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Two-stage procedure for estimating a linear function of normal means under an asymmetric loss function

Yoshikazu Takada*

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Abstract. This paper considers a sequential point estimation procedure to achieve a bounded risk when estimating a linear function of normal means under an asymmetric loss function. A two-stage estimation procedure is introduced to achieve the goal and its asymptotic property is examined.

1. Introduction

We consider a sequential point estimation of a linear function of normal means under a LINEX (Linear Exponential) loss function. The LINEX loss function was first proposed by Varian [8] when it is appropriate to use an asymmetric loss function. See also Zellner [9] for its theoretical development. The sequential applications of the loss function were considered by Chattopadhyay [2], Chattopadhyay et al. [3], Takada [4],[5] and Takada and Nagao [7]. The present paper proposes a two-stage procedure to the multisample problem treated by Chattopadhyay et al. [3]. It turns out that the proposed proceure is more efficient than that of Chattopadhyay et al. [3] in terms of the total sample size. See Remark in Section 2.

Let π_i be a normal population with unknown mean μ_i and unknown variance $\sigma_i^2, i = 1, \ldots, k\, (k \geq 2)$. Having observed $X_{i1}, \ldots, X_{in_i}$ from each π_i, we want to estimate $\mu = \sum_{i=1}^{k} \lambda_i \mu_i$ by δ_n where λ_i's are known constants and $n = (n_1, \ldots, n_k)$. We suppose that the loss incurred is LINEX,

$$L(\delta_n, \mu) = \exp\left(a(\delta_n - \mu)\right) - a(\delta_n - \mu) - 1 \tag{1.1}$$

where $a (\neq 0)$ is a known constant. For a preassigned positive constant W, we want to determine the sample size n_i from π_i, $i = 1, \ldots, k$ so as to construct such

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an estimator δ_n that

$$E_\theta L(\delta_n, \mu) \le W \quad \text{for all } \theta = (\mu_1, \ldots, \mu_k, \sigma_1, \ldots, \sigma_k). \tag{1.2}$$

It is well known that if n_i's are predetermined before the experiment, then there is no estimator which satisfies (1.2). Hence it is necessary to consider a sequential procedure to meet the requirement. We propose a two-stage procedure to asymptotically satisfy (1.2) as $W \to 0$. For the squared error loss, see Aoshima and Takada [1].

If σ_i's were known, we would estimate μ by $\delta_n = \sum_{i=1}^k \lambda_i \bar{X}_{i(n_i)} - \frac{a}{2}\sum_{i=1}^k \frac{\lambda_i^2\sigma_i^2}{n_i}$ with $\bar{X}_{i(n_i)} = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij}$, since the risk of the usual estimator $\sum_{i=1}^k \lambda_i \bar{X}_{i(n_i)}$ is improved by δ_n under (1.1). It is easy to see that $E_\theta L(\delta_n, \mu) = \frac{a^2}{2}\sum_{i=1}^k \frac{\lambda_i^2\sigma_i^2}{n_i}$. Hence (1.2) would be satisfied if and only if

$$\frac{a^2}{2}\sum_{i=1}^k \frac{\lambda_i^2\sigma_i^2}{n_i} \le W. \tag{1.3}$$

It is not difficult to see that the values of n_i's which minimize $\sum_{i=1}^k n_i$ subject to (1.3) are given by

$$n_i^* = \frac{a^2}{2W}\left(\sum_{j=1}^k |\lambda_j|\sigma_j\right)|\lambda_i|\sigma_i, \quad i = 1, \ldots, k. \tag{1.4}$$

Motivated by (1.4), we propose the following two-stage procedure. Let $X_{i1}, \ldots, X_{im}$ be the initial sample of size $m(\ge 2)$ from π_i and let

$$V_{i(m)}^2 = \frac{1}{m-1}\sum_{j=1}^m \left(X_{ij} - \bar{X}_{i(m)}\right)^2, \; i = 1, \ldots, k.$$

Then the total sample size N_i from π_i is determined by

$$N_i = \max\left\{m, \left[\frac{a^2\ell_m\left(\sum_{j=1}^k |\lambda_j|V_{j(m)}\right)|\lambda_i|V_{i(m)}}{2W}\right] + 1\right\}, \quad i = 1, \ldots, k, \tag{1.5}$$

where $[x]$ denotes the largest integer less than x and $\ell_m(\ge 1)$ is called a design constant and satisfies

$$\ell_m = 1 + \frac{\ell_o}{m} + o\left(\frac{1}{m}\right) \text{ as } m \to \infty. \tag{1.6}$$

The design constant is used to asymptotically assure (1.2). If $N_i > m$, take $N_i - m$ additional observations from π_i. Then we estimate μ by

$$\delta_N = \sum_{i=1}^k \lambda_i \bar{X}_{i(N_i)} - \frac{a}{2}\sum_{i=1}^k \frac{\lambda_i^2 V_{i(N_i)}^2}{N_i} \tag{1.7}$$

with $N = (N_1, \ldots, N_k)$, which mimics δ_n used if σ_i's were known.

The asymptotic results of the two-stage procedure are contained in Section 2. Section 3 conducts simulations to see the moderate sample size performances of the proposed two-stage procedure.

2. Two-stage procedure

This section derives the asymptotic expansions of the risk function and the expected sample size, which enable us to determine the design constant asymptotically assuring (1.2) and to see if the asymptotically second-order efficiency of the sample size is satisfied.

Substituting (1.7) into (1.1) and using the fact that the event $\{N_i = n_i, i = 1, \ldots, k\}$ is independent $\{\bar{X}_{n_i}, i = 1, \ldots, k\}$, we have

$$\begin{aligned} E_\theta L(\hat{\delta}_N, \mu) &= E_\theta \left\{ \exp\left(-\frac{a^2}{2} \sum_{i=1}^{k} \frac{\lambda_i^2}{N_i} \left(V_{i(N_i)}^2 - \sigma_i^2 \right) \right) + \frac{a^2}{2} \sum_{i=1}^{k} \frac{\lambda_i^2 V_{i(N_i)}^2}{N_i} - 1 \right\} \\ &= E_\theta W_N + \frac{a^2}{2} \sum_{i=1}^{k} E_\theta \left(\frac{\lambda_i^2 \sigma_i^2}{N_i} \right), \end{aligned} \tag{2.1}$$

where

$$W_N = \exp\left(-\frac{a^2}{2} \sum_{i=1}^{k} \frac{\lambda_i^2}{N_i} \left(V_{i(N_i)}^2 - \sigma_i^2 \right) \right) + \frac{a^2}{2} \sum_{i=1}^{k} \frac{\lambda_i^2}{N_i} \left(V_{i(N_i)}^2 - \sigma_i^2 \right) - 1. \tag{2.2}$$

From (1.4)

$$\frac{\lambda_i^2 \sigma_i^2}{N_i} = \frac{4W^2}{a^4 \left(\sum_{j=1}^{k} |\lambda_j| \sigma_j \right)^2} \left\{ \frac{(N_i - n_i^*)^2}{N_i} - N_i + 2n_i^* \right\}, \quad i = 1, \ldots, k,$$

which is substituted in (2.1) to yield

$$\begin{aligned} E_\theta L(\hat{\delta}_N, \mu) &= W + \frac{2W^2}{a^2 \left(\sum_{j=1}^{k} |\lambda_j| \sigma_j \right)^2} \left\{ \sum_{i=1}^{k} E_\theta \frac{(N_i - n_i^*)^2}{N_i} - \sum_{i=1}^{k} E_\theta (N_i - n_i^*) \right\} \\ &\quad + E_\theta W_N. \end{aligned} \tag{2.3}$$

We assume that the initial sample size m is determined according to W such that for some $d (> 0)$

$$\lim_{W \to 0} m W^d = \frac{a^2 c}{2}, \tag{2.4}$$

where c is a positive constant. It is easy to see that if $d > 1$, then $\lim_{W \to 0} E_\theta(N_i)/n_i^* = \infty, i = 1, \ldots, k$, which implies that it is necesssry to assume $0 < d \le 1$ in order to satisfy the asymptotically second-order efficiency. We also assume that when taking $d = 1$, the constant c satisfies

$$c < \left(\sum_{j=1}^{k} |\lambda_j|\sigma_j\right) |\lambda_i|\sigma_i, \quad i = 1, \ldots, k. \tag{2.5}$$

The following three lemmas are proved in the appendix.

Lemma 2.1.

$$\lim_{W\to 0} \frac{m}{n_i^*} E_\theta \frac{(N_i - n_i^*)^2}{N_i} = \eta_i^2, \quad i = 1, \ldots, k,$$

where

$$\eta_i^2 = \frac{1}{2}\left(1 + \frac{\sum_{j=1}^k \lambda_j^2 \sigma_j^2}{\left(\sum_{j=1}^k |\lambda_j|\sigma_j\right)^2} + \frac{2|\lambda_i|\sigma_i}{\sum_{j=1}^k |\lambda_j|\sigma_j}\right), \quad i = 1, \ldots, k. \tag{2.6}$$

Lemma 2.2. *For* $0 < d < 1$

$$\lim_{W\to 0} W^{1-d} E_\theta (N_i - n_i^*) = \frac{1}{c}\left\{\ell_o \left(\sum_{j=1}^k |\lambda_j|\sigma_j\right) |\lambda_i|\sigma_i - \frac{1}{2}\left(\sum_{j\neq i}^k |\lambda_j|\sigma_j\right)|\lambda_i|\sigma_i\right\},$$

$i = 1, \ldots, k,$ *and for* $d = 1$

$$\lim_{W\to 0} E_\theta (N_i - n_i^*) = \frac{1}{c}\left\{\ell_o \left(\sum_{j=1}^k |\lambda_j|\sigma_j\right) |\lambda_i|\sigma_i - \frac{1}{2}\left(\sum_{j\neq i}^k |\lambda_j|\sigma_j\right)|\lambda_i|\sigma_i\right\} + \frac{1}{2},$$

$i = 1, \ldots, k.$

Lemma 2.3.

$$E_\theta W_N = O\left(W^3\right) \quad \textit{as } W \to 0.$$

Using (1.4) and (2.4), Lemma 2.1 yields that for $0 < d \leq 1$

$$\lim_{W\to 0} W^{1-d} E_\theta \frac{(N_i - n_i^*)^2}{N_i} = \frac{\left(\sum_{j=1}^k |\lambda_j|\sigma_j\right)|\lambda_i|\sigma_i}{c}\eta_i^2, \quad i = 1, \ldots, k,$$

so that

$$\lim_{W\to 0} W^{1-d} \sum_{i=1}^k E_\theta \frac{(N_i - n_i^*)^2}{N_i} = \frac{1}{2c}\left\{\left(\sum_{j=1}^k |\lambda_j|\sigma_j\right)^2 + 3\sum_{j=1}^k \lambda_j^2\sigma_j^2\right\}. \tag{2.7}$$

It follows from Lemma 2.2 that for $0 < d < 1$

$$\lim_{W\to 0} W^{1-d} \sum_{i=1}^k E_\theta (N_i - n_i^*) = \frac{1}{c}\left\{\ell_o \left(\sum_{j=1}^k |\lambda_j|\sigma_j\right)^2 - \frac{1}{2}\left(\sum_{j=1}^k |\lambda_j|\sigma_j\right)^2 + \frac{1}{2}\sum_{j=1}^k \lambda_j^2\sigma_j^2\right\} \tag{2.8}$$

and for $d = 1$

$$\lim_{W\to 0}\sum_{i=1}^{k} E_\theta(N_i - n_i^*) = \frac{1}{c}\left\{\ell_o\left(\sum_{j=1}^{k}|\lambda_j|\sigma_j\right)^2 - \frac{1}{2}\left(\sum_{j=1}^{k}|\lambda_j|\sigma_j\right)^2 + \frac{1}{2}\sum_{j=1}^{k}\lambda_j^2\sigma_j^2\right\} + \frac{k}{2}. \tag{2.9}$$

Hence substituting (2.7) and (2.8) ((2.9)) into (2.3) and using Lemma 2.3, we have the following result.

Theorem 2.4. *For* $0 < d < 1$

$$E_\theta L(\hat{\delta}_N, \mu) = W + \frac{2W^{1+d}}{ca^2\left(\sum_{j=1}^{k}|\lambda_j|\sigma_j\right)^2}\left\{(1-\ell_o)\left(\sum_{j=1}^{k}|\lambda_j|\sigma_j\right)^2 + \sum_{j=1}^{k}\lambda_j^2\sigma_j^2\right\} + o\left(W^{1+d}\right) \quad \text{as } W \to 0.$$

and for $d = 1$

$$E_\theta L(\hat{\delta}_N, \mu) = W + \frac{2W^2}{a^2\left(\sum_{j=1}^{k}|\lambda_j|\sigma_j\right)^2}\left\{\frac{1}{c}\left((1-\ell_o)\left(\sum_{j=1}^{k}|\lambda_j|\sigma_j\right)^2 + \sum_{j=1}^{k}\lambda_j^2\sigma_j^2\right) - \frac{k}{2}\right\} + o\left(W^2\right) \quad \text{as } W \to 0.$$

>From Theorem 2.4, it turns out that when taking $0 < d \le 1$, we have to choose the design constant ℓ_m with $\ell_o \ge 2$ in order to assure (1.2) for small W, that is,

$$E_\theta L(\hat{\delta}_N, \mu) \le W + o\left(W^{1+d}\right) \quad \text{as } W \to 0. \tag{2.10}$$

However, it follows from Lemma 2.2 that such a choice for $0 < d < 1$ implies

$$\lim_{W\to 0} E_\theta(N_i - n_i^*) = \infty, \quad i = 1, \ldots, k,$$

that is, the asymptotically second-order efficiency does not hold, but that for $d = 1$ satisfies the asymptotically second-order efficiency. The condition (2.5) is essential to assure the asymptotically second-order efficiency when taking $d = 1$.

In order to make the two-stage procedure with $d = 1$ valid in real applications, we have to specify a lower bound $\sigma_{i*} (> 0)$ for each σ_i such that $\sigma_i > \sigma_{i*}, i = 1, \ldots, k$. Letting

$$\tau_* = \min(|\lambda_1|\sigma_{1*}, \ldots, |\lambda_k|\sigma_{k*})\sum_{j=1}^{k}|\lambda_j|\sigma_{j*},$$

we propose the two-stage procedure with the design constant ℓ_m with $\ell_o \geq 2$ and the following initial sample size from each population

$$m = \max\left(m_0, \left[\frac{a^2\tau_*}{2W}\right] + 1\right), \tag{2.11}$$

where $m_0(\geq 2)$ is any positive integer. Then the conditions (2.4) and (2.5) hold with $d = 1$ and $c = \tau_*$, and hence (2.10) holds with $d = 1$ and the asymptotically second-order efficiency is satisfied.

Remark Chattopadhyay et al.[3] proposed a sequential procedure to asymptotically satisfy (1.2). However, their procedure is constructed so as to mimic $\tilde{n}_i^* = ka^2\lambda_i^2\sigma_i^2/(2\log(1+W))$ instead of n_i^*, $i = 1, \ldots, k$. It is easy to see that

$$\lim_{W\to 0} \frac{\sum_{i=1}^k \tilde{n}_i^*}{\sum_{i=1}^k n_i^*} = \frac{k\sum_{i=1}^k \lambda_i^2\sigma_i^2}{\left(\sum_{i=1}^k |\lambda_i|\sigma_i\right)^2} > 1, \tag{2.12}$$

unless $|\lambda_1|\sigma_1 = \cdots = |\lambda_k|\sigma_k$. Let $\tilde{N}_i$ be their sample size from π_i, $i = 1, \ldots, k$. Then it holds

$$\lim_{W\to 0} \frac{E_\theta\left(\sum_{i=1}^k \tilde{N}_i\right)}{\sum_{i=1}^k \tilde{n}_i} = 1.$$

It follows from Lemma 2.2 that the two-stage procedure with $d(0 < d \leq 1)$ satisfies

$$\lim_{W\to 0} \frac{E_\theta\left(\sum_{i=1}^k N_i\right)}{\sum_{i=1}^k n_i^*} = 1.$$

Hence unless $|\lambda_1|\sigma_1 = \cdots = |\lambda_k|\sigma_k$, from (2.12)

$$\lim_{W\to 0} \frac{E_\theta\left(\sum_{i=1}^k \tilde{N}_i\right)}{E_\theta\left(\sum_{i=1}^k N_i\right)} > 1,$$

which implies

$$\lim_{W\to 0} \left\{E_\theta\left(\sum_{i=1}^k \tilde{N}_i\right) - E_\theta\left(\sum_{i=1}^k N_i\right)\right\} = \infty.$$

This shows that their procedure is less efficient than ours in terms of the total number of observations required for small W.

3. Simulations

We conducted simulations to see moderate sample size performances of the proposed two-stage procedure for $k = 3$. We considered the problem of estimating of $\mu = (\mu_1 + \mu_2)/2 - \mu_3$ under the LINEX loss function with $a = \sqrt{2}$. We chose

$\ell_m = 1 + 2/m$ and $\sigma_{1*} = \sigma_{2*} = \sigma_{3*} = 1$ as a lower bound for each unknown σ_i. The values of W were chosen as $W = 0.10(0.02)0.02, 0.01$. The initial sample size m is determined by (2.11) with $m_0 = 2$. The simulation results in Tables 1 and 2, which estimated the risk function R_N and $E_\theta(N_i - n_i^*), i = 1, 2, 3$, are based on $10,000$ replications under $\sigma_1 = 1.2$, $\sigma_2 = 1.4$ and $\sigma_3 = 1.6$ in Table 1 and $\sigma_1 = 1.6$, $\sigma_2 = 1.4$ and $\sigma_3 = 1.2$ in Table 2. We also provided their respective standard errors within parentheses. The values in the last row of each Table are derived from the asymptotic ones in Lemma 2.2 with $d = 1$ and $c = \tau_*$.

Table 1: $\sigma_1 = 1.2$, $\sigma_2 = 1.4$, $\sigma_3 = 1.6$

W	R_N	$E_\theta(N_1 - n_1^*)$	$E_\theta(N_2 - n_2^*)$	$E_\theta(N_3 - n_3^*)$
0.10	0.098	2.94	3.33	7.82
	(0.002)	(0.06)	(0.07)	(0.19)
0.08	0.077	3.08	3.53	8.20
	(0.001)	(0.07)	(0.08)	(0.21)
0.06	0.057	3.22	3.72	8.60
	(0.001)	(0.08)	(0.09)	(0.24)
0.04	0.040	2.98	3.56	8.03
	(0.001)	(0.09)	(0.11)	(0.28)
0.02	0.020	3.21	3.62	8.25
	(0.000)	(0.12)	(0.14)	(0.38)
0.01	0.010	3.17	3.50	8.90
	(0.000)	(0.17)	(0.20)	(0.54)
		3.29	3.79	8.74

Table 2: $\sigma_1 = 1.6$, $\sigma_2 = 1.4$, $\sigma_3 = 1.2$

W	R_N	$E_\theta(N_1 - n_1^*)$	$E_\theta(N_2 - n_2^*)$	$E_\theta(N_3 - n_3^*)$
0.10	0.094	3.53	3.21	5.51
	(0.001)	(0.08)	(0.07)	(0.13)
0.08	0.077	3.74	3.18	5.62
	(0.001)	(0.09)	(0.07)	(0.14)
0.06	0.059	3.69	3.35	5.66
	(0.001)	(0.10)	(0.08)	(0.16)
0.04	0.038	3.81	3.32	5.62
	(0.001)	(0.11)	(0.10)	(0.18)
0.02	0.020	4.02	3.67	6.41
	(0.000)	(0.17)	(0.13)	(0.25)
0.01	0.010	4.06	3.33	5.77
	(0.000)	(0.22)	(0.19)	(0.35)
		4.06	3.58	6.08

It turns out that the risk functions seem to satisfy the bounded risk condition (1.2) and that the values of $E_\theta(N_i - n_i^*)$ seem to be approximated fairly well by these asymptotic ones.

4. Appendix

In order to prove Lemmas 2.1 to 2.3, we need several lemmas. Let

$$Y_i = \left(\sum_{j=1}^{k} |\lambda_j| V_{j(m)}\right) |\lambda_i| V_{i(m)}, \quad i = 1, \ldots, k. \tag{4.1}$$

Lemma 4.1.

$$E_\theta \left| Y_i - \left(\sum_{j=1}^{k} |\lambda_j| \sigma_j\right) |\lambda_i| \sigma_i \right|^p = O\left(m^{-p/2}\right) \quad \text{as } m \to \infty, \quad i = 1, \ldots, k.$$

Proof. See Takada [6, p. 110]. □

Lemma 4.2. *For any $p > 1$,*

$$P_\theta(N_i = m) = O\left(m^{-p/2}\right) \quad \text{as } m \to \infty, \quad i = 1, \ldots, k.$$

Proof. From (1.5) we have

$$P_\theta(N_i = m) = P_\theta\left(Y_i < 2Wm/(a^2 \ell_m)\right).$$

It follows from (1.6), (2.4) and (2.5) that there exists $\epsilon\,(0 < \epsilon < 1)$ such that for small W

$$\begin{aligned} P_\theta(N_i = m) &\le P_\theta\left(Y_i < \epsilon \left(\sum_{j=1}^{k} |\lambda_j| \sigma_j\right) |\lambda_i| \sigma_i\right) \\ &\le P_\theta\left(\left|Y_i - \left(\sum_{j=1}^{k} |\lambda_j| \sigma_j\right) |\lambda_i| \sigma_i\right| > (1-\epsilon)\left(\sum_{j=1}^{k} |\lambda_j| \sigma_j\right) |\lambda_i| \sigma_i\right) \\ &\le \left((1-\epsilon)\left(\sum_{j=1}^{k} |\lambda_j| \sigma_j\right) |\lambda_i| \sigma_i\right)^{-p} E_\theta \left| Y_i - \left(\sum_{j=1}^{k} |\lambda_j| \sigma_j\right) |\lambda_i| \sigma_i \right|^p \end{aligned}$$

Then using Lemma 4.1, the proof is completed. □

Lemma 4.3.

$$N_i/n_i^* \to 1 \quad \text{a.e.} \quad \text{as } W \to 0, \quad i = 1, \ldots, k.$$

Proof. From (1.5) we have

$$\frac{Y_i}{\left(\sum_{j=1}^{k} |\lambda_j| \sigma_j\right) |\lambda_i| \sigma_i} \le \frac{N_i}{n_i^*} \le \frac{m I_{(N_i = m)}}{n_i^*} + \frac{Y_i}{\left(\sum_{j=1}^{k} |\lambda_j| \sigma_j\right) |\lambda_i| \sigma_i},$$

where I_A denotes the indicator function of the set A. Since $m \to \infty$ as $W \to 0$,

$$\frac{Y_i}{\left(\sum_{j=1}^{k} |\lambda_j|\sigma_j\right) |\lambda_i|\sigma_i} \to 1 \quad a.e. \quad \text{as } W \to 0, \quad i = 1, \ldots, k.$$

For $0 < d < 1$, from (2.4) $m/n_i^* \to 0$ as $W \to 0$, which completes the proof.

For $d = 1$, it suffices to show that $I_{(N_i=m)} \to 0$ a.e. as $W \to 0$. Let $A_m = \{I_{(N_i=m)} > \epsilon\} = \{N_i = m\}$ for any $\epsilon\,(0 < \epsilon < 1)$. Then from Lemma 4.2

$$P_\theta\left(\bigcup_{\ell=m}^{\infty} A_\ell\right) \le \sum_{\ell=m}^{\infty} P_\theta(A_\ell) = O\left(m^{-\frac{p-4}{2}}\right) \quad \text{as } m \to \infty$$

with $p > 4$. Hence

$$P_\theta\left(\limsup_{m\to\infty} A_m\right) = \lim_{m\to\infty} P_\theta\left(\bigcup_{\ell=m}^{\infty} A_\ell\right) = 0,$$

so that the Borel-Canteli Theorem implies that $I_{(N_i=m)} \to 0$ a.e. as $W \to 0$. □

Lemma 4.4. *For any* $\epsilon\,(0 < \epsilon < 1)$ *and any* $p > 1$

$$P_\theta(N_i \le \epsilon n_i^*) = O\left(m^{-p/2}\right) \quad \text{as } W \to 0, \quad i = 1, \ldots, k.$$

Proof. From (1.4) and (1.5) we have

$$P_\theta(N_i \le \epsilon n_i^*) \le P_\theta\left(Y_i \le \epsilon\left(\sum_{j=1}^{k} |\lambda_j|\sigma_j\right) |\lambda_i|\sigma_i\right).$$

The rest of the proof is the same as that of Lemma 4.2. □

Proof of Lemma 2.1

Let $W_i^2 = \nu V_{i(m)}^2/\sigma_i^2$, $i = 1, \ldots, k$ with $\nu = m - 1$. Then $W_1^2, \ldots, W_k^2$ are $i.i.d.$ chi-squred random variables with ν degrees of freedom. It is easy to see that $Z_i = \sqrt{2}(W_i - \sqrt{\nu})$ converges in distribution to $N(0,1)$ as $m \to \infty$, $i = 1, \ldots, k$. Substituting $V_{j(m)} = \sigma_j(\sqrt{\nu} + Z_j/\sqrt{2})/\sqrt{\nu}$, $j = 1, \ldots, k$ into (4.1), we have

$$\begin{aligned} Y_i \;=\; & \left(\sum_{j=1}^{k} |\lambda_j|\sigma_j\right) |\lambda_i|\sigma_i + \frac{1}{\sqrt{2\nu}}\left\{\left(\sum_{j=1}^{k} |\lambda_j|\sigma_j\right) |\lambda_i|\sigma_i Z_i + |\lambda_i|\sigma_i \sum_{j=1}^{k} |\lambda_j|\sigma_j Z_j\right\} \\ & + \frac{1}{2\nu}\left(\sum_{j=1}^{k} |\lambda_j|\sigma_j Z_j\right) |\lambda_i|\sigma_i Z_i, \end{aligned}$$

which yields from (1.6) that

$$\sqrt{\nu}\left(\frac{a^2\ell_m Y_i}{2} - \frac{a^2\left(\sum_{j=1}^k |\lambda_j|\sigma_j\right)|\lambda_i|\sigma_i}{2}\right)$$

converges in distribution to $N(0,\tau_i^2)$ as $m \to \infty$, where

$$\tau_i^2 = \frac{a^4\lambda_i^2\sigma_i^2}{8}\left(\left(\sum_{j=1}^k |\lambda_j|\sigma_j\right)^2 + \sum_{j=1}^k \lambda_j^2\sigma_j^2 + 2|\lambda_i|\sigma_i\sum_{j=1}^k |\lambda_j|\sigma_j\right), \quad i=1,\ldots,k.$$

Let

$$\begin{aligned} U_i &= \sqrt{\nu}\left(\frac{a^2\ell_m Y_i}{2} - \frac{a^2\left(\sum_{j=1}^k |\lambda_j|\sigma_j\right)|\lambda_i|\sigma_i}{2}\right) - \sqrt{\nu}W(N_i - n_i^*) \\ &= \sqrt{\nu}W\left(\frac{a^2\ell_m Y_i}{2} - N_i\right), \quad i=1,\ldots,k. \end{aligned}$$

It follows from (1.5), (2.4) and Lemma 4.2 that

$$E_\theta|U_i| \le \sqrt{\nu}W\left(mP_\theta(N_i = m) + 1\right) \to 0 \quad \text{as } W \to 0.$$

Hence $\sqrt{\nu}W(N_i - n_i^*)$ converges in distribution to $N(0,\tau_i^2)$ as $W \to 0, i = 1,\ldots,k$, which yields from (1.4) that $\sqrt{\nu}(N_i - n_i^*)/n_i^*$ converges in distribution to $N(0,\eta_i^2)$ as $W \to 0, i = 1,\ldots,k$, where η_i^2 is (2.6). Then it follows from Lemma 4.3 that $\sqrt{\nu/n_i^*}(N_i - n_i^*)/\sqrt{N_i}$ converges in distribution to $N(0,\eta_i^2)$ as $W \to 0, i = 1,\ldots,k$, so that from (2.4) in order to prove Lemma 2.1, it suffices to prove that $\{(N_i - n_i^*)^2/N_i\}$ is uniformly integrable ($u.i.$).

Let $A = \{N_i \le \epsilon n_i^*\}$ for some $\epsilon\,(0 < \epsilon < 1)$ and

$$\begin{aligned} \frac{(N_i - n_i^*)^2}{N_i} &= \frac{(N_i - n_i^*)^2}{N_i} I_A + \frac{(N_i - n_i^*)^2}{N_i} I_{\bar{A}} \\ &= I_1 + I_2 \quad \text{(say)}, \end{aligned}$$

where $\bar{A}$ denotes the compliment of the set A. Since

$$I_1 = \left(N_i - 2n_i^* + \frac{n_i^{*2}}{N_i}\right) I_A \le \left(\epsilon n_i^* + \frac{n_i^{*2}}{m}\right) I_A,$$

it follows from Lemma 4.4 that $\{I_1\}$ is $u.i.$. Since

$$I_2 \le \frac{(N_i - n_i^*)^2}{\epsilon n_i^*},$$

in order to prove that $\{I_2\}$ is *u.i.*, it suffices to show that $\{(N_i - n_i^*)^2/n_i^*\}$ is *u.i.*. Note that

$$\begin{aligned}\frac{(N_i - n_i^*)^2}{n_i^*} &\leq \frac{2\left(N_i - \frac{a^2\ell_m Y_i}{2W}\right)^2}{n_i^*} + \frac{2\left(\frac{a^2\ell_m Y_i}{2W} - n_i^*\right)^2}{n_i^*} \\ &= I_3 + I_4 \quad \text{(say)}.\end{aligned}$$

Since

$$I_3 \leq \frac{2}{n_i^*}\left(mI_{(N_i=m)} + 1\right)^2,$$

Lemma 4.2 shows that $\{I_3\}$ is *u.i.*. Write I_4 as

$$I_4 = \frac{\nu a^4}{2n_i^* W^2}\left(\ell_m Y_i - \left(\sum_{j=1}^{k}|\lambda_j|\sigma_j\right)|\lambda_i|\sigma_i\right)^2.$$

Then it follows from Lemma 4.1 that $\{I_4\}$ is *u.i.*, so that the proof is completed.

Proof of Lemma 2.2

Let $F_i = a^2\ell_m Y_i/(2W)$, $J_i = F_i - [F_i]$ and $R_i = F_i + 1 - J_i, i = 1, \ldots, k$. Then $N_i = \max(m, R_i), i = 1, \ldots, k$. From Lemma 4.2 we have

$$E_\theta|N_i - R_i| = o(1) \quad \text{as } W \to 0, \quad i = 1, \ldots, k,$$

which implies that as $W \to 0$,

$$\begin{aligned}E_\theta(N_i - n_i^*) &= E_\theta(R_i - n_i^*) + o(1) \\ &= E_\theta(F_i - n_i^*) + E_\theta(1 - J_i) + o(1) \quad i = 1, \ldots, k. \qquad (4.2)\end{aligned}$$

By the same method as the proof of (18) in Takada [6], we have

$$E_\theta(1 - J_i) = \frac{1}{2} + o(1), \quad i = 1, \ldots, k. \quad \text{as } W \to 0. \qquad (4.3)$$

Substituting (4.3) into (4.2) yields

$$E_\theta(N_i - n_i^*) = E_\theta(F_i - n_i^*) + \frac{1}{2} + o(1), \quad i = 1, \ldots, k, \qquad (4.4)$$

as $W \to 0$. Since

$$E_\theta V_j = \sigma_j\left(1 - \frac{1}{4\nu}\right) + o\left(\frac{1}{\nu}\right) \quad \text{as } m \to \infty,$$

it follows from (1.6) and (4.1) that

$$\begin{aligned} E_\theta F_i &= \frac{a^2}{2W}\left(1+\frac{\ell_o}{m}+o\left(\frac{1}{m}\right)\right)\left(\left(\sum_{j=1}^{k}|\lambda_j|\sigma_j\right)|\lambda_i|\sigma_i - \frac{1}{2\nu}\left(\sum_{j\neq i}^{k}|\lambda_j|\sigma_j\right)|\lambda_i|\sigma_i \right.\\ &\quad \left. +o\left(\frac{1}{\nu}\right)\right)\\ &= n_i^* + \frac{a^2}{2Wm}\left(\ell_o\left(\sum_{j=1}^{k}|\lambda_j|\sigma_j\right)|\lambda_i|\sigma_i - \frac{1}{2}\left(\sum_{j\neq i}^{k}|\lambda_j|\sigma_j\right)|\lambda_i|\sigma_i\right) + o\left(\frac{1}{W\nu}\right). \end{aligned}$$

Hence

$$\begin{aligned} &\lim_{W\to 0} W^{1-d}E_\theta(F_i - n_i^*)\\ &\quad = \frac{1}{c}\left(\ell_o\left(\sum_{j=1}^{k}|\lambda_j|\sigma_j\right)|\lambda_i|\sigma_i - \frac{1}{2}\left(\sum_{j\neq i}^{k}|\lambda_j|\sigma_j\right)|\lambda_i|\sigma_i\right), \end{aligned}$$

which completes the proof from (4.4).

Proof of Lemma 2.3

For any $\epsilon\,(0<\epsilon<1)$ and any $\delta(0<\delta<1)$ let

$$C = \bigcap_{i=1}^{k}\left(\{N_i \geq \epsilon n_i^*\}\bigcap\{|V_{i(N_i)}^2 - \sigma_i^2| < \delta\}\right).$$

Then

$$P_\theta(\bar{C}) \leq \sum_{i=1}^{k}\left\{P_\theta(N_i < \epsilon n_i^*) + P_\theta\left(\sup_{n>\epsilon n_i^*}|V_{i(n)}^2 - \sigma_i^2| > \delta\right)\right\}. \tag{4.5}$$

Since $\{|V_{i(n)}^2 - \sigma_i^2|^q\}$ with $q>1$ is a reverse submartimgale,

$$P_\theta\left(\sup_{n>\epsilon n_i^*}|V_{i(n)}^2 - \sigma_i^2| > \delta\right) \leq \frac{1}{\delta^q}E_\theta|V_{i(\epsilon n_i^*)} - \sigma_i^2|^q = O\left(n_i^{*-q/2}\right).$$

Hence it follows from Lemma 4.4 and (4.5) that for any $p(>1)$

$$P_\theta(\bar{C}) = O\left(W^p\right) \quad \text{as } W\to 0. \tag{4.6}$$

Let

$$\begin{aligned} E_\theta(W_N) &= \int_C W_N dP_\theta + \int_{\bar{C}} W_N dP_\theta\\ &= II_1 + II_2 \quad \text{(say)}. \end{aligned}$$

The Taylor expansion of e^x shows

$$\begin{aligned} II_1 &= \int_C \frac{1}{2}\left(\frac{a^2}{2}\sum_{i=1}^k \frac{\lambda_i^2}{N_i}(V_{i(N_i)}-\sigma_i^2)\right)^2 \exp\Delta_N dP_\theta \\ &\leq \frac{ka^2}{8}\int_C \sum_{i=1}^k \frac{\lambda_i^4}{N_i^2}(V_{i(N_i)}-\sigma_i^2)^2 \exp\Delta_N dP_\theta \\ &= \frac{2kW^4}{\left(\sum_{j=1}^k |\lambda_j|\sigma_j\right)^4 a^4}\sum_{i=1}^k \frac{n_i^*}{\sigma_i^4}\int_C \left(\frac{n_i^*}{N_i}\right)^3 N_i(V_{i(N_i)}-\sigma_i^2)^2 \exp\Delta_N dP_\theta \end{aligned}$$

where

$$|\Delta_N| \leq \frac{a^2}{2}\sum_{i=1}^k \frac{\lambda_i^2}{N_i}|V_{i(N_i)}-\sigma_i^2| \leq \frac{a^2\delta}{2m}\sum_{i=1}^k \lambda_i^2$$

on C. From Lemma 4.3 it is easy to see that $\left(\frac{n_i^*}{N_i}\right)^3 N_i(V_{i(N_i)}-\sigma_i^2)^2 \exp\Delta_N$ converges in distribution to $2\sigma_i^4\chi_1^2$ as $W \to 0$ and is $u.i.$ on C. Then we have

$$\int_C \left(\frac{n_i^*}{N_i}\right)^3 N_i(V_{i(N_i)}-\sigma_i^2)^2 \exp\Delta_N dP_\theta = 2\sigma_i^4 + o(1) \quad \text{as } W \to 0,$$

so that

$$II_1 = O\left(W^3\right) \quad \text{as } W \to 0.$$

Hence it is enough to show that

$$II_2 = O\left(W^3\right) \quad \text{as } W \to 0.$$

Since $N_i \geq m$, we have

$$\begin{aligned} 0 \leq II_2 &\leq \exp\left(\frac{a^2}{2m}\sum_{i=1}^k \lambda_i^2\sigma_i^2\right) P_\theta(\bar{C}) + \frac{a^2}{2m}\int_{\bar{C}}\sum_{i=1}^k \lambda_i^2|V_{i(N_i)}-\sigma_i^2| dP_\theta \\ &\leq \exp\left(\frac{a^2}{2m}\sum_{i=1}^k \lambda_i^2\sigma_i^2\right) P_\theta(\bar{C}) + \frac{a^2}{2m}K^{1/2}P_\theta(\bar{C})^{1/2}, \end{aligned}$$

where

$$K = \left(\sum_{i=1}^k \lambda_i^4\right)\sum_{i=1}^k E_\theta(V_{i(N_i)}-\sigma_i^2)^2,$$

which is finite from the reverse submartingale property of $\{(V_{i(n)}^2-\sigma_i^2)^2\}$. Then the result follows from (4.6).

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Yoshikazu Takada
Department of Computer Science
Kumamoto University
Kumamoto, 860-8555
Japan
e-mail: takada@kumamoto-u.ac.jp