb{C}^m$ for some m. The matrices $C = mc_\mu(A) := (\widetilde{B}_1, \ldots, \widetilde{B}_r) \in \mathbb{C}^{m\times m}$, where $\widetilde{B}_k$ is induced by the action of B_k on $\mathbb{C}^m \simeq \mathbb{C}^{nr}/(\mathscr{K} + \mathscr{L})$ are called the additive version of the middle convolution of $\mathbf{A}$ with parameter μ.

3. Main result

The sixth Painlevé equation (1) can be obtained [10] as the compatibility condition of the linear system

$$\partial_x Z = \left(\frac{A_0^0}{x} + \frac{A_1^0}{x-1} + \frac{A_t^0}{x-t}\right) Z, \quad \partial_t Z = -\frac{A_t^0}{z-t} Z, \tag{5}$$

where

$$A_0^0 = \begin{pmatrix} z_0 + \theta_0 & -uz_0 \\ u^{-1}(z_0+\theta_0) & -z_0 \end{pmatrix}, \quad A_1^0 = \begin{pmatrix} z_1 + \theta_1 & -vz_1 \\ v^{-1}(z_1+\theta_1) & -z_1 \end{pmatrix},$$

$$A_t^0 = \begin{pmatrix} z_t + \theta_t & -wz_t \\ w^{-1}(z_t+\theta_t) & -z_t \end{pmatrix}, \quad A_\infty^0 = -(A_0^0 + A_1^0 + A_t^0) = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

and $k_1 - k_2 = \theta_\infty$, $k_1 + k_2 = -(\theta_0 + \theta_1 + \theta_t)$, $\mathrm{Tr}(A_s) = \theta_s$, $s = 0, 1, t$.

To simplify computations we apply the following gauge transformation to the system above

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & v^{-1} \end{pmatrix} W,$$

and get the Fuchsian system which we denote by D_A following the notation in [5, 6]

$$\partial_x W = \left(\frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t}\right) W, \tag{6}$$

with

$$A_s = \begin{pmatrix} z_s + \theta_s & -u_s z_s \\ u_s^{-1}(z_s+\theta_s) & -z_s \end{pmatrix}, \quad s = 0, 1, t, \quad u_0 = \frac{u}{v}, \quad u_1 = 1, \quad u_t = \frac{w}{v}.$$

The formulas by which the solution $y = y(t)$ of the sixth Painlevé equation (1) is related to the coefficients of the Fuchsian system (5) can be found in [8, 10] and are given by

$$\begin{aligned}
\theta_\infty t z_0/y &= yF_1F_t\tilde{z}^2 + (\theta_1 F_t + t\theta_t F_1 - 2k_2F_1F_t)\tilde{z} + k_2^2(F_t - 1) - k_2(\theta_1 + t\theta_t), \\
-\theta_\infty T_1 z_1/F_1 &= yF_1F_t\tilde{z}^2 + ((\theta_1 + \theta_\infty)F_t + t\theta_t F_1 - 2k_2F_1F_t)\tilde{z} + k_2^2 F_t \\
&\quad - k_2(\theta_1 + t\theta_t) - k_1k_2, \\
\theta_\infty t T_1 z_t/F_t &= yF_1F_t\tilde{z}^2 + (\theta_1 F_t + t(\theta_t + \theta_\infty)F_1 - 2k_2F_1F_t)\tilde{z} + k_2^2 F_1 \\
&\quad - k_2(\theta_1 + t\theta_t) - tk_1k_2, \\
tz_0u &= k(t)y, \quad T_1z_1v = -k(t)F_1, \quad tT_1z_tw = k(t)F_t,
\end{aligned}$$

where

$$\begin{aligned}
&\tilde{z} = z - \theta_0/y - \theta_1/F_1 - \theta_t/F_t,\\
&tT_1 dy/dt = yF_1F_t(2z - \theta_0/y - \theta_1/F_1 - (\theta_t - 1)/F_t),\\
&tT_1 dz/dt = (-3y^2 + 2T_2y - t)z^2 + ((2y - T_2)\theta_0 + (2y - t)\theta_1 + (2y - 1)(\theta_t - 1))z\\
&\qquad\qquad - k_1(k_2 + 1),\\
&tT_1 d(\log k(t))/dt = (\theta_\infty - 1)F_t,\ F_1 = y - 1,\ F_t = y - t,\ T_1 = t - 1,\ T_2 = t + 1,
\end{aligned}$$

and the parameters are given by

$$\alpha = (\theta_\infty - 1)^2/2, \quad \beta = -\theta_0^2/2, \quad \gamma = \theta_1^2/2, \quad \delta = (1 - \theta_t^2)/2.$$

We first apply the additive version of the Dettweiler and Reiter algorithm to (6) and get 6×6 convolution matrices B_s of the new Fuchsian system with the same singularities $D_B = D_{c_\mu(A)}$ for any $\mu \in \mathbb{C}$ using formula (2). It is easy to check using (3) that there are the following invariant subspaces of the vector space spanned by A_s:

$$\mathscr{L}_s = \begin{pmatrix} 0 \\ 0 \\ (u_s z_s)/(\theta_s + z_s) \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{(s-th entry)},$$

where s corresponds to $0, 1, t$ in our numeration and by (4)

$$\mathscr{L} = \bigcap_s \mathrm{Ker}(B_s) = \mathrm{Ker}(B_0 + B_1 + B_t).$$

If $\mu \neq 0$, then it is not difficult to see that the subspace $\mathscr{L}$ is spanned by the following vector ${}^{\mathrm{tr}}(\ell, \ell, \ell)$, $\ell \in \mathrm{Ker}(A_0 + A_1 + A_t + \mu)$, where ${}^{\mathrm{tr}}\mathbf{v}$ stands for the transposed vector.

It is clear that the subspace $\mathscr{L}$ is not empty in case μ coincides with one of the eigenvalues of $A_\infty = -(A_0 + A_1 + A_t)$. We need to distinguish two cases when $\mu = k_1$ and $\mu \neq 0, k_1, k_2$. In the following we discuss in detail the case $\mu = k_1$, which leads to the Okamoto transformation, and at the end we briefly discuss the other cases.

Next we construct the quotient space $\mathbb{C}^6/(\mathscr{K} + \mathscr{L})$ by adding the vectors ${}^{\mathrm{tr}}(1,0,1,0,1,0), {}^{\mathrm{tr}}(0,0,1,0,0,0), {}^{\mathrm{tr}}(0,0,0,0,1,0)$ to the basis. We obtain the following Fuchsian system as the result of the middle convolution transformation

$D_C = D_{mc_{k_1}}(B)$ with 2×2 matrices C_s:

$$C_0 = \begin{pmatrix} c_0^{11} & c_0^{12} \\ c_0^{11} & c_0^{12} \end{pmatrix},$$
$$c_0^{11} = -\frac{(\theta_0 + z_0 - u_0 z_0)(\theta_1 + z_1)}{\theta_0 + z_0}, \quad c_0^{12} = -\theta_t - z_t + \frac{u_0 z_0(\theta_t + z_t)}{u_t(\theta_0 + z_0)},$$
$$C_1 = \begin{pmatrix} (-\theta_0 + \theta_1 + \theta_\infty - \theta_t)/2 & \theta_t + z_t - (z_1(\theta_t + z_t))/(u_t(\theta_1 + z_1)) \\ 0 & 0 \end{pmatrix},$$
$$C_t = \begin{pmatrix} 0 & 0 \\ \theta_1 + z_1 - (u_t z_t(\theta_1 + z_1))/(\theta_t + z_t) & (-\theta_0 - \theta_1 + \theta_\infty + \theta_t)/2 \end{pmatrix}.$$

We note that the eigenvalues of matrices C_s are 0 and $k_1 + \theta_s$. The eigenvalues of matrix $C_\infty = -(C_0 + C_1 + C_t)$ are $-k_1$ and $-k_1 + k_2$.

Let
$$\partial_x U = \left(\frac{C_0}{x} + \frac{C_1}{x-1} + \frac{C_t}{x-t}\right)U$$
be the Fuchsian system obtained after the application of the middle convolution transformation and $U = SY$ such that for a new system
$$\partial_x Y = \left(\frac{M_0}{x} + \frac{M_1}{x-1} + \frac{M_t}{x-t}\right)Y \tag{7}$$
the matrix at infinity $M_\infty = -S^{-1}(C_0+C_1+C_t)S = -(M_0+M_1+M_t)$ is diagonal. By direct computations we obtain that matrices M_s can be parametrized by
$$M_s = \begin{pmatrix} z_s^1 + \theta_s + k_1 & -u_s^1 z_s^1 \\ (u_s^1)^{-1}(z_s^1 + \theta_s + k_1) & -z_s^1 \end{pmatrix}, \quad s = 0, 1, t, \quad u_1^1 = -1,$$
coefficients being expressed in terms of rational functions in u_s, z_s, θ_s.

Theorem 3.1. *The isomonodromy deformation of middle convoluion system* (7) *gives the sixth Painlevé equation* (1) *for the function*
$$y_1(t) = mc_{k_1}(y(t)) = \frac{t u_0^1 z_0^1}{(t+1)u_0^1 z_0^1 - t z_1^1 + u_t^1 z_t^1}$$
$$= y - \frac{(\theta_0 + \theta_1 - \theta_\infty + \theta_t)(t-y)(y-1)y}{(\theta_0 + \theta_t - 1 + t(\theta_0 + \theta_1))y - (\theta_0 + \theta_1 + \theta_t - 1)y^2 - t(\theta_0 + (t-1)dy/dt)}$$
and parameters
$$\alpha_1 = \frac{1}{8}(\theta_0 + \theta_1 + \theta_\infty + \theta_t - 2)^2, \quad \beta_1 = -\frac{1}{8}(\theta_0 - \theta_1 + \theta_\infty - \theta_t)^2,$$
$$\gamma_1 = \frac{1}{8}(-\theta_0 + \theta_1 + \theta_\infty - \theta_t)^2, \quad \delta_1 = \frac{1}{2}(1 - (\theta_0 + \theta_1 - \theta_\infty - \theta_t)^2/4),$$
which coincides with Okamoto's birational transformation $x_2 w_2 x_2$.

Remark 1. Okamoto [14] found birational transformations by considering the Hamiltonian of (1). The action of w_2, one of the generators of $W(D_4)$, on parameters and a solution of the sixth Painlevé equation is given by

$$w_2(b_1,b_2,b_3,b_4)=(b_1,b_3,b_2,b_4),\quad w_2: y(t)\to y_2(t),$$

where

$$y_2=y-\frac{2(b_2-b_3)(t-y)(y-1)y}{(b_1+b_2-b_3-b_4+2b_2t)y+(-2b_2+b_3+b_4)y^2+t(-b_1-b_2+(t-1)y')}$$

and in our notation

$$b_1=\frac{\theta_0+\theta_1}{2},\quad b_2=\frac{\theta_0-\theta_1}{2},\quad b_3=\frac{\theta_\infty+\theta_t-2}{2},\quad b_4=\frac{\theta_t-\theta_\infty}{2}.$$

Thus, it is easy to check that $mc_{k_1}=x_2w_2x_2$, where

$$x_2(b_1,b_2,b_3,b_4)=(b_1-b_2+b_3-b_4,b_2-b_1+b_3-b_4,b_1+b_2+b_3+b_4,-b_1-b_2+b_3+b_4)/2$$

and

$$x_2: y(t)\to y^{-1}(1/t).$$

Remark 2. It is proved in [5] that the convolution is compatible with the Euler transformation where the integration is with respect to the Pochhammer contour. An interesting observation due to A.V. Kitaev is that the Laplace transformation and a simple gauge transformation studied in [12] give the same transformation $x_2w_2x_2$ for the solutions of the sixth Painlevé equation.

Remark 3. In case $\mu=0$ we also get 2×2 system. However, we do not obtain any shift of exponents and, hence, any transformation for the sixth Painlevé equation.

It remains to discuss the general case μ different from 0, k_1 and k_2.

Similar calculations of middle convolution transformation for system (5) show that we get 3×3 matrices of the Fuchsian system $D_G=D_{mc_\mu}(A^0)$ of the form

$$G_0=\begin{pmatrix}\theta_0+\mu & \theta_1+z_1-\frac{uz_0(\theta_1+z_1)}{v(\theta_0+z_0)} & \theta_t+z_t-\frac{uz_0(\theta_t+z_t)}{w(\theta_0+z_0)}\\ 0&0&0\\0&0&0\end{pmatrix},$$

$$G_1=\begin{pmatrix}0&0&0\\ \theta_0+z_0-\frac{vz_1(\theta_0+z_0)}{u(\theta_1+z_1)} & \theta_1+\mu & \theta_t+z_t-\frac{vz_1(\theta_t+z_t)}{w(\theta_1+z_1)}\\ 0&0&0\end{pmatrix},$$

$$G_t=\begin{pmatrix}0&0&0\\0&0&0\\ \theta_0+z_0-\frac{wz_t(\theta_0+z_0)}{u(\theta_t+z_t)} & \theta_1+z_1-\frac{wz_t(\theta_1+z_1)}{v(\theta_t+z_t)} & \theta_t+\mu\end{pmatrix}$$

and the residue matrix at infinity has the eigenvalues $-\mu$, $k_1 - \mu$, $k_2 - \mu$. The Schlesinger equations in this case are equivalent (see [1, 9] for details) to the following system:

$$\frac{dx}{dt} = \frac{f(x,t)}{t-1}, \quad \frac{dy}{dt} = -\frac{f(x,t)}{t},$$

where $x = \mathrm{Tr}(G_0G_t)$, $y = \mathrm{Tr}(G_1G_t)$, $f(x,y) = \mathrm{Tr}(G_0[G_1, G_t])$. Introducing a new variable by $H = x/t + y/(t-1)$ we reduce the system above to second order second degree differential equation. Comparing H with the Hamiltonian function given in [14] for the sixth Painlevé equation, we see that they coincide.

Acknowledgements. The author is grateful to Professors Yoshishige Haraoka and Hironobu Kimura for valuable discussions.

References

[1] P. Boalch, *From Klein to Painlevé via Fourier, Laplace and Jimbo*, Proc. London Math. Soc. (3) **90** (2005), 167–208.

[2] R. Conte and M. Musette, *A truncation for obtaining all the first degree birational transformations of the Painlevé transcendents*, J. Nonlinear Math. Phys. **9** (2002), 14–28.

[3] R. Conte and M. Musette, *First degree birational transformations of the Painlevé equations and their contiguity relations*, J. Phys. A **34** (2001), 10507–10522.

[4] R. Conte and M. Musette, *New contiguity relation of the sixth Painlevé equation from a truncation*, Phys. D **161** (2002), 129–141.

[5] M. Dettweiler and S. Reiter, *Middle convolution and Galois realizations*, Galois theory and modular forms. Dev. Math. **11** (2004), 143–158.

[6] M. Dettweiler and S. Reiter, *An algorithm of Katz and its application to the inverse Galois problem*, J. Symbolic Comput. **30** (2000), 761–798.

[7] M. Dettweiler and S. Reiter, *Painlevé equations and the middle convolution*, Preprint (2004).

[8] V. Gromak and G. Filipuk, *On the transformations of the sixth Painlevé equation*, J. Nonlinear Math. Phys. **10** (2003), 57–68.

[9] N. Hitchin, *Geometrical aspects of Schlesinger's equation*, J. Geom. Phys. **23** (1997), 287–300.

[10] M. Jimbo and T. Miwa, *Monodromy preserving deformations of linear ordinary differential equations with rational coefficients. II*, Phys. D **2** (1981), 407–448.

[11] N.M. Katz, *Rigid local systems*, Annals of Mathematics Studies, vol. 139, Princeton University Press, Princeton, 1996.

[12] M. Mazzocco, *Irregular isomonodromic deformations for Garnier systems and Okamoto's canonical transformations*, J. London Math. Soc. **70** (2004), 405–419.

[13] U. Mugan and A. Sakka, *Schlesinger transformations for Painlevé VI equation*, J. Math. Phys. **36** (1995), 1284–1298.

[14] K. Okamoto, *Studies on the Painlevé equations. I. Sixth Painlevé equation* P_{VI}, Ann. Mat. Pura Appl. **146** (1987), 337–381.

[15] L. Schlesinger, *Über eine Klasse von Differentialsystemen beliebiger Ordnung mit festen kritischen Punkten*, J. Reine Angew. Math. **141** (1912), 96–145.

Galina Filipuk
Department of Mathematics
Faculty of Science
Kumamoto University
Kumamoto, 860-8555 Japan
e-mail: filipuk@math.sci.kumamoto-u.ac.jp