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Relative norms of Gauss sums for characters of 2-power order

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Abstract. Let p be a prime number such that $p \equiv 9 \pmod{16}$ and t a natural number bigger than 3. We evaluate the relative norms of Gauss sums for characters of order $2^{t-s}, 0 \leq s \leq t-1$, on the finite field F_{p^f}, $f = 2^{t-3}$.

1. Introduction

In this note we treate to evaluate the relative norms of a special class of Gauss sums over finite fields. Let p be an odd prime number. Let m be a natural number bigger than 1 and $f = \mathrm{ord}_m p$ the multiplicative order of p modulo m. For a multiplicative character χ of order m on the finite field $F_q, q = p^f$, extended with $\chi(0) = 0$, the Gauss sums $G_f(\chi^r), 1 \leq r \leq m-1$, are defined by

$$G_f(\chi^r) = \sum_{\alpha \in F_q} \chi^r(\alpha)\zeta_p^{\mathrm{Tr}(\alpha)}$$

where $\zeta_p = e^{2\pi i/p}$ and Tr is the trace map from F_q onto F_p.

In general it is difficult to evaluate explicitly Gauss sums over finite fields, and only in the special cases the values of them are known. When $m = 2$, it is a classical theorem of Gauss that

$$G(\chi) = \sum_{a=1}^{p-1}\chi(a)\zeta_p^a = \begin{cases} \sqrt{p}, & p \equiv 1 \pmod 4, \\ i\sqrt{p}, & p \equiv 3 \pmod 4. \end{cases}$$

If $m = 4$ and $p \equiv 1 \pmod 4$, then $f = 1$ and χ is a quartic character on F_p (cf. [1, Chap.4]). If $m = 4$ and $p \equiv 3 \pmod 4$, then $f = 2$ and we have

$$G_2(\chi) = \sum_{\alpha \in F_{p^2}} \chi(\alpha)\zeta_p^{\mathrm{Tr}\alpha} = (-1)^{\frac{p+1}{4}} p$$

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by Stickelberger's theorem [1, Chap.11], since $-1 \in <p>$, the subgroup generated by p in the multiplicative group $(Z/4Z)^\times$. In the case $m = 2^t$ with $t \geq 3$ and $p \equiv 3$ or $5 \pmod 8$, the evaluations of $G_f(\chi^{2^s}), 0 \leq s \leq t-1$, have been recently given by Meijer and van der Vlught [4].

We consider the case $m = 2^t$ with $t \geq 4, p \equiv 9 \pmod{16}$ and hence $f = 2^{t-3}$. We see that $G_f(\chi)$ is in the $8p$-th cyclotomic number field K_1. So, we treate to evaluate the relative norms of $G_f(\chi^{2^s}), 0 \leq s \leq t-1$, from K_1 to its subfield K_0, the $4p$-th cyclotomic number field.

2. Preminaries

Let k be the m-th cyclotomic number field; then k has $k_0 = Q(i)$ and $k_1 = Q(i, \sqrt{2})$ as subfields, Q being the field of rational numbers. Let $K = k(\zeta_p)$, which has $K_i = k_i(\zeta_p), i = 0, 1$, as subfields. We see that

$$[k : k_1] = [K : K_1] = f = \frac{m}{8},$$
$$[k : k_0] = [K : K_0] = 2f = \frac{m}{4}.$$

Let $G \cong (Z/mZ)^\times$ be the Galois group of k over Q; for a rational integer a prime to m, let σ_a denote the automorphism of k sending ζ_m to ζ_m^a where $\zeta_m = e^{2\pi i/m}$. The order of σ_p is equal to f. We write σ simply for σ_5; then the order of σ is equal to $2f$, since $<5^2> = <p> \subset (Z/mZ)^\times$, and k_0 and k_1 correspond to the subgroups $<\sigma>$ and $<\sigma^2> = <\sigma_p>$ of G, respectively.

Since $p \equiv 1 \pmod 8), (p)$ splits completely in k_1, and hence so in k_0. Let $\mathfrak{p}_0 = (a + bi)$, where a and b are rational integers such that $p = a^2 + b^2$, be a prime divisor of (p) in k_0, and further let $\mathfrak{p}$ and $\mathfrak{p}^\sigma$ be a prime divisor of $\mathfrak{p}_0$ in k_1 and its conjugate by σ, respectively: $(p) = \mathfrak{p}_0 \overline{\mathfrak{p}}_0$ in k_0 and $\mathfrak{p}_0 = \mathfrak{p}\,\mathfrak{p}^\sigma, \overline{\mathfrak{p}}_0 = \overline{\mathfrak{p}}\,\overline{\mathfrak{p}}^\sigma$ in k_1, the bar indicating the complex conjugate. All of these prime divisors of (p) in k_1 remain prime in k, and may be considered as prime ideals of k, and all prime divisors of (p) in k_i are totally ramified in $K_i, i = 0, 1$: $\mathfrak{p}_0 = \mathfrak{P}_0^{p-1}$ in K_0 and $\mathfrak{p} = \mathfrak{P}^{p-1}$ in K_1 etc.

We identify F_q with the residue class field $O_k/\mathfrak{p}$, O_k being the integer ring of k, and define a multiplicative character χ of order m on F_q by

$$\chi(\alpha) \bmod \mathfrak{p} = \alpha^{\frac{q-1}{m}}, \qquad \alpha \in F_q. \tag{1}$$

Let χ^* denote the restriction of χ on F_p; then, since the multiplicative groups of finite fields are cyclic, the order of χ^* is equal to $m/(m, (q-1)/(p-1)) = 8$. For $1 \leq r \leq m-1$ we have

$$G_f(\chi^r) = \sum_{a=0}^{p-1}\Big(\sum_{\substack{\alpha\in F_q\\ \mathrm{Tr}\alpha=a}} \chi^r(\alpha)\Big)\zeta_p^a = \sum_{a=1}^{p-1}\chi^{*r}(a)\zeta_p^a \sum_{\substack{\alpha\in F_q\\ \mathrm{Tr}\alpha=1}} \chi^r(\alpha) + \sum_{\substack{\alpha\in F_q\\ \mathrm{Tr}\alpha=0}} \chi^r(\alpha)$$

and

$$\sum_{\substack{\alpha\in F_q\\ \mathrm{Tr}\alpha=0}} \chi^r(\alpha) = -\sum_{a=1}^{p-1}\sum_{\substack{\alpha\in F_q\\ \mathrm{Tr}\alpha=a}} \chi^r(\alpha) = -\sum_{a=1}^{p-1}\chi^{*r}(a)\sum_{\substack{\alpha\in F_q\\ \mathrm{Tr}\alpha=1}} \chi^r(\alpha)$$

$$= \begin{cases} 0, & \chi^{*r}\neq 1,\\ -(p-1)\sum\limits_{\substack{\alpha\in F_q\\ \mathrm{Tr}\alpha=1}}\chi^r(\alpha), & \chi^{*r}=1,\end{cases}$$

which implies

$$G_f(\chi^r) = \begin{cases} G(\chi^{*r})E_f(\chi^r), & \chi^{*r}\neq 1,\\ -pE_f(\chi^r), & \chi^{*r}=1,\end{cases} \tag{2}$$

where

$$E_f(\chi^r) = \sum_{\substack{\alpha\in F_q\\ \mathrm{Tr}\alpha=1}} \chi^r(\alpha).$$

Since $\mathrm{Tr}\alpha = \mathrm{Tr}\alpha^p$, it follows that $E_f(\chi^r) = E_f(\chi^{rp})$ is in k_1 and hence $G_f(\chi^r) = G_f(\chi^{rp})$ is in K_1. So, we evaluate the relative norm $\mathrm{N}_{K_1/K_0}G_f(\chi^r)$.

3. $\mathrm{N}_{K_1/K_0}G_f(\chi^2)$ and Lemma

For any rational integer a not divisible by m, we denote the least positive residue of a modulo m by $R_m(a) : a \equiv R_m(a) \pmod{m}, 1 \leq R_m(a) \leq m-1$. Define

$$s(a) = s_f(a) = a_0 + a_1 + \cdots + a_{f-1},$$

$$t(a) = t_f(a) = a_0!a_1!\ldots a_{f-1}!$$

with rational integers $a_j, 0 \leq j \leq f-1$, which appear in the p-adic expansion

$$R_m(a)\frac{q-1}{m} = a_0 + a_1p + \cdots + a_{f-1}p^{f-1}.$$

It follows that

$$s(a) = \frac{p-1}{m}\sum_{j=0}^{f-1} R_m(ap^j). \tag{3}$$

Stickebrger's congruence theorem [1, Chap.11] shows that

$$G_f(\chi^{-a}) \equiv -\frac{(\zeta_p-1)^{s(a)}}{t(a)} \pmod{\mathfrak{P}^{s(a)+1}} \tag{4}$$

and in particular $G_f(\chi^{-a})$ is divisible exactly by $\mathfrak{P}^{s(a)}$: $\mathfrak{P}^{s(a)} \parallel G_f(\chi^{-a})$.

Now, we have

$$\mathfrak{P}^{s(-2)} \parallel G_f(\chi^2),\ \mathfrak{P}^{s(-10)} \parallel G_f(\chi^{10}),\ \mathfrak{P}^{s(2)} \parallel G_f(\chi^{-2}),\ \mathfrak{P}^{s(10)} \parallel G_f(\chi^{-10}).$$

Appling $\sigma^{-1}, \sigma_{-1}, \sigma_{-5}^{-1} = \sigma_{-1}\sigma^{-1}$ respectively to these last three divisibility relations by prime ideals yields

$$(\mathfrak{P}^{\sigma})^{s(-10)} \parallel G_f(\chi^2), \qquad \overline{\mathfrak{P}}^{s(2)} \parallel G_f(\chi^2), \qquad (\overline{\mathfrak{P}}^{\sigma})^{s(10)} \parallel G_f(\chi^2).$$

Therefore the prime ideal factorization of $(G_f(\chi^2))$ is given as follows:

$$(G_f(\chi^2)) = \mathfrak{P}^{s(-2)}\,(\mathfrak{P}^{\sigma})^{s(-10)}\,\overline{\mathfrak{P}}^{s(2)}\,(\overline{\mathfrak{P}}^{\sigma})^{s(10)}.$$

Also, we have

$$(G_f(\chi^{10})) = \mathfrak{P}^{s(-10)}\,(\mathfrak{P}^{\sigma})^{s(-2)}\,\overline{\mathfrak{P}}^{s(10)}\,(\overline{\mathfrak{P}}^{\sigma})^{s(2)}.$$

These two factorizations imply

$$(\mathrm{N}_{K_1/K_0}G_f(\chi^2)) = (G_f(\chi^2)G_f(\chi^{10})) = (\mathfrak{P}\,\mathfrak{P}^{\sigma})^{s(-2)+s(-10)}\,(\overline{\mathfrak{P}}\,\overline{\mathfrak{P}}^{\sigma})^{s(2)+s(10)}. \quad (5)$$

Lemma. *For any rational integer a not divisible by $m = 2^t$, put*

$$S_f(a) = \sum_{j=0}^{f-1} R_m(ap^j).$$

If $a \not\equiv 0 \pmod 4$, then for $f = 2^{t-3}$ with $t \geq 4$

$$S_f(a) = \frac{f}{2}S_2(a) + (\frac{f}{2} - 1)m.$$

Proof. To obtain the desired equality it suffices to show the recurrence formula

$$S_f(a) = 2S_{\frac{f}{2}}(a) + \frac{f}{4}m$$

for $f = 2^{t-3}$ with $t \geq 5$. We have that

$$p^{\frac{f}{4}} \equiv 1 + 2f = 1 + \frac{m}{4} \pmod{\frac{m}{2}},$$

$$p^{\frac{f}{2}} \equiv 1 + 4f = 1 + \frac{m}{2} \pmod{m}.$$

If $a \not\equiv 0 \pmod 2$, then for $0 \leq j \leq f/2 - 1$

$$ap^j \equiv ap^{\frac{f}{2}+j} + \frac{m}{2} \pmod{m}$$

and hence

$$R_m(ap^j) + R_m(ap^{\frac{f}{2}+j}) = 2R_{\frac{m}{2}}(ap^j) + \frac{m}{2},$$

from which we obtain

$$S_f(a) = \sum_{j=0}^{\frac{f}{2}-1}(R_m(ap^j) + R_m(ap^{\frac{f}{2}+j}))$$

$$= \sum_{j=0}^{\frac{f}{2}-1}(2R_{\frac{m}{2}}(ap^j) + \frac{m}{2})$$

$$= 2S_{\frac{f}{2}}(a) + \frac{f}{4}m.$$

If $a \equiv 2 \pmod 4$, then

$$ap^j \equiv ap^{\frac{f}{4}+j} + \frac{m}{2} \pmod m, \quad 0 \le j \le \frac{f}{4} - 1,$$

$$ap^j \equiv ap^{\frac{f}{2}+j} \pmod m, \qquad 0 \le j \le \frac{f}{2} - 1,$$

and hence

$$R_m(ap^j) + R_m(ap^{\frac{f}{4}+j}) = 2R_{\frac{m}{2}}(ap^j) + \frac{m}{2}, \quad 0 \le j \le \frac{f}{4} - 1,$$

$$R_m(ap^j) = R_m(ap^{\frac{f}{2}+j}), \qquad 0 \le j \le \frac{f}{2} - 1,$$

from which we obtain

$$S_f(a) = 2\sum_{j=0}^{\frac{f}{2}-1} R_m(ap^j)$$

$$= 2\sum_{j=0}^{\frac{f}{4}-1}(R_m(ap^j) + R_m(ap^{\frac{f}{4}+j}))$$

$$= 2\sum_{j=0}^{\frac{f}{4}-1}(2R_{\frac{m}{2}}(ap^j) + \frac{m}{2})$$

$$= 4\sum_{j=0}^{\frac{f}{4}-1} R_{\frac{m}{2}}(ap^j) + \frac{f}{4}m$$

$$= 2S_{\frac{f}{2}}(a) + \frac{f}{4}m.$$

□

Putting Lemma together with (3) gives

$$s(a) = \frac{p-1}{m}(\frac{f}{2}S_2(a) + (\frac{f}{2} - 1)m)$$

$$= (p-1)(\frac{f}{2} + \frac{1}{16}S_2(a) - 1).$$

Note

$$S_2(a) = R_{16}(a) + R_{16}(ap).$$

Hence,

$$s(-2) + s(-10) = (p-1)(\frac{f}{2} + \frac{7}{4} - 1) + (p-1)(\frac{f}{2} + \frac{3}{4} - 1)$$

$$= (p-1)(f + \frac{1}{2}). \tag{6}$$

Since $s(a) = (p-1)f - s(-a)$, we also see that

$$s(2) + s(10) = (p-1)(f - \frac{1}{2}). \tag{7}$$

It then follows from (5) that

$$(\mathrm{N}_{K_1/K_0}G_f(\chi^2)) = (\mathfrak{P}\mathfrak{P}^\sigma)^{(p-1)(f+\frac{1}{2})}\,(\overline{\mathfrak{P}}\overline{\mathfrak{P}}^\sigma)^{(p-1)(f-\frac{1}{2})} = \left(\frac{p^f}{\sqrt{p}}\right)\mathfrak{p}_0. \tag{8}$$

Since $G_f(\chi^{*2})$ is in K_0 and $E_f(\chi^2)$ is in k_1, from (2) we have

$$\mathrm{N}_{K_1/K_0}G_f(\chi^2) = G_f(\chi^{*2})^2\mathrm{N}_{k_1/k_0}E_f(\chi^2). \tag{9}$$

For the quartic character χ^{*2} on F_p we know that

$$G_f(\chi^{*2})^2 = J(\chi^{*2},\chi^{*2})G_f(\chi^{*4}) = J(\chi^{*2},\chi^{*2})\sqrt{p}\,, \tag{10}$$

where

$$J(\chi^{*2},\chi^{*2}) = \sum_{a=1}^{p-1}\chi^{*2}(a(p+1-a))$$

is in k_0. It follows from (8), (9), (10) that there exists a unit η of k_0 such that

$$\mathrm{N}_{K_1/K_0}G_f(\chi^2) = p^f\,\frac{\eta(a+bi)}{\sqrt{p}}\,. \tag{11}$$

Remark.

$$J(\chi^{*2},\chi^{*2}) = 2\sum_{a=1}^{\frac{p-3}{2}} \chi^{*2}(a(p+1-a)) + \chi^{*2}((\frac{p+1}{2})^2)$$

$$\equiv 2\frac{p-3}{2} + \left(\frac{2}{p}\right) = p-2 \equiv -1 \pmod{\mathfrak{z}^3},$$

where $\mathfrak{z} = (1+i)$ is a prime divisor of (2) in k_0 (cf. [1, Chap.3], [3, Chap.6]). By Stickelberger's congruence we have

$$(G_f(\chi^{*2})) = \mathfrak{P}_0^{\frac{3(p-1)}{4}} \overline{\mathfrak{P}}_0^{\frac{p-1}{4}}$$

and so

$$(G_f(\chi^{*2})^2) = \mathfrak{P}_0^{\frac{3(p-1)}{2}} \overline{\mathfrak{P}}_0^{\frac{p-1}{2}} = (\sqrt{p})\,\mathfrak{p}_0,$$

from which together with (10) $G_f(\chi^{*2})^2$ can be written as

$$G_f(\chi^{*2})^2 = \varepsilon(a+bi)\sqrt{p}$$

with a unit ε of k_0 satisfying $\varepsilon(a+bi) \equiv -1 \pmod{\mathfrak{z}^3}$.

Now, put

$$t_2 = t(-2)t(-10).$$

Then, from (4), (6) we have

$$\mathrm{N}_{K_1/K_0} G_f(\chi^2) = G_f(\chi^2)G_f(\chi^{10}) \equiv \frac{(\zeta_p - 1)^{(p-1)(f+\frac{1}{2})}}{t_2} \pmod{\mathfrak{P}_0^{(p-1)(f+\frac{1}{2})+1}}$$

and so from (11)

$$p^f\,\frac{\eta(a+bi)}{\sqrt{p}} \equiv \frac{(\zeta_p - 1)^{(p-1)(f+\frac{1}{2})}}{t_2} \pmod{\mathfrak{P}_0^{(p-1)(f+\frac{1}{2})+1}},$$

which implies

$$\frac{\eta i\sqrt{p}}{2b} \equiv \frac{(\zeta_p - 1)^{\frac{p-1}{2}}}{t_2} \pmod{\mathfrak{P}_0^{\frac{p-1}{2}+1}},$$

because $(\zeta_p-1)^{p-1} \equiv -p \pmod{\mathfrak{P}_0^p}$ and $p = (a+bi)(a-bi) \equiv -2bi(a+bi) \pmod{\mathfrak{p}_0^2}$. But noting $(\zeta_p - 1)^{(p-1)/2} \equiv \pm i\sqrt{p} \pmod{\mathfrak{P}_0^{(p-1)/2+1}}$, after change of a, b by multiplication of $\pm\eta$ to $a+bi$, we obtain that

$$\mathrm{N}_{K_1/K_0} G_f(\chi^2) = p^f\,\frac{a+bi}{\sqrt{p}} \tag{12}$$

with $p = a^2+b^2$, $2b + t_2 \equiv 0 \pmod{p}$, $\mathfrak{p}_0 = (a+bi)$.

4. $N_{K_1/K_0}G_f(\chi)$ and Theorem

Next, we evaluate $N_{K_1/K_0}G_f(\chi)$. By the product formula of Davenport-Hasse [1,Chap.11],[2] we have

$$G_f(\chi^2) = -\chi^{*2}(2)\ \frac{G_f(\chi)G_f(\chi^{1+\frac{m}{2}})}{G_f(1)G_f(\chi^{\frac{m}{2}})}.$$

Here $\chi^{*2}(2) = \pm 1$ [1, Chap.7], [3, Chap.5], $G_f(1) = -1, G_f(\chi^{m/2}) = -p^{f/2}$ [1, Chap.11] and, since $1+m/2 \in < p^{f/2} > \subset < p > \subset (Z/mZ)^\times$, $G_f(\chi^{1+m/2}) = G_f(\chi)$. Hence, we have

$$G_f(\chi)^2 = -\chi^{*2}(2)p^{\frac{f}{2}}G_f(\chi^2),$$

which implies from (12)

$$N_{K_1/K_0}G_f(\chi)^2 = p^f N_{K_1/K_0}G_f(\chi^2) = p^{2f}\ \frac{a+bi}{\sqrt{p}}$$

and so

$$N_{K_1/K_0}G_f(\chi) = p^f\ \frac{\sqrt{a+bi}}{\sqrt[4]{p}} \tag{13}$$

with ambiguity of sign. Put

$$t_1 = t(-1)t(-5).$$

Then, from (4)

$$N_{K_1/K_0}G_f(\chi) = G_f(\chi)G_f(\chi^5) \equiv \frac{(\zeta_p-1)^{(p-1)(f+\frac{1}{4})}}{t_1} \quad (\mathrm{mod}\ \mathfrak{P}_0^{(p-1)(f+\frac{1}{4})+1}),$$

because

$$\begin{aligned} s(-1)+s(-5) &= (p-1)(\frac{f}{2}+\frac{11}{8}-1)+(p-1)(\frac{f}{2}+\frac{7}{8}-1) \\ &= (p-1)(f+\frac{1}{4}). \end{aligned}$$

Hence, from (13) we have

$$p^f\ \frac{\sqrt{a+bi}}{\sqrt[4]{p}} \equiv \frac{(\zeta_p-1)^{(p-1)(f+\frac{1}{4})}}{t_1} \quad (\mathrm{mod}\ \mathfrak{P}_0^{(p-1)(f+\frac{1}{4})+1})$$

and so

$$\frac{\sqrt{a+bi}}{\sqrt[4]{p}} \equiv \frac{(\zeta_p-1)^{\frac{p-1}{4}}}{t_1} \quad (\mathrm{mod}\ \mathfrak{P}_0^{\frac{p-1}{4}+1}).$$

For $2 \leq s \leq t-3$ by the product formula of Davenport-Hasse we have

$$G_f(\chi^{2^s}) = -\chi^{*2^s}(2^s)\,\frac{\prod_{j=0}^{2^s-1} G_f(\chi^{1+2^{t-s}j})}{\prod_{j=0}^{2^s-1} G_f(\chi^{2^{t-s}j})}.$$

Here $\chi^{*2^s}(2^s) = 1, G_f(1) = -1, G_f(\chi^{m/2}) = -p^{f/2}, G_f(\chi^{2^{t-s}j})G_f(\chi^{2^{t-s}(2^s-j)}) = p^f, 1 \le j \le 2^{s-1}-1$, and, since $1 + 2^{t-s}j \in <p> \subset (Z/mZ)^\times, G_f(\chi^{1+2^{t-s}j}) = G_f(\chi), 0 \le j \le 2^s - 1$. Hence, we get

$$G_f(\chi^{2^s}) = -\frac{G_f(\chi)^{2^s}}{p^{2^{s-1}f-\frac{f}{2}}}$$

and so from (13)

$$\mathrm{N}_{K_1/K_0}G_f(\chi^{2^s}) = \frac{1}{p^{f(2^s-1)}}\,\mathrm{N}_{K_1/K_0}G_f(\chi)^{2^s} = p^{f-2^{s-2}}(a+bi)^{2^{s-1}}.$$

For $s = t-2$ we have also

$$G_f(\chi^{2^s}) = -\chi^{*2^s}(2^s)\,\frac{\prod_{j=0}^{2^s-1} G_f(\chi^{1+2^sj})}{\prod_{j=0}^{2^s-1} G_f(\chi^{2^sj})}.$$

Here $\chi^{*2^s}(2^s) = 1, G_f(1) = -1, G_f(\chi^{m/2}) = -p^{f/2}, G_f(\chi^{2^sj})G_f(\chi^{2^s(2^s-j)}) = p^f, 1 \le j \le 2^{s-1}-1$, and $G_f(\chi^{1+2^sj}) = G_f(\chi)$ or $G_f(\chi^5)$ according as $j \equiv 0$ or $1 \pmod 2$, because $1 + 2^2 j \in <p>$ or $<5> - <p>$ according as $j \equiv 0$ or 1 (mod 2). Hence, from (13)

$$G_f(\chi^{2^s}) = -\frac{\mathrm{N}_{K_1/K_0}G_f(\chi)^{2^{s-1}}}{p^{2^{s-1}f-\frac{f}{2}}} = -p^{\frac{f}{2}-2^{s-3}}(a+bi)^{2^{s-2}}.$$

Thus, our results can be summarized as follows:

Theorem. *Suppose that p is a prime number such that $p \equiv 9 \pmod{16}, m = 2^t$ with $t \ge 4$, and χ is a character of order m on the finite field $F_q, q = p^{2^{t-3}}$, defined by (1). Then the relative norms $\mathrm{N}_{K_1/K_0}G_f(\chi^{2^s})$ of the Gauss sums $G_f(\chi^{2^s})$ are given by*

(i) $\mathrm{N}_{K_1/K_0}G_f(\chi) = p^{2^{t-3}}\,\dfrac{\sqrt{a+bi}}{\sqrt[4]{p}}$,

(ii) $\mathrm{N}_{K_1/K_0}G_f(\chi^2) = p^{2^{t-3}}\,\dfrac{a+bi}{\sqrt{p}}$,

(iii) $\mathrm{N}_{K_1/K_0}G_f(\chi^{2^s}) = p^{2^{t-3}-2^{s-2}}(a+bi)^{2^{s-1}}, \quad 2 \le s \le t-3$, and further

(iv) $G_f(\chi^{2^{t-2}}) = -p^{2^{t-5}}(a+bi)^{2^{t-4}}$,

(v) $G_f(\chi^{2^{t-1}}) = -p^{2^{t-4}}$.

Herein $p = a^2 + b^2, 2b + t_2 \equiv 0 \pmod{p}, \mathfrak{p}_0 = (a + bi)$, *and* $\sqrt{a+bi}/\sqrt[4]{p} \equiv (\zeta_p - 1)^{(p-1)/4}/t_1 \pmod{\mathfrak{P}_0^{(p-1)/4+1}}$.

References

[1] B.C. Bernet, R.J. Evans and K.S. Williams, *Gauss sums and Jacobi sums*, Canad. Math. Soc. Ser. Adv. Texts 21, John Wiley, 1998.

[2] H. Davenport and H. Hasse, *Die Nulstellen der Kongruenzzetafunktion in gewissen zyklischen Fällen*, J. Reine Angew. Math. **172** (1934), 151-182.

[3] F. Lemmermeyer, *Reciprocity laws: From Euler to Eisenstein*, Springer, 2000.

[4] P. Meijer and M. van der Vlug, *The evaluation of Gauss sums for characters of 2-power order*, J. Number Theory **100** (2003), 381-395.

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