Kumamoto J. Math.
Vol. 19 (2006), 37–57

On the singular Fano threefold V_{22}^* with a small Gorenstein singularity : (an example)

Mikio Furushima*

(Received December 28, 2005)

Abstract. Let V be a Fano 3-fold of $\rho(V) = 1$ with at most Gorenstein terminal singularities. Assume that V is indecomposable (see [11]). Then one has the genus $g \leq 10$ ($g \leq 12$ if V is smooth). On the other hand, in the case that V is decomposable, there can be a Fano 3-fold V_{22}^* of $g = 12$. In this paper, from the viewpoint of compactifications of $\mathbb{C}^3$, we shall construct a Fano 3-fold V_{22}^* of genus $g = 12$ with

(i) V_{22}^* has a small Gorenstein singularity of A_1-type.

(ii) $b_2(V_{22}^*) = 1$ and $b_4(V_{22}^*) = 2$.

(iii) There exists a reducible Cartier divisor $\Delta_{22} \subset V_{22}^*$ such that $W := V_{22}^* - \Delta_{22} \cong (\mathbb{C}^* \times \mathbb{C}^2) \dot{\cup} \mathbb{C}^2$ (disjoint union).

Notation

$N_{Y|X}$: normal bundle of Y in X
$c_1(\mathcal{F})$: first Chern class of $\mathcal{F}$
$h^i(\mathcal{L}) := \dim_{\mathbb{C}} H^i(\cdot;\mathcal{L})$
$\mathrm{Bs}\,|\mathcal{L}|$: base locus of the linear system $|\mathcal{L}|$
$b_i(X) := \dim_{\mathbb{R}} H_i(X;\mathbb{R})$
$\rho(X)$: Picard number of X
$\mathrm{mult}_A X$: multiplicity of X at a general point of A
K_X : canonical divisor of X
$\sim$: linear equivalence
$\cong$: isomorphism
$\mathbb{F}_n$: Hirzebruch surface of degree n
$\mathbb{Q}_0^2$: quadric cone in $\mathbb{P}^3$

Mathematics Subject Classification (2000): 14J45 , 32J05
Keywords: Fano threefold, Small Gorenstein singularity
*Partly supported by the Grant-in-Aid for Scientific Research (C) (2) no. 16540074, Japan Society for the Promotion of Science. A part of this work is done when the author stayed in Max-Planck-Institut für Mathematik in Bonn from Oct. 2002 to Sep. 2003. He thanks the M.P.I. for the financial support and hospitality. Especially he thanks Professor Dr. F. Hirzebruch for the encouragement.

$\mathbb{Q}_0^3$: quadric hypersurface in $\mathbb{P}^4$ with an isolated singular point
V_5 : smooth Fano threefold of index two and degree 5 in $\mathbb{P}^6$
V_5^*: singular Fano threefold of index two and degree 5 in $\mathbb{P}^6$ with small Gorenstein singularities
V_{22} : smooth Fano threefold of index one and degree 22 in $\mathbb{P}^{13}$
V_{22}^*: singular Fano threefold of index one and degree 22 in $\mathbb{P}^{13}$ with small Gorenstein singularities

1. Introduction

Let (X, Y) the analytic compactification of $\mathbb{C}^3$, that is, X is a 3-dimensional compact complex manifold and Y an analytic subset of X such that $X - Y$ is biholomorphic to $\mathbb{C}^3$. The compactification (X, Y) is said to be projective (resp. Moishezon) if X is projective (resp. Moishezon). Then we have the following

Theorem 1.1 (cf.[1],[2]). *Let (X, Y) be a projective compactification of $\mathbb{C}^3$ with the second Betti number $b_2(X) = 1$. Then Y is an ample divisor and $-K_X \sim rY$ $(r \in \mathbb{N}, 1 \leq r \leq 4)$, that is, X is a smooth Fano threefold with $\rho(X) = 1$. Moreover,*

(1) $r = 4 \Longrightarrow (X, Y) \cong (\mathbb{P}^3, \mathbb{P}^2)$.

(2) $r = 3 \Longrightarrow (X, Y) \cong (\mathbb{Q}^3, \mathbb{Q}_0^2)$.

(3) $r = 2 \Longrightarrow (X, Y) \cong (V_5, H_5^0)$ *or* (V_5, H_5^∞).

(4) $r = 1 \Longrightarrow (X, Y) \cong (V_{22}, H_{22}^0)$ *or* (V_{22}, H_{22}^∞),

Notation:

- H_5^0 (resp. H_5^∞) is a normal (resp. non-normal) hyperplane section of V_5 such that $\mathrm{Sing}\, H_5^0 = \{$a rational double point of A_4 type$\}$ (resp. $\mathrm{Sing}\, H_5^\infty = \ell$ (a line) with the normal bundle $N_{\ell|V_5} \cong \mathcal{O}_\ell(-1) \oplus \mathcal{O}_\ell(1)$).
- H_{22}^0 and H_{22}^∞ are the non-normal hyperplane sections with $\mathrm{Sing}\, H_{22}^0 = \mathrm{Sing}\, H_{22}^\infty = L$, where L is a line on V_{22}, which has the normal bundle $N_{L|V_{22}} \cong \mathcal{O}_L(-2) \oplus \mathcal{O}_L(1)$. In particular, one has $\mathrm{mult}_L\, H_{22}^0 = 2$ and $\mathrm{mult}_L\, H_{22}^\infty = 3$.

On the other hand, in the case that X is non-projective, we have the following:

Theorem 1.2 (cf.[3],[4],[5],[7]). *Let (X, Y) be a smooth analytic compactification of $\mathbb{C}^3$ with $\mathrm{b}_2(X) = 1$. Asuume that X is non-projective. Then*

(1) *X is Moishezon.*

(2) *Y is a non- projective non-normal irreducible divisor.*

(3) $-K_X \sim rY$ $(r = 1, 2)$.

Moreover assume that Y is nef. Then there exists a small birational contraction $\Phi : X \longrightarrow V^*$ of X onto a Fano threefold V^* with small Gorenstein singularities such that

(a) *the exceptional set of Φ consists of finitely many smooth rational curves C_i supported in Y, and the normal bundle $N_{C_i|X}$ of C_i in X is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, $\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}$ or $\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.*

(b) *$\Delta := \Phi_* Y \in \operatorname{Pic} V^*$ is ample and $-K_{V^*} \sim r\Delta$, in particular, $\operatorname{Pic} V^* \cong \mathbb{Z}\mathcal{O}_{V^*}(\Delta)$.*

(c) *$V^* - \Delta \cong X - Y \cong \mathbb{C}^3$ with $\mathrm{b}_2(V^*) = 1$.*

This yields the following problem.

Problem A. Determine all the singular compactifications (V^*, Δ) of $\mathbb{C}^3$ such that

(1) V^* is a Fano threefold with (non-empty) small Gorenstein singularities,

(2) $\mathrm{b}_2(V^*) = 1$, and

(3) $\Delta \in |\mathcal{O}_{V^*}(1)|$.

Let (V^*, Δ) be as above. Then we have $K_{V^*} \sim -rH$ $(0 < r \in \mathbb{Z})$ for $H \in |\mathcal{O}_{V^*}(1)|$. The integer r is called the Fano-index of V^*. If $r \geq 4$, then $V^* \cong \mathbb{P}^3$ (in fact, $r = 4$). Since V^* has singularities, this case can be excluded, that is, we have only to consider the case of $1 \leq r \leq 3$. We remark that the condition "$\mathrm{b}_2(V^*) = 1$" does not necessarily imply "$\mathrm{b}_4(V^*) = 1$" even if V^* has mild singularities. In fact, there exists an example such that $\mathrm{b}_2(V^*) = 1$ and $\mathrm{b}_4(V^*) = 2$. We note that he fourth Betti number $\mathrm{b}_4(V^*)$ is equal to the number of irreducible components of the boundary divisor Δ.

On Problem A, we obtain the following.

Theorem 1.3 ([4],[7]). *Assume that $r \geq 2$. Then*

(I) *If $r = 3$, then $(V^*, \Delta) \cong (\mathbb{Q}_0^3, \Delta_2)$, where $\Delta_2 \sim \Delta_2^1 + \Delta_2^2$ (as a Weil divisor) is a hyperplane section consisting of two planes such that $\Delta_2^1 \cap \Delta_2^2$ is a generating line passing through the vertex of $\mathbb{Q}_0^3$.*

(II) *If $r = 2$, then $d = (\Delta)^3 = 4, 5$. Moreover,*

(a) *$d = 4 \Longrightarrow (V^*, \Delta) \cong (V_4^*, \Delta_4)$, and $\mathrm{b}_2(V_4^*) = \mathrm{b}_4(V_4^*) = 1$.*

(b) $d = 5 \Longrightarrow (V^*, \Delta) \cong (V_5^*, \Delta_5)$, *where* V_5^* *has a small hypersurface singularity of* A_1*-type,and* $b_2(V_5^*) = 1$ *and* $b_4(V_5^*) = 2$. $\Delta_5 = \Delta_5^1 + \Delta_5^2$ *is a reducible hyperplane section. Moreover, there is a compactification* (V_5^*, Δ_5) *of* $\mathbb{C}^3$ *such that* $\Delta_5^1 \cong \mathbb{F}_1$, Δ_5^2 *is a normal rational surface with a rational double point of* A_1*-type.*

(III) *If* $r = 1$, *then there exists a Fano threefold* $V_{18}^* \subset \mathbb{P}^{11}$ *of degree 18 (that is, the genus* $g = \frac{1}{2}(-K_{V_{18}^*})^3 + 1 = 10$ *and a non-normal hyperplane section* Δ_{18} *of* V_{18}^* *such that*

(c) $b_2(V_{18}^*) = b_4(V_{18}^*) = 1$.

(d) V_{18}^* *a small Gorenstein singularity* $p \in \Delta_{18}$.

(e) $V_{18}^* - \Delta_{18} \cong \mathbb{C}^3$.

Remark 1.1. Now let V be a normal Gorenstein Fano threefold, that is, $-K_V$ is ample. We call the integer $g := \frac{1}{2}(-K_V)^3 + 1$ the "genus" of V. Then V is decomposable if $|-K_V|$ is a sum of two movable Weil divisors, i.e. $-K_V \sim H_1 + H_2$ with $\dim |H_i| > 0$ for $i = 1, 2$. V is indecomposable if V is not decomposable. Then Mukai [11] classifies the indecomposable Fano threefold with at most Gorenstein terminal singularities by the vector bundle method. He also proves that the genus g of V satisfies $g \leq 10$ if V is singular and indecomposable. On the other hand, if V is smooth, then it is shown that $g \leq 12$, $(\neq 11)$. It is known that Fano threefold V_{22} of $g = 12$ actually exists and is a compactification of $\mathbb{C}^3$. Thus the problem will be the existence of singular and decomposable Fano threefold of the genus $g = 12$. Now, in this note, we shall construct such a decomposable Fano threefold V_{22}^* $(g = 12)$ with a small Gorenstein terminal singularity.

Finally we shall propose the following

Conjecture 1. Let (V^*, Δ) be as in Problem A and assume that $r = 1$. Then $b_4(V^*) \leq 2$ and

(1) $V^* \cong V_{18}^*$ if $b_4(V^*) = 1$.

(2) $V^* \cong V_{22}^*$ if $b_4(V^*) = 2$ and $\mathbb{C}^3 \subset V_{22}^*$.

2. Singular Fano threefolds $\mathbb{Q}_0^3$ and V_5^* as a compactification of $\mathbb{C}^3$

2.1.

We recall the Fano threefolds $\mathbb{Q}_0^3$ and V_5^* constructed in the paper [7]. First let $\mathbb{P}^4$ be the 4-dimensional complex projective space with the homogeneous coordinate system $(x_0 : x_1 : x_2 : x_3 : x_4)$ and $\mathbb{Q}_0^3$ a quadric cone defined by $\mathbb{Q}_0^3 := \{x_0x_3 = x_1x_2\}$ in $\mathbb{P}^4$. Then the singular point of $\mathbb{Q}_0^3$ is the vertex $p := (0 : 0 : 0 : 0 : 1) \in \mathbb{P}^4$ of the cone. Let Q_∞ be the hyperplane section defined by $Q_\infty := \{x_4 = 0\}$.

Then Q_∞ ($\cong \mathbb{P}^1 \times \mathbb{P}^1$) is a smooth quadric hypersurface in $\mathbb{P}^3$. Let $\Delta_2 := \{x_0 = 0\}$ be the hyperplane section of $\mathbb{Q}_0^3$. Then we have $\Delta_2 := \Delta_2^{(1)} + \Delta_2^{(2)}$, where $\Delta_2^{(i)} := \{x_0 = x_i = 0\}$ $(i = 1, 2)$ is a smooth $\mathbb{Q}$-Cartier divisor isomorphic to $\mathbb{P}^2$. We put $g := \Delta_2^{(1)} \cap \Delta_2^{(2)} \cong \mathbb{P}^1$ (a generating line of $\mathbb{Q}_0^3$). It is easy to see that $\mathbb{Q}_0^3 - \Delta_2 \cong \mathbb{C}^3$. Let $\Phi : X \longrightarrow \mathbb{Q}_0^3$ be a small resolution with the exceptional set $C := \Phi^{-1}(p) \cong \mathbb{P}^1$. Let $\alpha : B_p(\mathbb{P}^4) \longrightarrow \mathbb{P}^4$ be the blowing up of $\mathbb{P}^4$ with the center p and $\overline{\mathbb{Q}_0^3}$ the proper transform of $\mathbb{Q}_0^3$. Then the restriction $\alpha : \overline{\mathbb{Q}_0^3} \longrightarrow \mathbb{Q}_0^3$ is a resolution of the singularity p with exceptional set $\alpha^{-1}(p) = \overline{Q}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $\beta : B_C(X) \longrightarrow X$ be the blowing up of X with the center C. Then we have $B_C(X) \cong \overline{\mathbb{Q}_0^3}$ and the birational morphism α is factorized as $\alpha = \Phi \circ \beta$, that is, $\alpha : \overline{\mathbb{Q}_0^3} \xrightarrow{\beta} X \xrightarrow{\Phi} \mathbb{Q}_0^3$. In particular, one has the normal bundle $N_{C|X} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. From $\mathbb{Q}_0^3 - \Delta_2 \cong \mathbb{C}^3$, one has $\mathrm{b}_i(\mathbb{Q}_0^3) = \mathrm{b}_i(\Delta_2)$ for $i > 0$. On the other hand, since $\mathrm{b}_i(\Delta_2) = \mathrm{b}_i(\Delta_2^{(1)}) + \mathrm{b}_i(\Delta_2^{(2)}) - \mathrm{b}_i(g)$, one has easily $\mathrm{b}_1(\mathbb{Q}_0^3) = \mathrm{b}_3(\mathbb{Q}_0^3) = 0$, $\mathrm{b}_2(\mathbb{Q}_0^3) = 1$ and $\mathrm{b}_4(\mathbb{Q}_0^3) = 2$. Thus we have

Theorem 2.1. $(\mathbb{Q}_0^3, \Delta_2)$ *is a singular Fano compactification of* $\mathbb{C}^3$ *of index* $r = 3$ *with a small hypersurface singularity of* A_1*-type, in particular,* $\mathrm{b}_2(\mathbb{Q}_0^3) = 1$ *and* $\mathrm{b}_4(\mathbb{Q}_0^3) = 2$.

2.2.

Next we shall give a construction of a singular Fano threefold V_5^* as a compactification of $\mathbb{C}^3$. Let $(\mathbb{Q}_0^3, \Delta_2, \Delta_2^{(i)}, Q_\infty, g)$ be as above. Let us consider a twisted cubic curve $\gamma : \mathbb{P}^1 \longrightarrow Q_\infty \subset \mathbb{P}^3$ defined by $\gamma(u : v) = (u^3 : u^2v : uv^2 : v^3)$. We set $s := \{x_0 = x_1 = x_4 = 0\}$, $f := \{x_0 = x_2 = x_4 = 0\}$ and $\gamma := \gamma(\mathbb{P}^1) \subset Q_\infty$. One sees that s, f are two different rulings of $Q_\infty \cong \mathbb{P}^1 \times \mathbb{P}^1$ with $\operatorname{Pic} Q_\infty \cong \mathbb{Z}f \oplus \mathbb{Z}s$. Then we have a linear equivalence $\gamma \sim s + 2f$. By construction one has $\Delta_2^{(2)} \cap \gamma = f \cap \gamma = q$ and $\Delta_2^{(1)} \cap \gamma = s \cap \gamma = 2q$, where $q = (0:0:0:1:0) = s \cap f \cap g$. Let $\pi : \widehat{\mathbb{Q}_0^3} \longrightarrow \mathbb{Q}_0^3$ be the blowing up of $\mathbb{Q}_0^3$ along γ with the exceptional set $\widehat{S} := \pi^{-1}(\gamma)$.

Then we can prove that

Proposition 2.2. (1) $\mathrm{h}^0(\mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\widehat{S} + 2\widehat{Q}_\infty)) = 7$

(2) $\operatorname{Bs}|\widehat{S} + 2\widehat{Q}_\infty| = \emptyset$

Let $\widehat{\Phi} : \widehat{\mathbb{Q}_0^3} \longrightarrow \mathbb{P}^6$ be a morphism defined by the linear system $|\widehat{S} + 2\widehat{Q}_\infty|$. We put $V_5^* := \widehat{\Phi}(\widehat{\mathbb{Q}_0^3}) \subset \mathbb{P}^6$ and $S := \widehat{\Phi}_* \widehat{S}$. Since $(\widehat{S} + 2\widehat{Q}_\infty)^3 = 5$, one has $\deg V_5^* = 5$ in $\mathbb{P}^6$. We can see that the exceptional set $\operatorname{Exc}(\widehat{\Phi}) = \widehat{Q}_\infty$. Then we also have

Proposition 2.3. V_5^* *is a Fano threefold with a small Gorenstein singularity* $p := \widehat{\Phi}(\pi^{-1}(p))$ *of* A_1*-type and smooth along the line* $E := \Phi(\widehat{Q}_\infty)$, *in particular,* $\operatorname{Sing} S = E$ *and* $\widehat{\Phi} : \widehat{\mathbb{Q}_0^3} \longrightarrow V_5^*$ *is the blowing up of* V_5^* *with the center* E.

Remark 2.1 It is easy to see that the restriction $\nu := \widehat{\Phi}|_{\widehat{S}} : \widehat{S} \longrightarrow S$ is the normalization with $\nu^{-1}(E) = \Sigma$. By construction $\nu|_{\Sigma} : \Sigma \longrightarrow E$ is a double covering. It is easy to verify that $b_2(V_5^*) = 1$, $b_3(V_5^*) = 0$, $b_4(V_5^*) = 2$, in particular, $\operatorname{Pic} V_5^* \cong \mathbb{Z}\mathcal{O}_{V_5^*}(S)$. The birational morphism $\overline{\Phi} := \widehat{\Phi} \circ \pi^{-1} : \mathbb{Q}_0^3 \dashrightarrow V_5^*$ is given by the linear system $|\mathcal{O}_{\mathbb{Q}_0^3}(2) - \gamma|$ since $\widehat{S} + 2\widehat{Q}_\infty \sim 2\pi^*(Q_\infty) - \widehat{S}$. We set $\Delta_5^{(i)} := \overline{\Phi}(\Delta_2^{(i)})$ for $i = 1, 2$ and $\Delta_5 = \Delta_5^{(1)} \cup \Delta_5^{(2)}$. Then $\Delta_5^{(i)}$ is an effective $\mathbb{Q}$-Cartier divisor on V_5^*. By construction one sees that $\Delta_5^{(1)}$(resp. $\Delta_5^{(2)}$) is a smooth rational surface (resp. rational surface with a rational double point of A_1-type). In particular, Δ_5 is a hyperplane section of V_5^*. In fact, we obtain $\widehat{\Phi}^*\Delta_5 \sim \widehat{\Phi}^* S \sim \widehat{S} + 2\widehat{Q}_\infty$.

Finally we have the following

Theorem 2.4 ([7]). *(V_5^*, Δ_5) is a singular Fano compactification of $\mathbb{C}^3$ of index $r = 2$ with a small Gorenstein singularity of A_1-type, in particular,* $b_2(V_5^*) = 1$ *and* $b_4(V_5^*) = 2$.

Remark 2.2. Let V be a Fano threefold of degree 5 in $\mathbb{P}^6$ with at most Gorenstein terminal singularities. Then one can prove that V is smooth if $b_2(V) = b_4(V) = 1$.

3. Fano threefold V_{22}^* of degree 22 in $\mathbb{P}^{13}$ with one small singularity of A_1-type.

First we shall study the detailed structure of the non-normal del Pezzo surface S constructed in the section **2**. Let us recall the normailization $\nu := \widehat{\Phi}|_{\widehat{S}} : \widehat{S} \longrightarrow S$ and the analytic inverse image $\nu^{-1}(E) = \Sigma$. We have the anti-dualizing sheaf $\omega_S^{-1} = \mathcal{O}_S(S)$, which is an ample invertible sheaf.

Lemma 3.1. **(1)** $h^0(\mathcal{O}_\Sigma) = 1$, $h^1(\mathcal{O}_\Sigma) = 0$.

(2) $\chi(\mathcal{O}_{\widehat{S}}) = 1$, $(\nu^*\omega_Y \cdot \Sigma) = -2$.

(3) $(\omega_S \cdot E) = -1$ *and E is irreducible reduced, in particular,* $E \cong \mathbb{P}^1$.

Lemma 3.2. $h^0(\nu^*\omega_S^{-1}) = h^0(\omega_S^{-1}) + 1$.

Proof. Let us consider an exact sequence (cf.[10, (3.34.2)]):

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \nu_*\mathcal{O}_{\widehat{S}} \longrightarrow \omega_S^{-1} \otimes \omega_E \longrightarrow 0$$

By operating $\otimes\omega_S^{-1}$, we obtain

$$0 \longrightarrow \omega_S^{-1} \longrightarrow \nu_*\mathcal{O}_{\widehat{S}} \otimes \omega_S^{-1} \longrightarrow \omega_S^{-1} \otimes \omega_E \longrightarrow 0$$

By the projection formula and the Serre duality theorem, we have:

$$\begin{aligned} H^0(S; \nu_*\mathcal{O}_{\widehat{S}} \otimes \omega_S^{-1}) &\cong H^0(S; \nu_*\mathcal{O}_{\widehat{S}}(\nu^*\omega_S^{-1})) \\ &\cong H^0(\widehat{S}; \nu^*\omega_S^{-1}) \end{aligned}$$

and

$$
\begin{aligned}
\mathrm{H}^0(S;\omega_S^{-1}\otimes\omega_E) &\cong \mathrm{H}^1(E;\mathcal{O}_E\otimes\omega_S)\\
&\cong \mathrm{H}^1(E;\mathcal{O}_E(-1))\\
&\cong \mathrm{H}^0(E;\mathcal{O}_E)\\
&\cong \mathrm{H}^0(\mathbb{P}^1;\mathcal{O}_{\mathbb{P}^1})\\
&\cong \mathbb{C}.
\end{aligned}
$$

Since $\mathrm{H}^1(S;\omega_S^{\otimes -n}) = 0$ for $n > 0$ by Goto-Mori-Reid (cf.[10]), we have the following:

$$0 \longrightarrow \mathrm{H}^0(S;\omega_S^{-1}) \xrightarrow{\nu^*} \mathrm{H}^0(\widehat{S};\nu^*\omega_S^{-1}) \longrightarrow \mathrm{H}^0(E;\mathcal{O}_E) \longrightarrow 0$$

This proves the lemma. □

We set $\mathcal{L} := 2\widehat{Q}_\infty + \widehat{S}$. From the following exact sequence

$$0 \longrightarrow \mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(2\widehat{Q}_\infty) \xrightarrow{\iota} \mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\mathcal{L}) \longrightarrow \mathcal{O}_{\widehat{S}}(\mathcal{L}) \longrightarrow 0,$$

we obtain the following:

$$0 \longrightarrow \iota^*\,\mathrm{H}^0(\widehat{\mathbb{Q}_0^3},\mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\mathcal{L})) \longrightarrow \mathrm{H}^0(\widehat{S},\mathcal{O}_{\widehat{S}}(\mathcal{L})) \longrightarrow \mathrm{H}^1(\widehat{\mathbb{Q}_0^3},\mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(2\widehat{Q}_\infty)) \longrightarrow 0 \quad (\#)$$

Claim (3.a.) $\mathrm{H}^0(\widehat{S},\mathcal{O}_{\widehat{S}}(\mathcal{L})) = \mathrm{H}^0(\widehat{S},\mathcal{O}_{\widehat{S}}(\Sigma+3F)) = \mathrm{H}^0(\widehat{S},\nu^*\omega_S^{-1}) = \mathbb{C}^7$.

In fact, one has $\mathcal{L}|_{\widehat{S}} = (2\widehat{Q}_\infty + \widehat{S})|_{\widehat{S}} \sim \Sigma + 3F$. On the other hand, since $\nu^*\omega_S^{-1}$ is ample on $\widehat{S} \cong \mathbb{F}_1$ with $\deg\nu^*\omega_S^{-1} = 5$, one has $\nu^*\omega_S^{-1} \sim \mathcal{O}_{\widehat{S}}(\Sigma+3F)$. Since $\mathrm{H}^1(\widehat{S},\mathcal{O}_{\widehat{S}}(\sigma+3F)) = 0$ for $i > 0$, by Riemann-Roch theorem, we have the claim. □

Claim (3.b.) $\mathrm{H}^1(\widehat{\mathbb{Q}_0^3},\mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(2\widehat{Q}_\infty)) \cong \mathrm{H}^0(E,\mathcal{O}_E)$.

In fact, let us consider the following exact sequences:

$$
\begin{aligned}
&0 \longrightarrow \mathcal{O}_{\widehat{\mathbb{Q}_0^3}} \longrightarrow \mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\widehat{Q}_\infty) \longrightarrow \mathcal{O}_{\widehat{Q}_\infty}(-\widehat{f}) \longrightarrow 0\\
&0 \longrightarrow \mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\widehat{Q}_\infty) \longrightarrow \mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(2\widehat{Q}_\infty) \longrightarrow \mathcal{O}_{\widehat{Q}_\infty}(-2\widehat{f}) \longrightarrow 0\\
&0 \longrightarrow \mathcal{O}_{\widehat{Q}_\infty}(-\widehat{f}) \longrightarrow \mathcal{O}_{\widehat{Q}_\infty} \longrightarrow \mathcal{O}_{\widehat{f}} \longrightarrow 0\\
&0 \longrightarrow \mathcal{O}_{\widehat{Q}_\infty}(-2\widehat{f}) \longrightarrow \mathcal{O}_{\widehat{Q}_\infty}(-\widehat{f}) \longrightarrow \mathcal{O}_{\widehat{f}} \longrightarrow 0
\end{aligned}
$$

Since $\widehat{Q}_\infty \cong \mathbb{P}^1 \times \mathbb{P}^1$, we have

$$\mathrm{H}^1(\widehat{Q}_\infty;\mathcal{O}_{\widehat{Q}_\infty}(-2\widehat{f})) \cong \mathrm{H}^0(\widehat{f};\mathcal{O}_{\widehat{f}}) \overset{(\Phi|_{\widehat{f}})_*}{\cong} \mathrm{H}^0(E;\mathcal{O}_E).$$

On the other hand, since $\mathrm{H}^1(\widehat{\mathbb{Q}_0^3}, \mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\widehat{Q}_\infty)) = 0$, we have

$$\mathrm{H}^1(\widehat{\mathbb{Q}_0^3}, \mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(2\widehat{Q}_\infty) \cong \mathrm{H}^1(\widehat{Q}_\infty; \mathcal{O}_{\widehat{Q}_\infty}(-2\widehat{f})).$$

This proves the claim. □

Thus by (#) ,(3.a) and (3.b) , we obtain the following

Lemma 3.3. $\iota^* \mathrm{H}^0(\widehat{\mathbb{Q}_0^3}, \mathcal{O}_{\widehat{\mathbb{Q}_0^3}}(\mathcal{L})) \cong \nu^* \mathrm{H}^0(S; \omega_S^{-1}) \cong \mathbb{C}^6$

Finally we shall give a basis $\{\nu^* h_0, \nu^* h_1, \ldots, \nu^* h_5\}$ of $\nu^* \mathrm{H}^0(S; \omega_S^{-1}) \cong \mathbb{C}^6$ explicitly below. Let $U_i := \{(u_i, v_i) \in \mathbb{C}^2\}$ $(i = 0, 1, 2, 3)$ be coordinates covering of $\widehat{S} \cong \mathbb{F}_1$ with

$$\begin{cases} u_1 = u_0^{-1} \\ v_1 = u_1 v_0, \end{cases} \quad \begin{cases} v_2 = v_1^{-1} \\ u_2 = u_1, \end{cases} \quad \begin{cases} u_3 = u_2^{-1} \\ v_3 = u_2^{-1} v_2, \end{cases} \quad \begin{cases} v_0 = v_3^{-1} \\ u_0 = u_3, \end{cases}$$

on $U_i \cap U_{i+1} \cong \mathbb{C} \times \mathbb{C}^*$, $(i = 0, 1, 2)$ and $U_3 \cap U_0 \cong \mathbb{C} \times \mathbb{C}^*$ respectively.

Let F, F_∞ (resp. Σ, Σ_∞) be the fibers (resp. sections) defined as follows:

$$F \cap U_i = \{u_i = 0\}\ (i = 0, 3) \text{ and } F_\infty \cap U_i = \{u_i = 0\}\ (i = 1, 2).$$
$$\Sigma \cap U_i = \{v_i = 0\}\ (i = 0, 1) \text{ and } \Sigma_\infty \cap U_i = \{v_i = 0\}\ (i = 2, 3).$$

Taking into account that $\nu^* \mathrm{H}^0(S; \omega_S^{-1}) \subset \mathrm{H}^0(\widehat{S} : \mathcal{O}(\Sigma + 3F))$, we may assume that $\nu^* h_i$'s are given by

$$\begin{cases} \nu^* h_0 & = u_0^3 v_0 = v_1 = 1 = u_3^3 \\ \nu^* h_1 & = u_0^2 v_0 = u_1 v_1 = u_2 = u_3^2 \\ \nu^* h_2 & = u_0 v_0 = u_1^2 v_1 = u_2^2 = u_3 \\ \nu^* h_3 & = v_0 = u_1^3 v_1 = u_2^3 = 1 \\ \nu^* h_4 & = u_0^2 = 1 = v_2 = u_3^2 v_3 \\ \nu^* h_5 & = 1 = u_1^2 = u_2^2 v_2 = v_3 \end{cases}$$

Then the normalization map $\nu = \widehat{\Phi}|_{\widehat{S}} : \widehat{S} \longrightarrow S = V_5^* \cap \{z_6 = 0\} \subset \mathbb{P}^5$ is given by $(\nu^* h_0 : \nu^* h_0 : \nu^* h_1 : \ldots : \nu^* h_5 : 0)$, where $(z_0 : z_1 : \ldots : z_5 : z_6)$ is the homogeneous coordinates of $\mathbb{P}^6$. We set $x_i := \dfrac{z_i}{z_5}$ $(0 \le i \le 4)$. Then we have the local defining equation

$$S^{(0)} := S \cap \{z_5 \neq 0\} \cong \{(x_2, x_3 . x_4) \in \mathbb{C}^3 | x_2^2 = x_3^2 x_4\}$$
$$E^{(0)} := E \cap \{z_5 \neq 0\} \cong \{(x_2, x_3 . x_4) \in \mathbb{C}^3 | x_2 = x_3 = 0\}$$

First we take a smooth rational curve $\Gamma \subset S \subset V_5^* \subset \mathbb{P}^6$ of degree 5 satisfying $\Gamma \cap E = \{t^*\}$ (double points), where $\nu^{-1}(t^*) \cap U_0 = \{(-1,0),(1,0)\}$. Such a rational curve Γ always exists. In fact, take a smooth rational curve $\widehat{\Gamma} \sim \Sigma + 3F$ on $\widehat{S}$ defined by

$$\begin{aligned}
\widehat{\Gamma} \cap U_0 &= \{v_0 = (u_0+1)^2\} \\
\widehat{\Gamma} \cap U_1 &= \{u_1^3 v_1 = (u_1+1)^2\} \\
\widehat{\Gamma} \cap U_2 &= \{u_2^3 = v_2(u_2+1)^2\} \\
\widehat{\Gamma} \cap U_3 &= \{1 = v_3(u_3+1)^2\}
\end{aligned}$$

and set $\Gamma := \nu(\widehat{\Gamma}) \subset S$. The local defining equation of Γ is given by

$$\Gamma^{(0)} := \Gamma \cap U_0 \cong \{x_2 = \frac{1}{2}(x_2 - x_3 - 1)x_3, x_4 = \frac{1}{4}(x_3 - x_4 - 1)^2\}$$

in the affine part $U_0 \cong \mathbb{C}^2$. Take the coordinate transformation below:

$$\theta_2 : \begin{cases} x = x_2 - \frac{1}{2}(x_3 - x_4 - 1)x_3 \\ y = \frac{1}{2}(x_3 - x_4 - 1) \\ z = x_4 - \frac{1}{4}(x_3 - x_4 - 1)^2 , \end{cases}$$

on $\mathbb{C}^3$, then the defining equations can be written as follows:

$$\begin{aligned}
S^{(0)} &= \{(x,y,z) \in \mathbb{C}^3 | x^2 + 2xy\{z + (y+1)^2\} - \{z + (y+1)^2\}z = 0\} \\
E^{(0)} &= \{(x,y,z) \in \mathbb{C}^3 | x = z + (y+1)^2 = 0\} \\
\Gamma^{(0)} &= \{(x,y,z) \in \mathbb{C}^3 | x = z = 0\} \\
t^* &= (0,-1,0)
\end{aligned} \qquad (\#\#)$$

Now, there exist two lines F_1, F_2 in S passing through t^* given by

$$(F_1) : \begin{cases} x = 2(1-y^2) \\ y = 1 - y^2 , \end{cases} \qquad (F_2) : \begin{cases} x = -2(1+y)^2 \\ y = 1 - y^2 \end{cases}$$

Then one sees that $F_1 \cap \Gamma = \{(0,1,0),(0,-1,0) = t^*\}$ and $F_2 \cap \Gamma = \{t^*\}$.

Let $\sigma : \overline{V_5}^* \longrightarrow V_5^*$ be the blowing up with center Γ and $\overline{D} = \mathrm{Exc}(\sigma)$ the exceptional set. Let $\overline{S}$ (resp. $\overline{E}$) be the proper transform of S (resp. E) in $\overline{V}_5^*$ and let $\overline{F}_i$ $(i = 1,2)$ be the proper transform of F_i in $\overline{S}$.

Lemma 3.4. $\mathrm{Sing}\,\overline{S} = \overline{E} \cup \{\overline{t}^*\}$, *where the isolated singular point* $\overline{t}^* \in \overline{S}$ *is the rational double point of* A_1*-type.*

Proof. We look at the local defining equation (##). Let $\sigma_0 : B_{\Gamma^{(0)}}(\mathbb{C}^3) \longrightarrow \mathbb{C}^3$ be the blowing up of $\mathbb{C}^3$ along $\Gamma^{(0)} = \{x = z = 0\}$. Let $\{W_i \cong \mathbb{C}^3_{(u_i,v_i,w_i)}\}$ be a chart of $B_{\Gamma^{(0)}}(\mathbb{C}^3) = W_1 \cup W_2$ with the relation:

$$\begin{cases} x = u_1 = u_2 w_2 \\ y = v_1 = v_2 \\ z = u_1 w_1 = u_2 \\ w_1 w_2 = 1 \end{cases}$$

on $W_1 \cap W_2$. An easy computation shows that the local defining equations of $\overline{S}$ and $\overline{E}$ are given by

$$(\overline{S}) : \begin{cases} u_1 + 2v_1[u_1w_1 + (v_1+1)^2] = w_1[u_1w_1 + (v_1+1)^2]^2 \;\; on \;\; W_1 \\ [u_2 + (v_2+1)^2 - v_2w_2]^2 = w_2^2(u_2 + v_2^2) \;\; on \;\; W_2 \end{cases}$$

and

$$(\overline{E}) : \;\; w_2 = u_2 + (v_2+1)^2 = 0 \;\; on \;\; W_2$$

It is easy to check that

$$\operatorname{Sing} \overline{S} \cap W_2 = \{w_2 = u_2 + (v_2+1)^2 = 0\} \cup \{(u_2, v_2, w_2) = (0, -1, 2) := \overline{t^*}\}.$$

Next we shall show that $\overline{S}$ has an (isolated) rational double point of A_1-type. To prove this, we shall recall the local defining equation of $\overline{S}$ on W_2. The following coordinates transformation

$$\begin{cases} u = \dfrac{u_2 + (v_2+1)^2}{w_2} \\ v = v_2 \\ w = w_2 \end{cases}$$

yields the following:

$$w^2(x-y)^2 = w^2(uw - 2v - 1).$$

Hence the defining equation of $\overline{S}$ near the point $(u, v, w) = (0, -1, 2)$ is given by

$$(u - v - 1)^2 = u(w - 2)$$

This shows that $\overline{S}$ has a rational double point of A_1-type at $\overline{t^*} \in W_1$. This proves the lemma. □

Corollary 3.5. *$\overline{S}|_{\overline{D}} = \overline{\Gamma} + 2\overline{G}_0$, where $\overline{\Gamma}$ is the closure of $\overline{S} \cap \sigma^{-1}(\Gamma - \{t^*\})$ in $\overline{D}$, and $\overline{G}_0$ is a fiber of $\mathbb{P}^1$-bundle $\overline{D} \longrightarrow \Gamma$ over a smooth rational curve Γ.*

Proof. Set $u_2 = 0$ in the above defining equation $(\overline{S})$ in W_2. □

Lemma 3.6. *On the surface $\overline{S}$ one has*

(1) $\overline{F}_1 \cap \overline{G}_0 = \{\overline{t^*}\}$, *in particular* $(\overline{F}_1 \cdot \overline{G}_0) = \frac{1}{2}$

(2) $(\overline{F}_1 \cdot \overline{\Gamma}) = 1$,

(3) $\overline{E} \cap \overline{F}_1 = \emptyset$.

(4) $\overline{t^*} \notin \overline{F}_2$.

Proof. The defining equation of the pull back $F_i^* := \sigma^* F_i$ of F_i is given by

$$(F_1^*) : \begin{cases} u_1 = 2(1 - v_1^2) \\ w_1 = \frac{1}{2} \end{cases} \cup \begin{cases} u_1 = 0 \\ v_1 = -1 \end{cases}$$

$$(F_2^*) : \begin{cases} u_2 = 2(1 - v_1^2) \\ w_2(1 - v_2) + 2(1 + v_2) = 0 \end{cases} \cup \begin{cases} u_2 = 0 \\ v_2 = -1, \end{cases}$$

where

$$\begin{aligned} \overline{G}_0 &= \{u_i = v_i + 1 = 0\}, \\ \overline{F}_1 &= \{u_1 - 2(1 - v_1^2) = w_1 - \frac{1}{2} = 0\} = \{w_2 - 2 = u_2 - 1 + v_2^2 = 0\}, \\ \overline{F}_2 &= \{u_1 + 2(v_1 + 1)^2 = 2w_1(v_1 + 1) + v_1 - 1 = 0\} \\ &= \{u_2 - 1 + v_2^2 = w_2(1 - v_2) + 2(1 + v_2) = 0\}. \end{aligned}$$

Since $\overline{F}_1$ and $\overline{G}_0$ intersects at the rational double point $\overline{t^*} \in \overline{S} - \overline{E}$ of A_1-type, one has $(\overline{F}_1 \cdot \overline{G}_0) = \frac{1}{2}$ via the minimal resolution. The rest follows directly from these defining equations. □

Lemma 3.7. $\overline{D} \cong \mathbb{F}_2$, $N_{\overline{\Gamma}|\overline{V}_5^*} \cong \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6)$.

Proof. Since $\overline{S} \sim \sigma^* S - \overline{D}$, one has $K_{\overline{V}_5^*} = \sigma^* K_{V_5^*} + \overline{D} \sim -2\overline{S} - \overline{D}$. By (3.5) one has $\overline{S}|_{\overline{D}} \sim 2\overline{G}_0 + \overline{\Gamma}$. This implies that

$$K_{\overline{D}} = (K_{\overline{V}_5^*} + \overline{D})|_{\overline{D}} = -2\overline{S}|_{\overline{D}} = -2(\overline{\Gamma} + 2\overline{G}_0).$$

Since $\overline{\Gamma}$ is a smooth rational curve in $\overline{D}$, by the adjunction formula, we obtain $-2 = (K_{\overline{D}} + \overline{\Gamma}) \cdot \overline{\Gamma} = -\overline{\Gamma}^2 - 4$, that is, $\overline{\Gamma}^2 = -2$. Thus we obtain $\overline{D} \cong \mathbb{F}_2$. On the other hand, since $c_1(N_{\overline{\Gamma}|\overline{V}_5^*}) = 10$, we get the normal bundle $N_{\overline{\Gamma}|V_5^*} = \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6)$. □

Lemma 3.8. (1) $(\overline{D} \cdot \overline{E}) = 2$

(2) $(\overline{S} \cdot \overline{E}) = -1$

(3) $(-K_{\overline{V}_5^*} \cdot \overline{E}) = (-K_{\overline{V}_5^*} \cdot \overline{F}_1) = 0$

(4) $(\overline{D})^3 = -8$

(5) $(\overline{S})^3 = -2$

(6) $(\overline{S})^2 \cdot \overline{D} = 2$

(7) $\overline{S} \cdot (\overline{D})^2 = 3$

(8) $(-K_{\overline{V}_5^*})^3 = 18$

Proof. (1): Since Γ is tangent to E at the point t^*, we have the claim.

(2): $(\overline{S} \cdot \overline{E}) = (\sigma^* S - \overline{D}) \cdot \overline{E} = 1 - 2 = -1$.

(3): Since $(-K_{\overline{V}_5^*} \cdot \overline{E}) = (2\sigma^* S - \overline{D}) \cdot \overline{E} = 2 - (\overline{D} \cdot \overline{E}) = 0$, we have the claim. On the other hand, one has

$$\begin{aligned}(-K_{\overline{V}_5^*} \cdot \overline{F}_1) &= (2\sigma^* S - \overline{D}) \cdot \overline{F}_1 = 2 - (\overline{D} \cdot \overline{F}_1) \\ &= 2 - (\overline{D}|_{\overline{S}} \cdot \overline{F}_1)_{\overline{S}} = 2 - (\overline{\Gamma} + 2\overline{G}_0 \cdot \overline{F}_1)_{\overline{S}} \\ &= 2 - (1 + 2(\frac{1}{2})) = 0\end{aligned}$$

(4): $c_1(N_{\overline{\Gamma}|V_5^*}) = (-K_{V_5^*} \cdot \Gamma) - 2 = (2S \cdot \Gamma) - 2 = 10 - 2 = 8$.

(5): $(\overline{S})^3 = (\sigma^* S - \overline{D})^3 = (\sigma^* S)^3 + (\sigma^* D \cdot \overline{D} \cdot \overline{D}) - (\overline{D})^3 = 5 - 15 + 8 = -2$.

(6): $(\overline{S})^2 \cdot \overline{D} = (\sigma^* S - \overline{D})^2 \cdot \overline{D} = -2(\sigma^* S \cdot \overline{D} \cdot \overline{D}) + (\overline{D})^3 = 10 - 8 = 2$.

(7): $\overline{S} \cdot (\overline{D})^2 = (\sigma^* S - \overline{D}) \cdot \overline{D}^2 = -5 + 8 = 3$.

(8): $(-K_{\overline{V}_5^*})^3 = (2\overline{S} + \overline{D})^3 = 18$. □

Proposition 3.9. $\mathrm{Bs}\,|2\sigma^* S - \overline{D}| = \emptyset$.

To prove Proposition (3.9), we need sublemmas. Let $\lambda : \tilde{V}_5^* \longrightarrow \overline{V}_5^*$ be the blowing up with center $\overline{E}$ and $\lambda^{-1}(\overline{E}) := \tilde{L}$ the exceptional divisor. Let $\tilde{S}$ (resp. $\tilde{D}$) be the proper transform of $\overline{S}$ (resp. $\overline{D}$).

Sublemma 3.10. *$\tilde{S}$ has a unique isolated singularity $\tilde{t^*} = \lambda^{-1}(\overline{t}^*)$, which is rational double point of A_1-type.*

Proof. $\mathrm{Sing}\,\overline{S} = \overline{E} \cup \{\overline{t^*}\}$. We have only to show that $\mathrm{Sing}\,\tilde{S} = \{\tilde{t^*}\}$. In fact, in the chart $W_2 \cong \mathbb{C}^3_{(u_2, v_2, w_2)}$, we put

$$\begin{cases} u_2 := u_2 + (v_2 + 1)^2 \\ v_2 := v_2 \\ w_2 := w_2. \end{cases}$$

then the local defining equations of $\overline{S}$ and $\overline{E}$ are written as follows:

$$\begin{aligned}(\overline{S}) :=& \{u_2 - v_2w_2)^2 = w_2^2(u_2 - 2v_2 - 1)\},\\ (\overline{E}) :=& \{u_2 = w_2 = 0\}.\end{aligned}$$

Thus the proper transform $\tilde{S}$ of $\overline{S}$ can be written as follows:

$$(u_2 - v_2)^2 = u_2w_2 - v_2 - 1.$$

This shows that $\tilde{S}$ has only a rational double point of A_1-type as an isolated singularity. We note that $\tilde{S}$ is smooth near $\tilde{L} \cap \tilde{S}$. □

Sublemma 3.11. Bs $|2\sigma^*S - \overline{D}| = \emptyset$ *on* $\overline{S}$.

Proof. Let $\mu : M \longrightarrow \tilde{S}$ be the minimal resolution with $\mathrm{Exc}(\mu) = B$, where B is a smooth rational curve with $B^2 = -2$. We set $\lambda := \lambda|_{\tilde{S}} : \tilde{S} \longrightarrow \overline{S}$ and $\tau := \lambda \circ \mu : M \longrightarrow \overline{S}$. We put $\mathcal{L} := (2\sigma^*S - \overline{D})|_{\overline{S}}$. There exists a $\mathbb{P}^1$-ruling $\psi : M \longrightarrow \mathbb{P}^1$ over $\mathbb{P}^1$ with only one singular fiber $\psi^{-1}(0) = \widehat{F}_1^+ \cup B \cup \widehat{G}_0^+$. We have

$$\mathrm{Pic}\, M \cong \mathbb{Z}[\widehat{E}^+] \oplus \mathbb{Z}[\widehat{G}_0^+] \oplus \mathbb{Z}[\widehat{F}_1^+] \oplus \mathbb{Z}[B],$$

where $\widehat{G}_0^+$, $\widehat{F}_1^+$, $\widehat{\Gamma}^+$, $\widehat{E}^+$ are the proper transforms of $\overline{G}_0$, $\overline{F}_1$, $\overline{\Gamma}$, $\overline{E}$ in M respectively. An easy computation yields that

$$(\widehat{F}_1^+)^2 = (\widehat{G}_0^+)^2 = -1,\ (B)^2 = -2,\ (B \cdot \widehat{F}_1^+) = (B \cdot \widehat{G}_0^+) = 1,\ (\widehat{F}_1^+ \cdot \widehat{G}_0^+) = 0.$$

Now we have a linear equivalence $\mathcal{L} \sim 2\overline{\Theta} - (2\overline{G}_0 + \overline{\Gamma})$, where we may assume that $\overline{\Theta} := \sigma^*S|_{\overline{S}}$ is irreducible. Let $\widehat{\Theta}^+$ be the proper transform of $\overline{\Theta}$. Then we have a linear equivalence $\tau^*\mathcal{L} \sim 2\widehat{\Theta}^+ - 2\widehat{G}_0^+ - B - \widehat{\Gamma}^+$. Since $\widehat{\Theta}^+ \sim \widehat{\Gamma}^+ \sim \widehat{E}^+ + 5\widehat{G}_0^+ + 4B + 3\widehat{F}_1^+$, we have $\tau^*\mathcal{L} \sim \widehat{E}^+ + 3\widehat{G}_0^+ + 3B + 3\widehat{F}_1^+$, and $(\tau^*\mathcal{L} \cdot \widehat{E}^+) = (\tau^*\mathcal{L} \cdot \widehat{F}_1^+) = (\tau^*\mathcal{L} \cdot B) = 0$, $(\tau^*\mathcal{L} \cdot \widehat{G}_0^+) = 1$. This shows that Bs $|\tau^*\mathcal{L}| = \emptyset$ on M, hence Bs $|\mathcal{L}| = \emptyset$ on $\overline{S}$.

Next let us consider the following exact sequence:

$$0 \longrightarrow \mathcal{O}(\sigma^*S) \longrightarrow \mathcal{O}(2\sigma^*S - \overline{D}) \longrightarrow \mathcal{O}_{\overline{S}}(2\sigma^*S - \overline{D}) \longrightarrow 0$$

Since $\mathrm{H}^1(\overline{V}_5^*; \mathcal{O}(\sigma^*S)) = 0$, one has a surjection

$$\mathrm{H}^0(\overline{V}_5^*; \mathcal{O}(2\sigma^*S - \overline{D}) \longrightarrow \mathrm{H}^0(\overline{S}; \mathcal{O}(2\sigma^*S - \overline{D})) \longrightarrow 0$$

Since Bs $|\mathcal{L}| =$ Bs $|\mathcal{O}_{\overline{S}}(2\sigma^*S - \overline{D})| = \emptyset$, we have the claim. □

Sublemma 3.12. $-K_{\overline{V}_5^*} \sim 2\sigma^*S - \overline{D}$ *is nef and big on* $\overline{V}_5^*$.

Proof. Assume that there is a curve R on $\overline{V}_5^*$ with $(2\sigma^*S - \overline{D} \cdot R) < 0$. Since $2\sigma^*S - \overline{D} = \sigma^* + \overline{S}$, one has $(\overline{S} \cdot R) < 0$, that is, $R \subset \overline{S}$. By (3.11), $\mathcal{L}$ is semi-ample on $\overline{S}$, hence $0 \leq (\mathcal{L} \cdot R)_{\overline{S}} = (2\sigma^*S - \overline{D}) \cdot R$, which is a contradiction. Thus $2\sigma^*S - \overline{D}$ is nef. On the other hand, one has $(2\sigma^*S - \overline{D})^3 = 18$ by (3.8)-(8), hence $2\sigma^*S - \overline{D}$ is big. □

We continue the proof of Proposition (3.9).

First we have $h^0(\mathcal{O}_{\overline{V}_5^*}(2) \otimes \mathcal{I}_\Gamma) \geq 17$ by the counting method. In fact, take general 11 points of Γ. Then any smooth quadric hypersurface in $\mathbb{P}^6$ passing through these 11 points always contain Γ, since Γ is of degree 5 in $\mathbb{P}^6$. The general member $\overline{Z}$ of the linear system $|2\sigma^*S - \overline{D}|$ is a smooth K-3 surfaces. We may assume that $|2\sigma^*S - \overline{D}|$ has no fixed component on $\overline{Z}$. Thus the nef big divisor $(2\sigma^*S - \overline{D})|_{\overline{Z}}$ has no base points on $\overline{Z}$. Since $H^1(\overline{V}_5^*; \mathcal{O}_{\overline{V}_5^*}) = 0$, from the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(2\sigma^*S - \overline{D}) \longrightarrow \mathcal{O}_{\overline{Z}}(2\sigma^*S - \overline{D}) \longrightarrow 0,$$

we have an surjection

$$H^0(\overline{V}_5^*; \mathcal{O}(2\sigma^*S - \overline{D})) \longrightarrow H^0(\overline{Z}; \mathcal{O}_{\overline{Z}}(2\sigma^*S - \overline{D})) \longrightarrow 0.$$

Thus $|2\sigma^*S - \overline{D}|$ has no base points on $\overline{Z}$. Since $\overline{Z}$ is general, we have finally $\mathrm{Bs}\,|2\sigma^*S - \overline{D}| = \emptyset$. □

Proposition 3.13. (1) $N_{\overline{E}|\overline{V}_5^*} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$

(2) $N_{\overline{F}_1|\overline{V}_5^*} \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}$

Proof. (1): We have $\tilde{S} \sim \lambda^*\overline{S} - 2\tilde{L}$, $\tilde{D} \sim \lambda^*\overline{D}$ and $K_{\tilde{V}_5^*} \sim \lambda^* K_{\overline{V}_5^*} + \tilde{L} \sim -2\tilde{S} - 3\tilde{L} - \tilde{D}$. We set $\tilde{\Sigma} := \tilde{S}|_{\tilde{L}} \subset \tilde{S} \cap \tilde{L}$, which is a smooth rational curve. Since $c_1(N_{\overline{E}|\overline{V}_5^*}) = -2 + (-K_{\overline{V}_5^*} \cdot \overline{E}) = -2$, one has $(\tilde{L})^3 = -2$. Thus one has

$$\begin{aligned}(\tilde{\Sigma})^2_{\tilde{L}} &= (\tilde{S}|_{\tilde{L}})^2 = (\lambda^*\overline{S} - 2\tilde{L})^2 \cdot \tilde{L} \\ &= -4\lambda^*\overline{S} \cdot (\tilde{L})^2 + 4(\tilde{L})^3 = -4(\overline{S} \cdot \overline{E})(\tilde{T} \cdot \tilde{L}) + 8 = 4,\end{aligned}$$

where $\tilde{T}$ is a general fiber of the $\mathbb{P}^1$-bundle $\tilde{L} \longrightarrow \overline{E}$. On the other hand, since $K_{\tilde{V}_5^*} + \tilde{L} = -2\tilde{S} - 2\tilde{L} - \tilde{D} = -(\lambda^*\sigma^*S + \tilde{S})$, one has

$$K_{\tilde{L}} = -\lambda^*\sigma^*S|_{\tilde{L}} - \tilde{S}|_{\tilde{L}} = -(\sigma^*S \cdot \overline{E})\tilde{T} - \tilde{\Sigma} = -\tilde{T} - \tilde{\Sigma}.$$

In particular, we have $(\tilde{T} \cdot \tilde{\Sigma}) = -\tilde{T} \cdot (K_{\tilde{L}} + \tilde{T}) = 2$.

Let $\tilde{\Delta}_0$ be the negative section of the $\mathbb{P}^1$-bundle $\tilde{L} \longrightarrow \overline{E}$. Then we can write as $\tilde{\Sigma} \sim 2\tilde{\Delta}_0 + a\tilde{T}$ for some $0 < a \in \mathbb{Z}$. Since $(\tilde{\Sigma})^2_{\tilde{L}} = 4$, one has $\tilde{\Delta}_0^2 + a = 1$. From the relation $(\tilde{\Sigma} \cdot \tilde{\Delta}_0) = 2\tilde{\Delta}_0^2 + a \geq 0$ and $0 \geq \tilde{\Delta}_0^2 = 1 - a$, one has $a = 1, 2$, hence

$$\tilde{L} \cong \begin{cases} \mathbb{P}^1 \times \mathbb{P}^1 & \text{if } a = 1 \\ \mathbb{F}_1 & \text{if } a = 2 \end{cases}$$

Since $c_1(N_{\overline{E}|\overline{V}_5^*}) \cong \mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(n)$, $(m \leq n \in \mathbb{Z})$ with $m+n = -2$, $\tilde{L} \cong \mathbb{F}_{n-m} \cong \mathbb{F}_0 := \mathbb{P}^1 \times \mathbb{P}^1$, hence $m = n = -1$ and $a = 1$. This proves the (1).

(2): Now $\overline{F}_1$ passes through the rational double point (of A_1-type) $\bar{t}^* \in \overline{S} - \overline{E} \subset \overline{V}_5^* - \{p^*\}$ and $(-K_{\overline{V}_5^*} \cdot \overline{F}_1) = 0$. Thus by the similar arguments in [8], we have the claim. □

Let $\tilde{F}_1$ (resp. $\tilde{G}_0$, $\tilde{\Gamma}$) be the proper transform of $\overline{F}_1$ (resp. $\overline{G}_0$, $\overline{\Gamma}$). Then we have the following

Corollary 3.14. (1) $(\tilde{\Sigma}^2)_{\tilde{S}} = -3$

(2) $(\tilde{F}_1^2)_{\tilde{S}} = (\tilde{G}_0^2)_{\tilde{S}} = -\frac{1}{2}$

(3) $(\tilde{F}_1 \cdot \tilde{G}_0)_{\tilde{S}} = \frac{1}{2}$

(4) $\tilde{D}|_{\tilde{S}} \sim \tilde{\Gamma} + 2\tilde{G}_0$

Proof. We have $(\tilde{\Sigma}^2)_{\tilde{S}} = (\tilde{L}|_{\tilde{S}} \cdot \tilde{L}|_{\tilde{S}})_{\tilde{S}} = (\tilde{L}^2 \cdot \tilde{S}) = \lambda^*\overline{S} - 2\tilde{L}(\tilde{L})^2 = -(\overline{S} \cdot \overline{E}) - 2(\tilde{L})^3 = 1 - 4 = -3$. This shows (1). By construction there is a $\mathbb{P}^1$-ruling $\tilde{\psi} : \tilde{S} \longrightarrow \mathbb{P}^1$ which has a unique singular fiber $\tilde{\psi}^{-1}(0) := \tilde{F}_1 \cup \tilde{G}_0$. Since $\operatorname{Sing} \tilde{S} = \tilde{t}^* = \tilde{F}_1 \cap \tilde{G}_0$ is the rational bouble point of A_1-type, one has the claims (2) and (3) via minimal resolution. The claim (4) follows from the fact $\tilde{D} \cap \tilde{S} = \tilde{\Gamma} \cup \tilde{G}_0$.

Corollary 3.15. *There is a birational contraction $\varphi : \tilde{V}_5^* \longrightarrow \overline{V}_5^+$ of $\tilde{L}$ to a smooth rational curve $\overline{E}^+ = \varphi(\tilde{L})$.*

Proof. Since $K_{\tilde{V}_5^*} \sim -2\tilde{S} - 3\tilde{L} - \tilde{D}$, we obtain that

$$\begin{aligned} &-2\tilde{\Delta}_0 - 2\tilde{T} \sim K_{\tilde{L}} \sim (K_{\tilde{V}_5^*} + \tilde{L})|_{\tilde{L}} = -2\tilde{S}|_{\tilde{L}} - 2\tilde{L}|_{\tilde{L}} - 2\tilde{T} \\ &= -4\tilde{\Delta}_0 - 4\tilde{T} - 2\tilde{L}|_{\tilde{L}} - \tilde{D}|_{\tilde{L}} = -2\tilde{\Sigma} - 2\tilde{L}|_{\tilde{L}} \end{aligned}$$

This yields $\tilde{L}|_{\tilde{L}} \sim -\tilde{\Delta}_0 - \tilde{T}$. Thus $\tilde{L} \cong \mathbb{P}^1 \times \mathbb{P}^1$ contracts to another direction, that is, $\tilde{\Delta}_0$-direction. This contraction morphism φ is desired one. □

We set $\overline{S}^+ = \varphi(\tilde{S})$, $\overline{D}^+ = \varphi(\tilde{D})$, $\overline{E}^+ := \varphi(\tilde{L})$, $\overline{F}_1^+ = \varphi(\tilde{F}_1)$ and $\overline{G}_0^+ := \varphi(\tilde{G}_0)$. We set $\chi_1 := \varphi \circ \lambda^{-1} : \overline{V}_5^* \dashrightarrow \overline{V}_5^+$, which is a birational map with $\overline{V}_5^* - \overline{E} \overset{\chi_1}{\cong} \overline{V}_5^+ - \overline{E}^+$. On the other hand we have easily $\tilde{S} \overset{\varphi}{\cong} \overline{S}^+$. Thus we have

Lemma 3.16. (1) $(K_{\overline{V}_5^+} \cdot \overline{E}^+) = (K_{\overline{V}_5^+} \cdot \overline{F}^+) = 0$.

(2) $K_{\overline{V}_5^+} \sim -2\overline{S}^+ - \overline{D}^+$.

(3) $(\overline{S}^+ \cdot \overline{E}^+) = 1$.

(4) $(\overline{D}^+ \cdot \overline{E}^+) = -2$, *hence* $\overline{E}^+ \subset \overline{D}^+$.

(5) $\mathrm{mult}_{\overline{E}^+} \overline{D}^+ = 2$.

(6) $\overline{D}^+|_{\overline{S}^+} = 2\overline{E}^+ + 2\overline{G}_0^+ + \overline{\Gamma}^+$, *where* $\overline{\Gamma}^+ = \overline{E}^+ + 5\overline{G}_0^+ + 3\overline{F}_1^+$ *in* $\overline{S}^+$.

(7) $\overline{S}^+|_{\overline{S}^+} = -(\overline{E}^+ + 2\overline{G}_0^+)$

(8) $(\overline{G}_0^+)^2_{\overline{S}^+} = (\overline{F}_1^+)^2_{\overline{S}^+} = -\frac{1}{2}$, $(\overline{G}_0^+ \cdot \overline{F}_1^+)_{\overline{S}^+} = \frac{1}{2}$, $(\overline{G}_0^+ \cdot \overline{E}^+) = 1$, $(\overline{E}^+)^2_{\overline{S}^+} = -3$.

(9) $(\overline{S}^+ \cdot \overline{F}_1^+) = -1$, $(\overline{D}^+ \cdot \overline{F}_1^+) = 2$.

Proof. (1): This follows from (3.8)-(3). (2): This follows from the fact that $K_{\overline{V}_5^*} \sim -2\overline{S} - \overline{D}$. (3): $(\overline{S}^+ \cdot \overline{E}^+) = -(\overline{S} \cdot \overline{E}) = 1$ by (3.8)-(2). (4): $0 = (-K_{\overline{V}_5^+} \cdot \overline{E}^+) = (\overline{S}^+ \cdot \overline{E}^+) + (\overline{D}^+ \cdot \overline{E}^+) = 2 + (\overline{D}^+ \cdot \overline{E}^+)$.

(5): $\varphi^* K_{\overline{V}_5^+} = K_{\tilde{V}_5^*} - \tilde{L} = -2\tilde{S} - 4\tilde{L} - \tilde{D}$. On the other hand, since $K_{\overline{V}_5^+} = -2\overline{S}^+ - \overline{D}^+$ and $\varphi^*\overline{S}^* = \tilde{S} + \tilde{L}$, one has $\varphi^*\overline{D}^+ = -2\varphi^*\overline{S}^+ - \varphi^* K_{\overline{V}_5^+} = \tilde{D} + 2\tilde{L}$. This shows the claim.

(6),(7),(8): The first part of (6) follows from (5), (3.14)-(4) and the fact that $\overline{S}^+$ is smooth along $\overline{E}^+$. Now let $\mu : M \longrightarrow \overline{S}^+$ be the minimal resolution with the exceptional set $B := \mu^{-1}(\overline{t}^+)$, where $\overline{t}^+ := \varphi(\tilde{t}^*)$. Since $\overline{S}^+ \overset{\varphi}{\cong} \tilde{S}$, M can be considered as the same one as in (3.11), via, $M \overset{\mu}{\longrightarrow} \overline{S}^+ \overset{\varphi}{\cong} \tilde{S}$. So we use the same notation as in (3.11). Let $\widehat{F}_1^+$, $\widehat{G}_0^+$, $\widehat{E}^+$, $\widehat{\Gamma}^+$ be the proper transforms of $\overline{F}_1^+$, $\overline{G}_0^+$, $\overline{E}^+$, $\overline{\Gamma}^+$ in M respectively. Then M is a ruled surface over $\mathbb{P}^1$ with only one singular fiber $\widehat{F}_1^+ \cup B \cup \widehat{G}_0^+$ and has $\mathrm{Pic}\, M \cong \mathbb{Z}[\widehat{E}^+] \oplus \mathbb{Z}[\widehat{G}_0^+] \oplus \mathbb{Z}[\widehat{F}_1^+] \oplus \mathbb{Z}[B]$, where

$$(\widehat{F}_1^+)^2 = (\widehat{G}_0^+)^2 = -1,\ (B)^2 = -2,\ (B \cdot \widehat{F}_1^+) = (B \cdot \widehat{G}_0^+) = 1,\ (\widehat{F}_1^+ \cdot \widehat{G}_0^+) = 0.$$

Since $\tilde{S} \overset{\varphi}{\cong} \overline{S}^+$, we have also $(\overline{E}^+)^2_{\overline{S}^+} = -3$ by (3.14)-(1). Moreover, taking into an account that $\mu^*\overline{F}_1^+ = \widehat{F}_1^+ + \frac{1}{2}B$, $\mu^*\overline{G}_0^+ = \widehat{G}_0^+ + \frac{1}{2}B$, one has the claim (8). Next, since $\mu^* K_{\overline{S}^+} = K_M \sim -2\widehat{E}^+ - 5\widehat{G}_0^+ - 4C - 3\widehat{F}_1^+$, one has $K_{\overline{S}^+} \sim -2\overline{E}^+ - 5\overline{G}_0^+ - 3\overline{F}_1^+$. On the other hand, one has easily that $K_{\overline{S}^+} = (K_{\overline{V}_5^+} + \overline{S}^+)|_{\overline{S}^+} = -(\overline{S}^+|_{\overline{S}^+} + \overline{D}^+|_{\overline{S}^+})$. Since $\overline{D}^+|_{\overline{S}^+} \sim 2\overline{E}^+ + 2\overline{G}_0^+ + \overline{\Gamma}^+$, we have $\overline{S}^+|_{\overline{S}^+} \sim 3\overline{G}_0^+ + 3\overline{F}_1^+ - \overline{\Gamma}^+$. From the relation $\widehat{\Gamma}^+ = \widehat{E}^+ + 5\widehat{G}_0^+ + 4B + 3\widehat{F}_1^+$, one gets $\overline{\Gamma}^+ \sim \overline{E}^+ + 5\overline{G}_0^+ + 3\overline{F}_1^+$. This proves the second part of (6) and (7).

(9): One has

$$(\overline{S}^+ \cdot \overline{F}_1^+) = (\overline{S}^+|_{\overline{S}^+} \cdot \overline{F}_1^+) = -(\overline{E}^+ + 2\overline{G}_0^+) \cdot \overline{F}_1^+ = -2(\overline{G}_0^+ \cdot \overline{F}_1^+) = -1$$

and

$$(\overline{D}^+ \cdot \overline{F}_1^+) = (\overline{D}^+|_{\overline{S}^+} \cdot \overline{F}_1^+) = (3\overline{E}^+ + 7\overline{G}_0^+ + 3\overline{F}_1^+) \cdot \overline{F}_1^+ = \frac{7}{2} - \frac{3}{2} = 2.$$

□

Lemma 3.17. (1) $(\overline{S}^+)^3 = -1$.

(2) $(\overline{D}^+)^2 \cdot \overline{S}^+ = 7$.

(3) $(\overline{S}^+)^2 \cdot \overline{D}^+ = 0$.

(4) $(\overline{D}^+)^3 = -16$.

(5) $(-K_{\overline{V}_5^+})^3 = 18$.

Proof. (1):$(\overline{S}^+)^3 = (\overline{E}^+ + 2\overline{G}_0^+)^2 = -1$ by (3.16)-(8).
(2):$(\overline{D}^+)^2 \cdot \overline{S}^+ = (2\overline{E}^+ + 2\overline{G}_0^+ + \overline{\Gamma}^+)^2 = 7$ by (3.16)-(7),-(8).
(3):$(\overline{S}^+)^2 \cdot \overline{D}^+ = (\overline{D}^+|_{\overline{S}^+} \cdot \overline{S}^+|_{\overline{S}^+}) = -(3\overline{E}^+ + 7\overline{G}_0^+ + 3\overline{F}_1^+)(\overline{E}^+ + \overline{G}_0^+) = 0$.
(4),(5): Since

$$18 = (-K_{\overline{V}_5^*})^3 = (-K_{\overline{V}_5^+})^3 = 8(\overline{S}^+)^3 + 12(\overline{S}^+)^2 \cdot \overline{D}^+ + 6\overline{S}^+ \cdot (\overline{D}^+)^2 + (\overline{D}^+)^3,$$

we have $(\overline{D}^+)^3 = -16$.

□

By an argument similar to (3.13)-(2), we obtain

Lemma 3.18. $N_{\overline{F}^+|\overline{V}_5^+} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$.

Thus we have a birational map $\chi_2 : \overline{V}_5^+ \dashrightarrow \overline{V}_5^{++}$ and a smooth rational curve $\overline{F}_1^{++}$ such that

(1) $\chi_2 : \overline{V}_5^+ - \overline{F}_1^+ \cong \overline{V}_5^{++} - \overline{F}_1^{++}$

(2) $\overline{V}_5^{++}$ is smooth near the curve $\overline{F}_1^{++}$

The birational map χ_2 is called the flop (or $\overline{D}^+$-flop, [9]) along $\overline{F}_1^+$. Let S^{++}, D^{++}, be the proper transform of $\overline{S}^+$, $\overline{D}^+$ in V_5^{++}. We set $\chi_2(\overline{F}_1^+) = F_1^{++}$, $\chi_2(\overline{E}^+) = E^{++}$. Then one can see that S^{++} is smooth and $S^{++} \cong \mathbb{F}_3$ (see Pagoda in [8], [12]). By (3.16), taking an account of elementary properties of the flop , we obtain the following

Lemma 3.19. (1) $K_{V_5^{++}} = -2S^{++} - D^{++}$.

(2) $D^{++}|_{S^{++}} = 3E^{++} + 7G_0^{++}$.

(3) $S^{++}|_{S^{++}} = -E^{++} - 2G_0^{++}$.

(4) $(S^{++} \cdot F_1^{++}) = 1$.

(5) $(D^{++} \cdot F_1^{++}) = -2$.

(6) $(S^{++} \cdot E^{++}) = 1$.

(7) $(S^{++} \cdot F_1^{++}) = 1$, *in particular,* S^{++} *intersects with* F_1^{++} *transversally at one point.*

We put $\chi := \chi_2 \circ \chi_1$. Then we obtain finally the following:

Proposition 3.20. *There is a projective threefold* V_5^{++} *and disjoint smooth rational curves* E^{++}, F_1^{++} *and a birational map* $\chi : V_5^* \dashrightarrow V_5^{++}$ *such that* $V_5^* - (\overline{F}_1 \cup \overline{E}) \overset{\chi}{\cong} V_5^{++} - (F_1^{++} \cup E^{++})$, *the birational map* χ *is called the* $\overline{D}$-*flop along* $\overline{E}$, $\overline{F}_1$.

Corollary 3.21. $\mathrm{Bs}\,|D^{++} + 2S^{++}| = \emptyset$.

Proof. By (3.12), $\mathrm{Bs}\,|-K_{\overline{V}_5^*}| = |\overline{D} + 2\overline{S}| = \emptyset$. Since $\chi_*(-K_{\overline{V}_5^*}) = -K_{V_5^{++}} \sim D^{++} + 2S^{++}$. Thus we have the claim. □

Lemma 3.22. (1) $(S^{++})^3 = 1$.

(2) $(D^{++})^2 \cdot S^{++} = 15$.

(3) $(S^{++})^2 \cdot D^{++} = -4$.

(4) $(K_{V_5^{++}})^3 = -18$.

(5) $(D^{++})^3 = -32$.

Proof. (1): $(S^{++})^3 = (S^{++}|_{S^{++}})^2 = (-E^{++} - 2G_0^{++})^2 = 1$. (2): $(D^{++})^2 \cdot S^{++} = (D^{++}|_{S^{++}})^2 = (3E^{++} + 7G_0^{++})^2 = 15$.

(3): $(S^{++})^2 \cdot D^{++} = (S^{++}|_{S^{++}} \cdot D^{++}|_{S^{++}}) = -(E^{++} + 2G_0^{++})(3E^{++} + 7G_0^{++}) = -4$. (4): $(K_{V_5^{++}})^3 = (K_{\overline{V}_5^+})^3 = -18$ by (3.17)-(5). (5): $18 = (-K_{V_5^{++}})^3 = (D^{++} + 2S^{++})^2 = 50 + (D^{++})^3$. □

Proposition 3.23. *There is a birational contraction* $\Psi : V_5^{++} \longrightarrow V_{22}^*$ *with* $\mathrm{Exc}(\Psi) = S^{++}$ *such that*

(1) V_{22}^* *is smooth near* $\ell := \Psi(S^{++}) \cong \mathbb{P}^1$,

(2) $(-K_{V_{22}^*} \cdot \ell) = 1$,

(3) $\mathrm{mult}_\ell\, D^* = 3$,

(4) $-K_{V_{22}^*} = -D^*$, *where* $D^* := \Psi_* D^{++}$,

(5) $(-K_{V_{22}^*})^3 = 22$.

In particular $\Psi : V_5^{++} \longrightarrow V_{22}^*$ *is a blowing up of* V_{22}^* *with the center* ℓ.

Proof. First we recall that E^{++} (resp. G_0^{++}) is the negative section (resp. a fiber) of the Hirzebruch surface $S^{++} \cong \mathbb{F}_3$. By (3.19)-(3), we have $(S^{++} \cdot G_0^{++}) = -1$. Thus we have the contraction $\Psi : V_5^{++} \longrightarrow V_{22}^*$ of S^{++} in the fiber direction. In particular, V_{22}^* is smooth neat $\ell := \Phi(S^{++})$. Since

$$-2S^{++} - D^{++} = K_{V_5^{++}} = \Phi^* K_{V_{22}^*} + S^{++} = -\Phi^* D^* + S^{++},$$

one has $\Phi^* D^* = D^{++} + 3S^{++}$, which says $\operatorname{mult}_\ell D^* = 3$. We also have $(-K_{V_{22}^*})^3 = (\Phi^* D^*)^3 = (D^{++} + 3S^{++})^3 = 22$. Since

$$\begin{aligned}-1 &= (D^{++} + 3S^{++}) \cdot (S^{++})^2 = \Phi^* D^* \cdot (S^{++})^2 = (D^* \cdot \ell) \cdot (S^{++} \cdot G_0^{++}) \\ &= -(D^* \cdot \ell),\end{aligned}$$

we obtain $(-K_{V_{22}^*} \cdot \ell) = -(D^* \cdot \ell) = 1$. □

Lemma 3.24. (1) $\dim \mathrm{H}^0(V_5^{++}; \mathcal{O}(D^{++} + 3S^{++})) = 14$

(2) $\operatorname{Bs}|D^{++} + 3S^{++}| = \emptyset$. *In fact, the birational morphism* $\Psi : V_5^{++} \longrightarrow V_{22}^*$ *is given by the linear system* $|D^{++} + 3S^{++}|$.

Proof. (1): First we will show that $D^{++} + 3S^{++}$ is nef and big. In fact, assume that there is a curve R such that $(D^{++} + 3S^{++}) \cdot R < 0$. Since $D^{++} + 2S^{++}$ is semi-ample, one has $(S^{++} \cdot R) < 0$, that is, $R \subset S^{++}$. On the other hand, since $(D^{++} + 3S^{++})|_{S^{++}} = G_0^{++}$ on $S^{++} \cong \mathbb{F}_3$, one has $(S^{++} \cdot R) = (G_0^{++} \cdot R) \geq 0$, which is a contradiction. Hence $D^{++} + 3S^{++}$ is nef. Since $(D^{++} + 3S^{++}) = 22$, one sees that it is big.

Take a general member $Z^{++} \in |D^{++} + 2S^{++}|$, which is a smooth K-3 surface. Consider the following exact sequence:

$$0 \longrightarrow \mathcal{O}(S^{++}) \longrightarrow \mathcal{O}(D^{++} + 3S^{++}) \longrightarrow \mathcal{O}_{Z^{++}}(D^{++} + 3S^{++}) \longrightarrow 0$$

Since $\mathrm{h}^0(\mathcal{O}_{V_5^{++}}(S^{++})) = 1$, $\mathrm{h}^1(\mathcal{O}_{V_5^{++}}(S^{++})) = 0$, one has

$$\mathrm{h}^0(\mathcal{O}_{V_5^{++}}(D^{++} + 3S^{++})) = 1 + \mathrm{h}^0(\mathcal{O}_{Z^{++}}(D^{++} + 3S^{++})).$$

Now since $D^{++} + 3S^{++}$ is nef big on Z^{++}, $\mathrm{h}^i(\mathcal{O}_{Z^{++}}(D^{++} + 3S^{++})) = 0$ for $i > 0$. The Riemann-Roch theorem says

$$\mathrm{h}^0(\mathcal{O}_{Z^{++}}(D^{++} + 3S^{++})) = \frac{1}{2}(D^{++} + 3S^{++})^2(D^{++} + 2S^{++}) + 2 = 13,$$

hence $\mathrm{h}^0(\mathcal{O}_{V_5^{++}}(D^{++} + 3S^{++})) = 14$.

(2): Since $D^{++} + 2S^{++}$ is semi ample, one can take a general member $Z^{++} \in |D^{++} + 2S^{++}|$ such that Z^{++} contains no fixed component of $|D^{++} + 2S^{++}|$. Thus the nef and big line bundle $\mathcal{O}_{Z^{++}}(D^{++} + 3S^{++})$ has no fixed component on Z^{++}, hence has no basepoints. A surjection

$$\mathrm{H}^0(\mathcal{O}(D^{++} + 3S^{++})) \longrightarrow \mathrm{H}^0(\mathcal{O}_{Z^{++}}(D^{++} + 3S^{++})) \longrightarrow 0$$

implies the claim. □

Thus we have finally the following

Theorem 3.25. *There exists the Fano threefold* $V_{22}^* \subset \mathbb{P}^{13}$ *of index* $r = 1$ *and the genus* $g = 12$ *with a small Gorenstein singularity* p *of* A_1*-type. In particular,* $\mathrm{b}_2(V_{22}^*) = 1$ *and* $\mathrm{b}_2(V_{22}^*) = 2$.

Remark 3.1. $\operatorname{Sing} V_{22}^* = p := \Psi(\chi(\sigma^{-1}(p))$. We set $\Delta_{22} := \Psi(\chi(\sigma^{-1}(\Delta_5)))$ and $W := V_{22}^* - \Delta_{22}$, which is an smooth affine threefold. By construction one sees that $V_5 - \Delta_5 - F_1 \cong V_{22}^* - \Delta_{22} - F_1^{++}$. In particular, $W \supset \mathbb{C}^2 \times \mathbb{C}^*$ and W has a decomposition $W = (\mathbb{C}^2 \times \mathbb{C}^*) \dot{\cup} \mathbb{C}^2$ (disjoint union).

Conjecture 2. $W \cong \mathbb{C}^3$.

References

[1] M. Furushima, *Singular del Pezzo surfaces and analytic compactifications of 3-dimensional complex affine space* $\mathbb{C}^3$, Nagoya Math. J.,**104** (1986),1–28.

[2] M. Furushima, *The complete classification of compactifications of* $\mathbb{C}^3$ *which are projective manifoldss with second Betti number one*, Math. Ann.,**297** (1993),627–662.

[3] M. Furushima, *Non-projective compactifications of* $\mathbb{C}^3$ *(I)*, Kyushu J. Math.,**50** (1996),221–239.

[4] M. Furushima, *Non-projective compactifications of* $\mathbb{C}^3$ *(II)(New Examples)*, Kyushu J. Math.,**52** (1998),149–162.

[5] M. Furushima, *Non-projective compactifications of* $\mathbb{C}^3$ *III: A remark on indices*, Hiroshima Math. J.,**29** (1999),295–298.

[6] M. Furushima, *Birational construction of projective compactifications of* $\mathbb{C}^3$ *with second Betti number equal to one*, Ann.di Mat. pura ed appl.,**178** (2000),115–128.

[7] M. Furushima, *Singular Fano compactifications of* $\mathbb{C}^3$ *(I)*, Math. Z.,**248** (2004),709–723.

[8] M. Furushima and N. Nakayama, *New construction of a compactification of* $\mathbb{C}^3$, Tohoku Math. J.,**41** (1989),543–560.

[9] J. Kollãr, *Flips, Flops, Minimal models, etc.*, Survey in Differential Geometry,**1** (1991),113–199.

[10] S. Mori, *Threefolds whose canonical bundles are not numerically effective*, Ann. Math.,**116** (1982),133–176.

[11] S. Mukai, *New development in the theory of Fano manifold : vector bundle method and moduli problems*, Sugaku Exposition,**15** (2002),125–150.

[12] M. Reid, *Minimal model of canonical 3-folds*, Algebraic Varaieties and Analytic Varieties (Advanced Studies in pure Math.). Ed. S. Iitaka, Kinokuniya, Japan , 1 (1983),131–180.

Mikio Furushima
Department of Mathematics
Faculty of Science
Kumamoto University
Kumamoto 860-8555, Japan
e-mail: wagami@math.sci.kumamoto-u.ac.jp