Kumamoto J. Math.
Vol. 19 (2006), 25-36

A generalization of Sándor's theorem using iterated logarithms *

Jaroslav Hančl, Pavel Rucki and Jan Šustek

(Received November 14, 2005)

Abstract. The paper deals with a criterion for certain sums of the series to be linearly independent numbers. The proof uses the idea of Sándor's theorem.

1. Introduction

There is a nice book of Nishioka [8] where there are many results concerning the irrationality, transcendence, linear and algebraic independence of the sums of infinite series. Other criteria for linear independence can be found in [1], [3] or [10].

For series consisting of rational numbers and converging very fast it is convenient to define linearly unrelated sequences.

Definition 1.1. *Let $\{a_{i,n}\}_{n=1}^{\infty}$ be sequences of positive real numbers for $i = 1,2,\ldots,K$. If for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers the numbers $\sum_{n=1}^{\infty} 1/(a_{1,n}c_n)$, $\sum_{n=1}^{\infty} 1/(a_{2,n}c_n)$, $\ldots$, $\sum_{n=1}^{\infty} 1/(a_{K,n}c_n)$ and 1 are linearly independent, then the sequences $\{a_{i,n}\}_{n=1}^{\infty}$ are said to be linearly unrelated for $i = 1,2,\ldots,K$.*

This definition is taken from [2]. In the same paper there is also a criterion for sequences of Cantor type to be linearly unrelated. Recently Hančl and Sobková [5] proved the following theorem.

Theorem 1.2. *Let K be a positive integer and let α and ε be positive real numbers such that $0 < \alpha < 1$. Let $\{a_{i,n}\}_{n=1}^{\infty}$ and $\{b_{i,n}\}_{n=1}^{\infty}$ be sequences of positive integers for $i = 1,2,\ldots,K$ with $\{a_{1,n}\}_{n=1}^{\infty}$ nondecreasing, such that*

$$\limsup_{n\to\infty} a_{1,n}^{\frac{1}{(K+1)^n}} = \infty,$$

Mathematics Subject Classification (2000): 11J72
Keywords: Linear independence, infinite series
*Supported by the grants no. 201/04/0381 and MSM6198898701

that

$$a_{1,n} \geq n^{1+\epsilon},$$

that

$$b_{i,n} \leq 2^{(\log_2 a_{1,n})^\alpha}, \quad i = 1, 2, \ldots, K,$$

that

$$\lim_{n\to\infty} \frac{a_{i,n} b_{j,n}}{b_{i,n} a_{j,n}} = 0, \quad i, j = 1, 2, \ldots, K, \quad i > j$$

and

$$a_{i,n} 2^{-(\log_2 a_{1,n})^\alpha} \leq a_{1,n} \leq a_{i,n} 2^{(\log_2 a_{1,n})^\alpha}$$

hold for every sufficiently large n. Then the sequences $\{a_{i,n}/b_{i,n}\}_{n=1}^\infty$ are linearly unrelated for $i = 1, 2, \ldots, K$.

Other criteria can be found in [6] and [7]. The main result of this paper is Theorem 2.2 which deals with linear independence of special infinite series. The terms of this series consist of rational numbers and converge very rapidly to zero in comparison with the terms of series in Theorem 1.2. The proof makes use the ideas of the proof of Sándor's theorem in [9] and of the proof of Theorem 1.2. This paper also includes two corollaries which are consequences of Theorem 2.2.

2. Main results

Definition 2.1. *Let t be a nonnegative integer. We define the function $L_t(x)$ by*

$$L_t(x) := \underbrace{\log\log\cdots\log}_{t\text{-times}} x \qquad \text{for } t > 0$$

$$L_0(x) := x.$$

Theorem 2.2. *Let K and s be positive integers. Let $\{\varepsilon_{i,j}\}_{j=1}^s$ be a sequence of positive real numbers. Let $\{t_{i,j}\}_{j=1}^s$, $\{a_{i,n}\}_{n=1}^\infty$ and $\{b_{i,n}\}_{n=1}^\infty$ be sequences of positive integers for $i = 1, 2, \ldots, K$. Suppose that*

$$\liminf_{n\to\infty} \left(\frac{\operatorname{lcm}(a_{1,1}, \ldots, a_{K,n})}{\min\limits_{j=1,\ldots,s} L_{t_{1,j}}^{\varepsilon_{1,j}}\left(\frac{a_{1,n+j}}{b_{1,n+j}}\right)} + \sum_{i=1}^{K-1} \frac{\min\limits_{j=1,\ldots,s} \frac{a_{i,n+j}}{b_{i,n+j}}}{\min\limits_{j=1,\ldots,s} L_{t_{i+1,j}}^{\varepsilon_{i+1,j}}\left(\frac{a_{i+1,n+j}}{b_{i+1,n+j}}\right)} \right) = 0 \tag{1}$$

and that

$$\frac{a_{i,n+s+j}}{b_{i,n+s+j} \prod_{k=1}^{t_{i,j}} L_k\left(\frac{a_{i,n+s+j}}{b_{i,n+s+j}}\right) \cdot L_{t_{i,j}}^{\varepsilon_{i,j}}\left(\frac{a_{i,n+s+j}}{b_{i,n+s+j}}\right)} \geq \frac{a_{i,n+j}}{b_{i,n+j} \prod_{k=1}^{t_{i,j}} L_k\left(\frac{a_{i,n+j}}{b_{i,n+j}}\right) \cdot L_{t_{i,j}}^{\varepsilon_{i,j}}\left(\frac{a_{i,n+j}}{b_{i,n+j}}\right)} + 1 \tag{2}$$

for $i = 1, 2, \ldots, K$, $j = 1, 2, \ldots, s$ and for every sufficiently large n. Then the numbers $\sum_{n=1}^{\infty} b_{1,n}/a_{1,n}$, $\sum_{n=1}^{\infty} b_{2,n}/a_{2,n}$, $\ldots$, $\sum_{n=1}^{\infty} b_{K,n}/a_{K,n}$ and the number 1 are linearly independent over rational numbers.

Example 2.3. *Let $\{a_{i,n}\}_{n=1}^{\infty}$ and $\{b_{i,n}\}_{n=1}^{\infty}$ be sequences of positive integers for $i = 1, 2$ such that $a_{1,1} = a_{2,1} = 1$, $a_{1,2} = a_{2,2} = 2$ and $b_{1,n} = b_{2,n} = 1$ for $n \in \mathbb{N}$. Also suppose*

$$a_{1,n+2} = \begin{cases} 3^{a_{1,1}\, a_{1,2} \cdots a_{2,n+1}} + 2, & \text{if } n = 2k, \quad k = 2^m \\ a_{1,n} + [5 \log^3 (a_{1,n} + 1)], & \text{if } n = 2k, \quad k \neq 2^m \\ \left[4^{2^{(a_{1,1}\, a_{1,2} \cdots a_{2,n}) \log a_{1,n}}}\right] + 1, & \text{if } n = 2k+1,\ k = 2^{2m} \\ a_{1,n} + [\log^2 \log^3 (a_n + \log a_{1,n})], & \text{if } n = 2k+1,\ k \neq 2^{2m} \end{cases}$$

and

$$a_{2,n+2} = \begin{cases} 7^{5^{3^{a_{1,1}\, a_{1,2} \cdots a_{2,n+1}}}} + 2, & \text{if } n = 2k, \quad k = 2^m \\ a_{2,n} + \left[\frac{3n}{2n+2} \log^4 \log(a_{2,n}^2 + \sqrt{a_{2,n}})\right] + 2, & \text{if } n = 2k, \quad k \neq 2^m \\ 3^{4^{5^{3^{a_{1,1}\, a_{1,2} \cdots a_{2,n+1}}}}} + 1, & \text{if } n = 2k+1,\ k = 2^{2m} \\ a_{2,n} + \left[\frac{1}{\sqrt{n^6+4}} \log^3 \log^{2n} \log(a_{2,n} + 5 \log\log a_{2,n})\right], & \text{if } n = 2k+1,\ k \neq 2^{2m} \end{cases}$$

for each $n = 1, 2, \ldots$. Let us take $K = 2$, $s = 2$, $\varepsilon_{1,1} = 2$, $\varepsilon_{1,2} = 1$, $\varepsilon_{2,1} = 3$, $\varepsilon_{2,2} = 2$, $t_{1,1} = 1$, $t_{1,2} = 2$, $t_{2,1} = 2$, $t_{2,2} = 3$ in Theorem 2.2. Then we obtain the fact that the numbers

$$\sum_{n=1}^{\infty} \frac{1}{a_{1,n}}, \quad \sum_{n=1}^{\infty} \frac{1}{a_{2,n}}$$

and the number 1 are linearly independent over rational numbers.

Taking the special values of the integers K and s we get the following corollaries.

Corollary 2.4. *Let K be a positive integer. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_K$ be positive real numbers and let $t_1, t_2, \ldots, t_K$ be positive integers. Let $\{a_{i,n}\}_{n=1}^{\infty}$ and $\{b_{i,n}\}_{n=1}^{\infty}$ be sequences of positive integers for $i = 1, 2, \ldots, K$. Suppose that*

$$\liminf_{n\to\infty} \left(\frac{\operatorname{lcm}(a_{1,1}, \ldots, a_{K,n})}{L_{t_1}^{\varepsilon_1}\left(\frac{a_{1,n+1}}{b_{1,n+1}}\right)} + \sum_{i=1}^{K-1} \frac{\frac{a_{i,n+1}}{b_{i,n+1}}}{L_{t_{i+1}}^{\varepsilon_{i+1}}\left(\frac{a_{i+1,n+1}}{b_{i+1,n+1}}\right)} \right) = 0$$

and that

$$\frac{a_{i,n+1}}{b_{i,n+1} \prod_{k=1}^{t_i} L_k\left(\frac{a_{i,n+1}}{b_{i,n+1}}\right) \cdot L_{t_i}^{\varepsilon_i}\left(\frac{a_{i,n+1}}{b_{i,n+1}}\right)} \geq \frac{a_{i,n}}{b_{i,n} \prod_{k=1}^{t_i} L_k\left(\frac{a_{i,n}}{b_{i,n}}\right) \cdot L_{t_i}^{\varepsilon_i}\left(\frac{a_{i,n}}{b_{i,n}}\right)} + 1$$

for $i = 1, 2, \ldots, K$ and for every sufficiently large n. Then the numbers $\sum_{n=1}^{\infty} b_{1,n}/a_{1,n}$, $\sum_{n=1}^{\infty} b_{2,n}/a_{2,n}$, $\ldots$, $\sum_{n=1}^{\infty} b_{K,n}/a_{K,n}$ and the number 1 are linearly independent over rational numbers.

Example 2.5. *Let $\{a_{i,n}\}_{n=1}^{\infty}$ and $\{b_{i,n}\}_{n=1}^{\infty}$ be sequences of positive integers for $i = 1, 2, 3$ such that $a_{1,1} = 1$, $a_{2,1} = 2$, $b_{1,n} = 1$, $b_{2,n} = 1$ for every $n = 1, 2, \ldots$. Let us define*

$$a_{1,n+1} = \begin{cases} a_{1,n} + [\log a_{1,n} \log^{3/2} \log a_{1,n} + \log^3 \log a_{1,n}], & \text{if } n \neq 7^m \\ 3^{4^{n(a_{1,1}a_{1,2}\cdots a_{3,n})^2}} + 1, & \text{otherwise,} \end{cases}$$

$$a_{2,n+1} = \begin{cases} a_{2,n} + [\log a_{2,n} \log\log a_{2,n} \log\log^2 \log a_{2,n} \log^2 \log\log\log a_{2,n}], & \text{if } n \neq 7^m \\ 2^{2^{2^{3^{4^{(a_{1,1}a_{1,2}\cdots a_{3,n})^3}}}}} + 3, & \text{otherwise} \end{cases}$$

and

$$a_{3,n+1} = \begin{cases} a_{3,n} + [3 \log^{5/7} a_{3,n}], & \text{if } n \neq 7^m \\ 2^{2^{2^{2^{3^{4^{n(a_{1,1}a_{1,2}\cdots a_{3,n})^3}}}}}} + 11, & \text{otherwise.} \end{cases}$$

Let us take $K = 3$, $\varepsilon_1 = 1/2$, $\varepsilon_2 = 1$, $\varepsilon_3 = 1/4$, $t_1 = 2$, $t_2 = 4$, $t_3 = 1$ in Corollary 2.4. Then we obtain the fact that the numbers

$$\sum_{n=1}^{\infty} \frac{1}{a_{1,n}}, \quad \sum_{n=1}^{\infty} \frac{1}{a_{2,n}}, \quad \sum_{n=1}^{\infty} \frac{1}{a_{3,n}}$$

and the number 1 are linearly independent over rational numbers.

Corollary 2.6. *Let s be a positive integer. Let $\{\varepsilon_j\}_{j=1}^{s}$ be a sequence of positive real numbers. Let $\{t_j\}_{j=1}^{s}$, $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of positive integers. Suppose that*

$$\liminf_{n\to\infty} \left(\frac{\operatorname{lcm}(a_1, \ldots, a_n)}{\min\limits_{j=1,\ldots,s} L_{t_j}^{\varepsilon_j}\left(\frac{a_{n+j}}{b_{n+j}}\right)} \right) = 0$$

and that

$$\frac{a_{n+s+j}}{b_{n+s+j} \prod_{k=1}^{t_j} L_k\left(\frac{a_{n+s+j}}{b_{n+s+j}}\right) \cdot L_{t_j}^{\varepsilon_j}\left(\frac{a_{n+s+j}}{b_{n+s+j}}\right)} \geq \frac{a_{n+j}}{b_{n+j} \prod_{k=1}^{t_j} L_k\left(\frac{a_{n+j}}{b_{n+j}}\right) \cdot L_{t_j}^{\varepsilon_j}\left(\frac{a_{n+j}}{b_{n+j}}\right)} + 1$$

for $i = 1, 2, \ldots, s$ and for every sufficiently large n. Then the series $\sum_{n=1}^{\infty} b_n/a_n$ is an irrational number.

Example 2.7. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of positive integers such* *that* $a_1 = 1$, $a_2 = 2$ *and* $b_n = 1$ *for every* $n = 1, 2, \ldots$. *Let us suppose that*

$$a_{n+3} = \begin{cases} \left[5\sqrt[3]{na_1a_2\ldots a_na_{n+1}a_{n+2}}\right] + 1, & \text{if } n = 3k, \quad 501 \mid k \\ a_n + [2\log^4 a_n], & \text{if } n = 3k, \quad 501 \nmid k \\ 3^{3^{a_1a_2\ldots a_na_{n+1}}} + 4, & \text{if } n = 3k+1, \ 503 \mid k \\ a_n + [\log a_n \log^3 \log^2 a_n], & \text{if } n = 3k+1, \ 503 \nmid k \\ 4^{4^{n^{a_1a_2\ldots a_n}}} + 5, & \text{if } n = 3k+2, \ 505 \mid k \\ a_n + [\log a_n \log\log^2 a_n \log^3 \log\log a_n], & \text{if } n = 3k+2, \ 505 \nmid k. \end{cases}$$

Let us take $s = 3$, $\varepsilon_1 = 3$, $\varepsilon_2 = 2$, $\varepsilon_3 = 1$, $t_1 = 1$, $t_2 = 2$, $t_3 = 3$ *in Corollary 2.6. Then we obtain the fact that the series*

$$\sum_{n=1}^{\infty} \frac{1}{a_n}$$

is an irrational number.

Remark 2.8. *If we put* $K = 1$ *in Corollary 2.4 then we obtain Theorem 2.2 from [4].*

3. Proofs

We need the following two lemmas to prove the Theorem 2.2. The first one gives us basic properties of the function $f(x)$ defined as a product of the functions $L_i(x)$ occuring in Theorem 2.2.

Lemma 3.1. *Let* t *be a positive integer and* ε *be a positive real number. Let* $f(x)$ *be a real function such that*

$$y = f(x) := \frac{x}{\prod_{k=1}^{t} L_k(x) \cdot L_t^{\varepsilon}(x)}.$$

Let $F(y)$ *denote the inverse function of* $f(x)$. *Then*

i) $F(y) > \prod_{k=0}^{t} L_k(y) \cdot L_t^{\varepsilon}(y)$ *for sufficiently large* x,

ii) $f(x) > x^{\beta}$, $\quad 0 \le \beta < 1$ *for sufficiently large* x *and*

iii) $L_t^{\varepsilon}(x) = o(x)$.

Proof. [Proof of Lemma 3.1] ad i) It is obvious that $y < x$. Then

$$y = \frac{x}{\prod_{k=1}^{t} L_k(x) \cdot L_t^{\varepsilon}(x)} < \frac{x}{\prod_{k=1}^{t} L_k(y) \cdot L_t^{\varepsilon}(y)}.$$

Multiplying both sides of the inequality by the product in the denominator we get

$$\prod_{k=0}^{t} L_k(y) \cdot L_t^{\varepsilon}(y) < x = F(y)\,.$$

ad ii) The inequality is an immediate consequence of the limit

$$\lim_{x\to\infty} \frac{f(x)}{x^{\beta}} = \infty, \qquad 0 \le \beta < 1.$$

ad iii) It is clear. □

The second lemma yields a special estimate of the relation between terms of a positive real sequences x_i and iterated logarithms $L_t^{\varepsilon}(x_i)$.

Lemma 3.2. *Let K and s be positive integers. Let $\{x_{i,n}\}_{n=1}^{\infty}$ and $\{\varepsilon_{i,j}\}_{j=1}^{s}$ be sequences of positive real numbers such that $\lim_{n\to\infty} x_{i,n} = \infty$. Let $\{t_{i,j}\}_{j=1}^{s}$ be a sequence of positive integers for $i = 1, 2, \ldots, K$. Suppose that there exists a real number $M \in (0,1)$ such that*

$$\frac{\min\limits_{j=1,\ldots,s} x_{i,n+j}}{\min\limits_{j=1,\ldots,s} L_{t_{i+1,j}}^{\varepsilon_{i+1,j}}(x_{i+1,n+j})} < M \tag{3}$$

for every i and every sufficiently large n. Then the inequality

$$\frac{\min\limits_{j=1,\ldots,s} x_{v,n+j}}{\min\limits_{j=1,\ldots,s} L_{t_{u,j}}^{\varepsilon_{u,j}}(x_{u,n+j})} < M$$

holds for each pair u, v satisfying $1 \le v < u \le K$.

Proof. [Proof of Lemma 3.2] Let $J(i)$ be a positive integer depending on i such that $x_{v+i,n+J(i)} = \min\limits_{j=1,\ldots,s} x_{v+i,n+j}$ for each $i = 1, 2, \ldots, K$. Then using (3) we get

$$\begin{aligned}
\frac{\min\limits_{j=1,\ldots,s} x_{v,n+j}}{\min\limits_{j=1,\ldots,s} L_{t_{u,j}}^{\varepsilon_{u,j}}(x_{u,n+j})} &= \prod_{i=1}^{u-v} \frac{\min\limits_{j=1,\ldots,s} x_{v+i-1,n+j}}{\min\limits_{j=1,\ldots,s} L_{t_{v+i,j}}^{\varepsilon_{v+i,j}}(x_{v+i,n+j})} \cdot \prod_{i=1}^{u-v-1} \frac{\min\limits_{j=1,\ldots,s} L_{t_{v+i,j}}^{\varepsilon_{v+i,j}}(x_{v+i,n+j})}{\min\limits_{j=1,\ldots,s} x_{v+i,n+j}} \\
&< M^{u-v} \cdot \prod_{i=1}^{u-v-1} \frac{\min\limits_{j=1,\ldots,s} L_{t_{v+i,j}}^{\varepsilon_{v+i,j}}(x_{v+i,n+j})}{x_{v+i,n+J(i)}} \\
&\le M^{u-v} \cdot \prod_{i=1}^{u-v-1} \frac{L_{t_{v+i,J(i)}}^{\varepsilon_{v+i,J(i)}}(x_{v+i,n+J(i)})}{x_{v+i,n+J(i)}}\,.
\end{aligned} \tag{4}$$

Using the assertion ii) of Lemma 3.1 we obtain that the last product in (4) tends to zero, thus

$$\frac{\min\limits_{j=1,\ldots,s} x_{v,n+j}}{\min\limits_{j=1,\ldots,s} L_{t_{u,j}}^{\varepsilon_{u,j}}(x_{u,n+j})} < M^{u-v} \cdot \prod_{i=1}^{u-v-1} \frac{L_{t_{v+i,J(i)}}^{\varepsilon_{v+i,J(i)}}(x_{v+i,n+J(i)})}{x_{v+i,n+J(i)}} \le M^{u-v} \le M.$$

□

Now we can prove Theorem 2.2.

Proof. [Proof of Theorem 2.2] Suppose to the contrary that there exists a K-tuple of integers $\alpha_1, \alpha_2, \ldots, \alpha_K$ not all equal to zero, an integer p and a positive integer q such that

$$\frac{p}{q} = \sum_{i=1}^{K} \alpha_i \sum_{k=1}^{\infty} \frac{b_{i,k}}{a_{i,k}}.$$

Let L be the first index with $\alpha_L \neq 0$. Without loss of generality assume that $\alpha_L > 0$. Then

$$\frac{p}{q} = \sum_{i=L}^{K} \alpha_i \sum_{k=1}^{\infty} \frac{b_{i,k}}{a_{i,k}} = \sum_{i=L}^{K} \sum_{k=1}^{\infty} \alpha_i \frac{b_{i,k}}{a_{i,k}}.$$

Multiplying both sides by $q \operatorname{lcm}(a_{1,1}, \ldots, a_{K,n})$ we obtain that

$$\begin{aligned} p \operatorname{lcm}(a_{1,1}, \ldots, a_{K,n}) &= q \operatorname{lcm}(a_{1,1}, \ldots, a_{K,n}) \sum_{i=L}^{K} \sum_{k=1}^{n} \alpha_i \frac{b_{i,k}}{a_{i,k}} \\ &+ q \operatorname{lcm}(a_{1,1}, \ldots, a_{K,n}) \sum_{i=L}^{K} \sum_{k=n+1}^{\infty} \alpha_i \frac{b_{i,k}}{a_{i,k}}. \end{aligned}$$

This implies that the number

$$B_n = q \operatorname{lcm}(a_{1,1}, \ldots, a_{K,n}) \sum_{i=L}^{K} \sum_{k=n+1}^{\infty} \alpha_i \frac{b_{i,k}}{a_{i,k}} \tag{5}$$

is an integer for each $n \in \mathbb{N}$. The idea of the proof is simple. Show that $0 < |B_n| < 1$ for infinitely many n. The proof consists of three parts.

Firstly, rearrange the assumption (2) in the following way

$$f_{i,j}\left(\frac{a_{i,n+s+j}}{b_{i,n+s+j}}\right) \geq f_{i,j}\left(\frac{a_{i,n+j}}{b_{i,n+j}}\right) + 1$$

where

$$f_{i,j}(x) := \frac{x}{\prod_{k=1}^{t_{i,j}} L_k(x) \cdot L_{t_{i,j}}^{\varepsilon_{i,j}}(x)}$$

for each $i = 1, 2, \ldots, K$, $j = 1, 2 \ldots, s$ and for every sufficiently large n. The inequality can be generalized using mathematical induction. So we get that

$$f_{i,j}\left(\frac{a_{i,n+rs+j}}{b_{i,n+rs+j}}\right) \geq f_{i,j}\left(\frac{a_{i,n+j}}{b_{i,n+j}}\right) + r \tag{6}$$

for every positive integer r and for sufficiently large n. It is easy to show that $f_{i,j}(x)$ is an increasing function for sufficiently large x and for each i and j. That means that there exists a function $F_{i,j}(x)$ which is inverse to $f_{i,j}(x)$. Note that the function $f_{i,j}(x)$ is the same as the function $f(x)$ defined in Lemma 3.1 for given numbers i and j. This implies that the assertion i) in Lemma 3.1 is valid for the function $F_{i,j}(x)$ as well. As well as $f_{i,j}(x)$, the function $F_{i,j}(x)$ is increasing for sufficiently large x. This fact, the relation (6) and the assertion i) in Lemma 3.1 yield that

$$\begin{aligned}\frac{a_{i,n+rs+j}}{b_{i,n+rs+j}} &\geq F_{i,j}\left(f_{i,j}\left(\frac{a_{i,n+j}}{b_{i,n+j}}\right)+r\right)\\ &> \prod_{k=0}^{t_{i,j}} L_k\left(f_{i,j}\left(\frac{a_{i,n+j}}{b_{i,n+j}}\right)+r\right)\cdot L_{t_{i,j}}^{\varepsilon_{i,j}}\left(f_{i,j}\left(\frac{a_{i,n+j}}{b_{i,n+j}}\right)+r\right) \qquad (7)\end{aligned}$$

for each $i = 1,2\dots,K$, $j = 1,2,\dots,s$ and for every sufficiently large n.

Secondly, prove that $B_n \neq 0$ for infinitely many n. To do this we need to find the upper bound and the lower bound of the series $\sum_{k=n+1}^{\infty} b_{i,k}/a_{i,k}$. Use the following integral inequality to obtain the upper bound.

$$\sum_{l=0}^{\infty} \frac{1}{\prod_{k=0}^{t} L_k(z+l)\cdot L_t^{\varepsilon}(z+l)} < \int_{z-1}^{\infty} \frac{\mathrm{d}x}{\prod_{k=0}^{t} L_k(x)\cdot L_t^{\varepsilon}(x)} = \frac{1}{\varepsilon\, L_t^{\varepsilon}(z-1)}$$

where $\varepsilon > 0$, t is a positive integer and z is a sufficiently large real number. Using this, (7) and the fact that $\lim_{k\to\infty} a_{i,k}/b_{i,k} = \infty$ for each $i = 1,2,\dots,K$, which is implied by assumption (2), we get that

$$\begin{aligned}\sum_{k=n+1}^{\infty} \frac{b_{i,k}}{a_{i,k}} &= \sum_{j=1}^{s}\sum_{m=0}^{\infty} \frac{b_{i,n+ms+j}}{a_{i,n+ms+j}}\\ &< \sum_{j=1}^{s}\sum_{m=0}^{\infty} \frac{1}{\prod_{k=0}^{t_{i,j}} L_k\left(f_{i,j}\left(\frac{a_{i,n+j}}{b_{i,n+j}}\right)+m\right)\cdot L_{t_{i,j}}^{\varepsilon_{i,j}}\left(f_{i,j}\left(\frac{a_{i,n+j}}{b_{i,n+j}}\right)+m\right)}\\ &< \sum_{j=1}^{s} \frac{1}{\varepsilon_{i,j}\, L_{t_{i,j}}^{\varepsilon_{i,j}}\left(f_{i,j}\left(\frac{a_{i,n+j}}{b_{i,n+j}}\right)-1\right)} < \sum_{j=1}^{s} \frac{c_i}{\varepsilon_{i,j}} \frac{1}{L_{t_{i,j}}^{\varepsilon_{i,j}}\left(f_{i,j}\left(\frac{a_{i,n+j}}{b_{i,n+j}}\right)\right)}\end{aligned}$$

for $i = 1,2,\dots,K$ where $c_1,c_2,\dots,c_K$ are suitable positive real constant not depending on n. As mentioned above, the assertion ii) in Lemma 3.1 holds also for the function $f_{i,j}(x)$ for given indices i and j. So we obtain that

$$\begin{aligned}\sum_{k=n+1}^{\infty} \frac{b_{i,k}}{a_{i,k}} &< \sum_{j=1}^{s} \frac{c_i}{\varepsilon_{i,j}} \frac{1}{L_{t_{i,j}}^{\varepsilon_{i,j}}\left(f_{i,j}\left(\frac{a_{i,n+j}}{b_{i,n+j}}\right)\right)} < \sum_{j=1}^{s} \frac{c_i}{\varepsilon_{i,j}} \frac{1}{L_{t_{i,j}}^{\varepsilon_{i,j}}\left(\left(\frac{a_{i,n+j}}{b_{i,n+j}}\right)^{\beta}\right)}\\ &< \sum_{j=1}^{s} \frac{c_i}{\varepsilon_{i,j}} \frac{1}{\gamma L_{t_{i,j}}^{\varepsilon_{i,j}}\left(\frac{a_{i,n+j}}{b_{i,n+j}}\right)} < R \max_{j=1,\dots,s} L_{t_{i,j}}^{-\varepsilon_{i,j}}\left(\frac{a_{i,n+j}}{b_{i,n+j}}\right) \qquad (8)\end{aligned}$$

where β and γ are real numbers with $0 < \beta,\gamma < 1$ and R is a suitable positive real constant not depending on n. This is the upper bound for the series $\sum_{k=n+1}^{\infty} b_{i,k}/a_{i,k}$. It is trivial to find the lower bound of the series, thus

$$\sum_{k=n+1}^{\infty} \frac{b_{i,k}}{a_{i,k}} > \max_{j=1,\dots,s} \frac{b_{i,n+j}}{a_{i,n+j}} \qquad \text{for } i = 1,2,\dots,K. \qquad (9)$$

We have defined the number α_L as the first index with $\alpha_L \neq 0$. To prove that $B_n \neq 0$ it suffices to show that B_n has the same sign as α_L for infinitely many n. It is obvious that the sign of B_n depends only on the sign of the double series

$$\sum_{i=L}^{K} \sum_{k=n+1}^{\infty} \alpha_i \frac{b_{i,k}}{a_{i,k}},$$

look at (5). Without loss of generality we have supposed that $\alpha > 0$. Then using (8), (9) and the fact that $(b_{i,k}/a_{i,k}) \to 0$ as $k \to \infty$ for $i = 1, 2, \dots, K$ we obtain that

$$\begin{aligned}
\sum_{i=L}^{K} \sum_{k=n+1}^{\infty} \alpha_i \frac{b_{i,k}}{a_{i,k}} &= \alpha_L \sum_{k=n+1}^{\infty} \frac{b_{L,k}}{a_{L,k}} + \sum_{i=L+1}^{K} \alpha_i \sum_{k=n+1}^{\infty} \frac{b_{i,k}}{a_{i,k}} \\
&\geq \alpha_L \sum_{k=n+1}^{\infty} \frac{b_{L,k}}{a_{L,k}} - \sum_{i=L+1}^{K} |\alpha_i| \sum_{k=n+1}^{\infty} \frac{b_{i,k}}{a_{i,k}} \\
&> \alpha_L \max_{j=1,\dots,s} \frac{b_{L,n+j}}{a_{L,n+j}} - R \sum_{i=L+1}^{K} |\alpha_i| \max_{j=1,\dots,s} L_{t_{i,j}}^{-\varepsilon_{i,j}} \left(\frac{a_{i,n+j}}{b_{i,n+j}} \right) \\
&= \max_{j=1,\dots,s} \frac{b_{L,n+j}}{a_{L,n+j}} \left(\alpha_L - R \sum_{i=L+1}^{K} |\alpha_i| \frac{\max_{j=1,\dots,s} L_{t_{i,j}}^{-\varepsilon_{i,j}} \left(\frac{a_{i,n+j}}{b_{i,n+j}} \right)}{\max_{j=1,\dots,s} \frac{b_{L,n+j}}{a_{L,n+j}}} \right) \\
&= \max_{j=1,\dots,s} \frac{b_{L,n+j}}{a_{L,n+j}} \left(\alpha_L - R \sum_{i=L+1}^{K} |\alpha_i| \frac{\min_{j=1,\dots,s} \frac{a_{L,n+j}}{b_{L,n+j}}}{\min_{j=1,\dots,s} L_{t_{i,j}}^{\varepsilon_{i,j}} \left(\frac{a_{i,n+j}}{b_{i,n+j}} \right)} \right) \qquad (10)
\end{aligned}$$

for sufficiently large n. Now we use the condition (1). It asserts that there exist infinitely many n such that

$$\frac{\operatorname{lcm}(a_{1,1}, \dots, a_{K,n})}{\min_{j=1,\dots,s} L_{t_{1,j}}^{\varepsilon_{1,j}} \left(\frac{a_{1,n+j}}{b_{1,n+j}} \right)} + \sum_{i=1}^{K-1} \frac{\min_{j=1,\dots,s} \frac{a_{i,n+j}}{b_{i,n+j}}}{\min_{j=1,\dots,s} L_{t_{i+1,j}}^{\varepsilon_{i+1,j}} \left(\frac{a_{i+1,n+j}}{b_{i+1,n+j}} \right)} < M$$

for every real number M with $0 < M < 1$. Hence

$$\frac{\operatorname{lcm}(a_{1,1}, \dots, a_{K,n})}{\min_{j=1,\dots,s} L_{t_{1,j}}^{\varepsilon_{1,j}} \left(\frac{a_{1,n+j}}{b_{1,n+j}} \right)} < M \qquad (11)$$

and

$$\frac{\min_{j=1,\dots,s} \frac{a_{i,n+j}}{b_{i,n+j}}}{\min_{j=1,\dots,s} L_{t_{i+1,j}}^{\varepsilon_{i+1,j}} \left(\frac{a_{i+1,n+j}}{b_{i+1,n+j}} \right)} < M \qquad \text{for } i = 1, 2, \dots, K-1.$$

Note the important fact, that the positive number M can be chosen arbitrarily small for infinitely many n. We will use this fact several times. The latter inequality, showing the relation between terms of the successive sequences $a_{i,n}/b_{i,n}$ and

$a_{i+1,n}/b_{i+1,n}$, can be generalized so that the relation holds for each two sequences $a_{u,n}/b_{u,n}$ and $a_{v,n}/b_{v,n}$ with $1 \le v < u \le K$. To do this we use Lemma 3.2 putting $x_{i,n} = a_{i,n}/b_{i,n}$ for $i = 1, 2, \dots, K$ and we obtain that

$$\frac{\min_{j=1,\dots,s} \frac{a_{v,n+j}}{b_{v,n+j}}}{\min_{j=1,\dots,s} L_{t_{u,j}}^{\varepsilon_{u,j}}\left(\frac{a_{u,n+j}}{b_{u,n+j}}\right)} < M \qquad \text{for } 1 \le v < u \le K. \tag{12}$$

Let us put $v = L$ in (12). Then this and (10) imply that

$$\begin{aligned}\sum_{i=L}^{K} \sum_{k=n+1}^{\infty} \alpha_i \frac{b_{i,k}}{a_{i,k}} &> \max_{j=1,\dots,s} \frac{b_{L,n+j}}{a_{L,n+j}} \left(\alpha_L - RM \sum_{i=L+1}^{K} |\alpha_i|\right) \\ &> \max_{j=1,\dots,s} \frac{b_{L,n+j}}{a_{L,n+j}} (\alpha_L - 1) \ge 0.\end{aligned}$$

If $\alpha_i = 0$ for $i = L+1, L+2, \dots, K$ then the second inequality is evident. Otherwise it suffices to take the number M sufficiently small, i.e. $M < \min(1, 1/(R\sum_{i=L+1}^{K} |\alpha_i|))$. From this and (5) it follows that B_n is a positive integer. In the case $\alpha_L < 0$, the procedure is analogous. So we have proved that B_n is a nonzero integer with the same sign as α_L.

Thirdly, prove now that $|B_n| < 1$. From (5) we obtain that

$$0 < |B_n| \le q \operatorname{lcm}(a_{1,1}, \dots, a_{K,n}) \sum_{i=L}^{K} \sum_{k=n+1}^{\infty} |\alpha_i| \frac{b_{i,k}}{a_{i,k}}. \tag{13}$$

Setting $v = 1$ in (12) and using the upper bound (8) we have that

$$\begin{aligned}\sum_{i=L}^{K} |\alpha_i| \sum_{k=n+1}^{\infty} \frac{b_{i,k}}{a_{i,k}} &< R \sum_{i=L}^{K} |\alpha_i| \max_{j=1,\dots,s} L_{t_{i,j}}^{-\varepsilon_{i,j}}\left(\frac{a_{i,n+j}}{b_{i,n+j}}\right) \\ &= \frac{R|\alpha_1|}{\min_{j=1,\dots,s} L_{t_{1,j}}^{\varepsilon_{1,j}}\left(\frac{a_{1,n+j}}{b_{1,n+j}}\right)} \\ &\quad + \max_{j=1,\dots,s} \frac{b_{1,n+j}}{a_{1,n+j}} \cdot R \sum_{i=\max(2,L)}^{K} |\alpha_i| \frac{\max_{j=1,\dots,s} L_{t_{i,j}}^{-\varepsilon_{i,j}}\left(\frac{a_{i,n+j}}{b_{i,n+j}}\right)}{\max_{j=1,\dots,s} \frac{b_{1,n+j}}{a_{1,n+j}}} \\ &= \frac{R|\alpha_1|}{\min_{j=1,\dots,s} L_{t_{1,j}}^{\varepsilon_{1,j}}\left(\frac{a_{1,n+j}}{b_{1,n+j}}\right)} \\ &\quad + \max_{j=1,\dots,s} \frac{b_{1,n+j}}{a_{1,n+j}} \cdot R \sum_{i=\max(2,L)}^{K} |\alpha_i| \frac{\min_{j=1,\dots,s} \frac{a_{1,n+j}}{b_{1,n+j}}}{\min_{j=1,\dots,s} L_{t_{i,j}}^{\varepsilon_{i,j}}\left(\frac{a_{i,n+j}}{b_{i,n+j}}\right)} \\ &< \frac{R|\alpha_1|}{\min_{j=1,\dots,s} L_{t_{1,j}}^{\varepsilon_{1,j}}\left(\frac{a_{1,n+j}}{b_{1,n+j}}\right)} + \max_{j=1,\dots,s} \frac{b_{1,n+j}}{a_{1,n+j}} \cdot RM \sum_{i=\max(2,L)}^{K} |\alpha_i|\end{aligned}$$

$$= \frac{R|\alpha_1|}{\min\limits_{j=1,\dots,s} L_{t_{1,j}}^{\varepsilon_{1,j}}\left(\frac{a_{1,n+j}}{b_{1,n+j}}\right)} + \frac{RM \sum\limits_{i=\max(2,L)}^{K} |\alpha_i|}{\min\limits_{j=1,\dots,s} \frac{a_{1,n+j}}{b_{1,n+j}}}$$
$$< \frac{R|\alpha_1|}{L_{t_{1,j''}}^{\varepsilon_{1,j''}}\left(\frac{a_{1,n+j''}}{b_{1,n+j''}}\right)} + \frac{1}{\frac{a_{1,n+j'}}{b_{1,n+j'}}}$$

where M is a sufficiently small positive real number and j' resp. j'' are indices with $1 \le j', j'' \le s$ in which the minima

$$\min_{j=1,\dots,s} \frac{a_{1,n+j}}{b_{1,n+j}} \qquad \text{resp.} \qquad \min_{j=1,\dots,s} L_{t_{1,j}}^{\varepsilon_{1,j}}\left(\frac{a_{1,n+j}}{b_{1,n+j}}\right)$$

occur. Using this and assertion iii) in Lemma 3.1 we get that

$$\sum_{i=L}^{K} \sum_{k=n+1}^{\infty} |\alpha_i| \frac{b_{i,k}}{a_{i,k}} < \frac{1}{\frac{a_{1,n+j'}}{b_{1,n+j'}}} + \frac{R|\alpha_1|}{L_{t_{1,j''}}^{\varepsilon_{1,j''}}\left(\frac{a_{1,n+j''}}{b_{1,n+j''}}\right)}$$
$$< \frac{1}{L_{t_{1,j'}}^{\varepsilon_{1,j'}}\left(\frac{a_{1,n+j'}}{b_{1,n+j'}}\right)} + \frac{R|\alpha_1|}{L_{t_{1,j''}}^{\varepsilon_{1,j''}}\left(\frac{a_{1,n+j''}}{b_{1,n+j''}}\right)}$$
$$\le \frac{1}{L_{t_{1,j''}}^{\varepsilon_{1,j''}}\left(\frac{a_{1,n+j''}}{b_{1,n+j''}}\right)} + \frac{R|\alpha_1|}{L_{t_{1,j''}}^{\varepsilon_{1,j''}}\left(\frac{a_{1,n+j''}}{b_{1,n+j''}}\right)} = \frac{1 + R|\alpha_1|}{\min\limits_{1,\dots,s} L_{t_{1,j}}^{\varepsilon_{1,j}}\left(\frac{a_{1,n+j}}{b_{1,n+j}}\right)}.$$

From this, (11) and (13), we obtain that

$$0 < |B_n| \le q \operatorname{lcm}(a_{1,1}, \dots, a_{K,n}) \sum_{i=L}^{K} \sum_{k=n+1}^{\infty} |\alpha_i| \frac{b_{i,k}}{a_{i,k}}$$
$$< \frac{(1 + R|\alpha_1|)\, q \operatorname{lcm}(a_{1,1}, \dots, a_{K,n})}{\min\limits_{j=1,\dots,s} L_{t_{1,j}}^{\varepsilon_{1,j}}\left(\frac{a_{1,n+j}}{b_{1,n+j}}\right)} < (1 + R|\alpha_1|)\, qM < 1$$

for infinitely many n. So we have finally proved that $0 < |B_n| < 1$. However, this contradicts the fact that B_n is an integer. □

Proof. [Proof of Corollary 2.4] Put $s = 1$ in Theorem 2.2. □

Proof. [Proof of Corollary 2.6] Put $K = 1$ in Theorem 2.2. □

Acknowledgement: We would like to thank Professor Radhakrishnan Nair from the Department of Mathematical Sciences, University of Liverpool, for his help with the presentation of this article.

References

[1] J. P. Bezivin, *Linear Independence of the Values of Transcendental Solutions of Some Functional Equations*, Manuscripta Math. 61(1988) no.1, 103–129.

[2] J. Hančl, *Linearly unrelated sequences*, Pacific J. Math., vol. 190, no. 2, 1999, 299–310.

[3] J. Hančl, *A criterion for linear independence of special series*, University of Istanbul Faculty of Science The Journal of Mathematics 60, (2001), 33–39.

[4] J. Hančl, P. Rucki, *The irrationality of certain infinite series*, Saitama Math. J., vol. 21, 2003, 1–8.

[5] J. Hančl, S. Sobková, *A general criterion for linearly unrelated sequences*, Tsukuba J. Math., vol. 27, no. 2, (2003), 341–357.

[6] J. Hančl, S. Sobková, *Criteria for rapidly convergent sequences to be linearly unrelated*, JP Journal Algebra Num. Theory Appl. vol. 5, no. 2, (2005), 205–219.

[7] J. Hančl, R. Tijdeman, *On the irrationality of factorial series*, Acta Arith. vol. 118, no. 4, (2005), 383–404.

[8] K. Nishioka, *Mahler Functions and Transcendence*, Lecture notes in mathematics 1631, Springer, (1996).

[9] J. Sándor, *Some classes of irrational numbers*, Stud. Univ. Babes-Bolyai, Math. XXIX, 1984, 3–12.

[10] V. N. Sorokin, *Linear Independence of Logarithm of Some Rational Numbers*, Mat. Zametki 46 (1989) no.3, 74–79, 127, translated in Math. Notes 46 (1989), no.3–4, 727–730.

Jaroslav Hančl
Department of Mathematics and Institute for Research and
Applications of Fuzzy Modeling, University of Ostrava,
30. dubna 22, 701 03 Ostrava 1,
Czech Republic
e-mail: hancl@osu.cz

Pavel Rucki
Department of Mathematics with Didactics, University of Ostrava,
Dvořákova 7, 701 03 Ostrava 1,
Czech Republic
e-mail: pavel.rucki@seznam.cz

Jan Šustek
Department of Mathematics and Institute for Research and
Applications of Fuzzy Modeling, University of Ostrava,
30. dubna 22, 701 03 Ostrava 1,
Czech Republic
e-mail: jan.sustek@seznam.cz