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A certain reduction of a single differential equation to a system of differential equations

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Abstract. When one attempts to establish some theory of differential equations, it seems to be very effective to treat systems of differential equations rather than single differential equations. In this short paper, we shall show a method of reduction of every linear differential equation with a finite number of regular singularities and one irregular singularity to a system of linear differential equations of the form $(t-B)\frac{dX}{dt}=(A+Ct)X$.

1. Introduction

In the note [3], one of authors considers the global analysis for the single linear differential equation

$$\left(t^2-1\right)t^2y''+\left(3t^2-1\right)ty'+\left\{\alpha^2t^4-\left(3\beta^2+\gamma\right)t^2+\beta^2\right\}y=0,$$

which has three regular singular points at $t=0,\pm1$ and an irregular singularity at infinity. And the study extends to the system of linear differential equations

$$(t-B)\frac{dX}{dt}=(A+Ct)X, \tag{1}$$

which is closely related with the above single linear differential equation. In fact, differentiating the single differential equation two times, he obtains the single linear differential equation

$$t^2\left(t^2-1\right)y^{(4)}=P_3(t)y^{(3)}+P_2(t)y''+P_1(t)y'+P_0(t)y,$$

where

$$\begin{aligned}
P_3(t)&=-11t^3+5t,\\
P_2(t)&=-\alpha^2t^4+\left(3\beta^2+\gamma-30\right)t^2-\beta^2+4,\\
P_1(t)&=-8\alpha^2t^3+2\left(6\beta^2+2\gamma-9\right)t,\\
P_0(t)&=-12\alpha^2t^2+2\left(3\beta^2+\gamma\right),
\end{aligned}$$

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and then reduces it to a system of linear differential equations of the form (1) by means of the transformation

$$\begin{cases} y_1 = y, \\ y_2 = \varphi_1 y' + e_{2,0} y, \\ y_3 = \varphi_2 y'' + e_{3,1} y' + e_{3,0} y, \\ y_4 = \varphi_3 y^{(3)} + e_{4,2} y'' + e_{4,1} y' + e_{4,0} y \end{cases}$$

with $\varphi_1 = (t-1)$, $\varphi_2 = \left(t^2-1\right)$ and $\varphi_3 = t\left(t^2-1\right)$.

Actually, it can be verified that the transformation

$$\begin{cases} e_{2,0}(t) = \alpha i\,(t-1)\,, \\ e_{3,0}(t) = \alpha^2 t^2 - \alpha i t - \left(2\beta^2 + \gamma - \alpha i\right), \\ e_{3,1}(t) = 2t, \\ e_{4,0}(t) = -3\alpha^2 t^2 + \left(\beta^2+6\right)\alpha i t + 8\beta^2 + 5\gamma - \left(\beta^2+6\right)\alpha i, \\ e_{4,1}(t) = \alpha^2 t^3 - \left(2\beta^2 + \gamma + 6\right) t, \\ e_{4,2}(t) = 4 \end{cases}$$

leads to a system of linear differential equations of the form (1) with

$$B = \operatorname{diag}(1,-1,0,0)\,,$$

$$A = \begin{pmatrix} \alpha i & 1 & 0 & 0 \\ 2\beta^2 + \gamma - \alpha^2 - \alpha i & \alpha i - 1 & 1 & 0 \\ \left(\beta^2+2\right)\alpha i - \gamma & 0 & 4 & 1 \\ \eta & 0 & \beta^2 - 36 & -8 \end{pmatrix}$$

and

$$C = \begin{pmatrix} -\alpha i & 0 & 0 & 0 \\ 3\alpha i & \alpha i & 0 & 0 \\ \alpha^2 - \left(\beta^2+3\right)\alpha i & -\alpha i & 0 & 0 \\ \xi & \left(\beta^2+6\right)\alpha i & 0 & 0 \end{pmatrix},$$

where

$$\xi = -\alpha^2\left(\beta^2+6\right) + 2\left(5\beta^2+9\right)\alpha i,$$
$$\eta = 2\beta^4 - 2\beta^2 + \left(\beta^2+6\right)\gamma - 3\left(3\beta^2+4\right)\alpha i.$$

In this short paper, we shall consider the reduction of any single linear differential equation of the following general form

$$\varphi_n(t) y^{(n)} = P_{n-1}(t) y^{(n-1)} + \cdots + P_1(t) y' + P_0(t) y, \tag{2}$$

where

$$\varphi_n(t) = \prod_{j=1}^{n} (t - \lambda_j)$$

and the coefficients $P_j(t)$ $(j = 0, 1, \ldots, n-1)$ are polynomials of degree at most n, to the system of linear differential equations (1) with

$$B = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)\,.$$

From now on, we assume that

$$\lambda_i \neq \lambda_j \quad (i \neq j).$$

Then, obviously, the single linear differential equation has n regular singularities at $t = \lambda_j$ and an irregular singularity at $t = \infty$, if at least one $P_j(t)$ is a polynomial of degree greater than $j+1$.

2. Transformation for Reduction

In order to attain such a reduction, we put

$$(3) \qquad \begin{cases} y_1 = y, \\ y_2 = \varphi_1 y' + e_{2,0} y, \\ \quad \vdots \\ y_j = \varphi_{j-1} y^{(j-1)} + e_{j,j-2} y^{(j-2)} + \cdots + e_{j,1} y' + e_{j,0} y, \\ \quad \vdots \\ y_n = \varphi_{n-1} y^{(n-1)} + e_{n,n-2} y^{(n-2)} + \cdots + e_{n,1} y' + e_{n,0} y, \end{cases}$$

where

$$\varphi_j = \prod_{k=1}^{j} (t - \lambda_k) \quad (j = 1, 2, \ldots, n-1), \quad \varphi_0 \equiv 1.$$

and attempt to derive a system of linear differential equations for the column vector $Y = (y_1, y_2, \ldots, y_n)_*$:

$$\begin{pmatrix} t-\lambda_1 & & & 0 \\ & t-\lambda_2 & & \\ & & \ddots & \\ 0 & & & t-\lambda_n \end{pmatrix} Y' = \begin{pmatrix} d_{1,1} & 1 & & & 0 \\ d_{2,1} & d_{2,2} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & 1 \\ d_{n,1} & d_{n,2} & \cdots & \cdots & d_{n,n} \end{pmatrix} Y,$$

where all $d_{j,i}(t)$ are polynomials of the first degree.

Now we shall derive relations which accomplish such a reduction. The first element is caluculated as follows:

$$(t-\lambda_1)\, y_1' = \varphi_1 y' = y_2 - e_{2,0} y_1 \equiv y_2 + d_{1,1} y_1,$$

in which we have put

$$d_{1,1} = -e_{2,0}.$$

Then, we calculate

$$\begin{aligned}(t-\lambda_2)\,y_2' &= (t-\lambda_2)\left(\varphi_1 y'' + \varphi_1' y' + e_{2,0} y' + e_{2,0}' y\right)\\ &= \underline{\varphi_2 y''} + \{(t-\lambda_2)(\varphi_1' + e_{2,0})\}\,y' + (t-\lambda_2)\,e_{2,0}' y\\ &= y_3 + \{(t-\lambda_2)(\varphi_1' + e_{2,0}) - e_{3,1}\}\,y' + \left\{(t-\lambda_2)\,e_{2,0}' - e_{3,0}\right\} y.\end{aligned}$$

Here we put

$$\underline{(t-\lambda_2)(\varphi_1' + e_{2,0}) - e_{3,1} = d_{2,2}\varphi_1}$$

and then proceed to the next step of calculation

$$\begin{aligned}(t-\lambda_2)\,y_2' &= y_3 + d_{2,2}\,(y_2 - e_{2,0} y) + \left\{(t-\lambda_2)\,e_{2,0}' - e_{3,0}\right\} y\\ &= y_3 + d_{2,2} y_2 + \left\{(t-\lambda_2)\,e_{2,0}' - e_{3,0} - d_{2,2} e_{2,0}\right\} y_1\\ &= y_3 + d_{2,2} y_2 + d_{2,1} y_1,\end{aligned}$$

where we have put

$$\underline{(t-\lambda_2)\,e_{2,0}' - e_{3,0} - d_{2,2} e_{2,0} = d_{2,1}}.$$

Hereafter we use the following notations:

(4) $$e_{1,0} \equiv 1, \quad e_{j,j-1} \equiv \varphi_{j-1}, \quad e_{j,-k} \equiv 0 \quad (k > 0).$$

In exactly the same manner as above, taking account of

$$y_j = \sum_{k=0}^{j-1} e_{j,k} y^{(k)}$$

or

$$\varphi_{j-1} y^{(j-1)} = y_j - \sum_{k=0}^{j-2} e_{j,k} y^{(k)},$$

we shall calculate the j-th element as follows:

$$\begin{aligned}(t-\lambda_j)\,y_j' &= (t-\lambda_j)\left\{\sum_{k=0}^{j-1} e_{j,k} y^{(k+1)} + \sum_{k=0}^{j-1} e_{j,k}' y^{(k)}\right\}\\ &= \underline{\varphi_j y^{(j)}} + (t-\lambda_j)\left\{\sum_{k=0}^{j-1}\left(e_{j,k-1} + e_{j,k}'\right) y^{(k)}\right\}\\ &= y_{j+1} + \sum_{k=0}^{j-1}\left\{(t-\lambda_j)\left(e_{j,k-1} + e_{j,k}'\right) - e_{j+1,k}\right\} y^{(k)}.\end{aligned}$$

Here we put

$$\underset{\sim\sim\sim}{(t-\lambda_j)\left(e_{j,j-2}+e'_{j,j-1}\right)-e_{j+1,j-1}=d_{j,j}\varphi_{j-1}}$$

and then obtain

$$(t-\lambda_j)\,y'_j=y_{j+1}+d_{j,j}y_j+\sum_{k=0}^{j-2}\left\{(t-\lambda_j)\left(e_{j,k-1}+e'_{j,k}\right)-e_{j+1,k}-d_{j,j}e_{j,k}\right\}y^{(k)}.$$

When we continue to calculate and obtain

$$\begin{aligned}(t-\lambda_j)\,y'_j&=y_{j+1}+d_{j,j}y_j+\cdots+d_{j,j-l}y_{j-l}\\&\quad+\sum_{k=0}^{j-l-2}\left\{(t-\lambda_j)\left(e_{j,k-1}+e'_{j,k}\right)-e_{j+1,k}-\sum_{h=0}^{l}d_{j,j-h}e_{j-h,k}\right\}y^{(k)},\end{aligned}$$

we put

$$\underset{\sim\sim\sim}{(t-\lambda_j)\left(e_{j,j-l-3}+e'_{j,j-l-2}\right)-e_{j+1,j-l-2}-\sum_{h=0}^{l}d_{j,j-h}e_{j-h,j-l-2}=d_{j,j-l-1}\varphi_{j-l-2},}$$

obtaining

$$\begin{aligned}(t-\lambda_j)\,y'_j&=y_{j+1}+d_{j,j}y_j+\cdots+d_{j,j-l}y_{j-l}+d_{j,j-l-1}y_{j-l-1}\\&\quad+\sum_{k=0}^{j-l-3}\left\{(t-\lambda_j)\left(e_{j,k-1}+e'_{j,k}\right)-e_{j+1,k}-\sum_{h=0}^{l+1}d_{j,j-h}e_{j-h,k}\right\}y^{(k)}.\end{aligned}$$

Consequently, the procedure yields a sequence of relations

$$\left\{\begin{aligned}&(t-\lambda_j)\left(e_{j,j-2}+e'_{j,j-1}\right)-e_{j+1,j-1}=d_{j,j}\varphi_{j-1},\\&\quad(t-\lambda_j)\left(e_{j,j-l-3}+e'_{j,j-l-2}\right)-e_{j+1,j-l-2}-\sum_{h=0}^{l}d_{j,j-h}e_{j-h,j-l-2}\\&=d_{j,j-l-1}\varphi_{j-l-2}\quad(l=0,1,\ldots,j-2).\end{aligned}\right.$$

In particular, for $j=n$, we have to use the single linear differential equation

$$\varphi_n y^{(n)}=\sum_{k=0}^{n-1}P_k y^{(k)}$$

in the first stage of calculation

$$\begin{aligned}(t-\lambda_n)\,y'_n&=\varphi_n y^{(n)}+(t-\lambda_n)\left\{\sum_{k=0}^{n-1}\left(e_{n,k-1}+e'_{n,k}\right)y^{(k)}\right\}\\&=\sum_{k=0}^{n-1}\left\{(t-\lambda_n)\left(e_{n,k-1}+e'_{n,k}\right)+P_k\right\}y^{(k)}\end{aligned}$$

and continue the above procedure.

We have thus derived the following relations for reduction.

Theorem. *For* $j = 1, 2, \ldots, n$ *and* $l = -1, 0, \ldots, j-2$,

$$(5) \qquad (t-\lambda_j)\left(e_{j,j-l-3} + e'_{j,j-l-2}\right) = e_{j+1,j-l-2} + \sum_{h=0}^{l+1} d_{j,j-h} e_{j-h,j-l-2},$$

where

$$e_{n+1,k} = -P_k \quad (k = 0, 1, \ldots, n-1)$$

together with (4).

3. Determination of the $d_{j,i}$

We shall now explain the determination of the $e_{j,i}$ and $d_{j,i}$ from (5).

First we consider the relation for $j = n$ and $l = -1$ in (5):

$$(t-\lambda_n)\left(e_{n,n-2} + \varphi'_{n-1}\right) = d_{n,n}\varphi_{n-1} - P_{n-1}.$$

Since the right hand side is a polynomial of degree at most n, it must include the divisor $(t-\lambda_n)$, and hence

$$d_{n,n}\varphi_{n-1} - P_{n-1} \,|_{t=\lambda_n} = 0.$$

Then we see that $e_{n,n-2}$ is determined as a polynomial of degree at most $(n-1)$. Hereinafter, we say "a polynomial of degree at most k" simply by "a polynomial of degree k". Similarly, from the relation for $j = n$ and $l = 0$ in (5), we have

$$d_{n,n}e_{n,n-2} + d_{n,n-1}\varphi_{n-2} - P_{n-2} \,|_{t=\lambda_n} = 0$$

and obtain the polynomial $e_{n,n-3}$ of degree $(n-1)$.

Next, we consider the relation for $j = n$ and $l = 1$ in (5):

$$(t-\lambda_n)\left(e_{n,n-4} + e'_{n,n-3}\right) = d_{n,n}e_{n,n-3} + d_{n,n-1}\underset{\sim\sim\sim}{e_{n-1,n-3}} + d_{n,n-2}\varphi_{n-3} - P_{n-3},$$

in which there is a polynomial $e_{n-1,n-3}$ not yet determined. One can determine it by the relation for $j = n-1$ and $l = -1$ in (5):

$$(t-\lambda_{n-1})\left(e_{n-1,n-3} + \varphi'_{n-2}\right) = e_{n,n-2} + d_{n-1,n-1}\varphi_{n-2}.$$

Again, the right hand side is a polynomial of degree $(n-1)$ and hence, by

$$e_{n,n-2} + d_{n-1,n-1}\varphi_{n-2} \,|_{t=\lambda_{n-1}} = 0,$$

$e_{n-1,n-3}$ can be seen to be determined as a polynomial of degree $(n-2)$. From this, putting

$$d_{n,n}e_{n,n-3} + d_{n,n-1}e_{n-1,n-3} + d_{n,n-2}\varphi_{n-3} - P_{n-3} \,|_{t=\lambda_n} = 0,$$

we can determine $e_{n,n-4}$ as a polynomial of degree $(n-1)$.

The relation (5) for $j = n-1$ and $l = 0$:

$$(t-\lambda_{n-1})\left(e_{n-1,n-4} + e'_{n-1,n-3}\right) = d_{n-1,n-1}e_{n-1,n-3} + d_{n-1,n-2}\varphi_{n-3} + e_{n,n-3}$$

leads to the determination of $e_{n-1,n-4}$ as a polynomial of degree $(n-2)$, since the right hand side in the above formula is of degree $(n-1)$.

Moreover, the relation for $j = n-2$ and $l = -1$:

$$(t-\lambda_{n-2})\left(e_{n-2,n-4} + \varphi'_{n-3}\right) = e_{n-1,n-3} + d_{n-2,n-2}\varphi_{n-3}$$

gives $e_{n-2,n-4}$ as a polynomial of degree $(n-3)$.

The above order of calculation is as follows:

$$\begin{array}{ccccc}
e_{n,n-2}\ (d_{n,n}) & & & & \\
\Downarrow & \searrow & & & \\
e_{n,n-3}\ (d_{n,n-1}) & & e_{n-1,n-3}\ (d_{n-1,n-1}) & & \\
\Downarrow & \swarrow & \downarrow & \searrow & \\
e_{n,n-4}\ (d_{n,n-2}) & \rightarrow & e_{n-1,n-4}\ (d_{n-1,n-2}) & \rightarrow & e_{n-2,n-4}\ (d_{n-2,n-2})
\end{array}$$

We shall now prove by mathematical induction that each $e_{j,i}$ can be determined as a polynomial of degree $(j-1)$.

Suppose that we have determined

$$e_{j,i} \quad (i = n-2, n-3, \ldots, n-k;\ j-i \leq 2)$$

till $(k-1)$-th line (row) from the first line $e_{n,n-2}$, where each $e_{j,i}$ is a polynomial of degree $(j-1)$.

Then the k-th line begins with

$$(t-\lambda_n)\left(e_{n,n-k-1} + e'_{n,n-k}\right) = \sum_{h=0}^{k-2} d_{n,n-h}e_{n-h,n-k} + d_{n,n-k+1}\varphi_{n-k} - P_{n-k},$$

from which it is easily seen that $e_{n,n-k-1}$ is determined as a polynomial of degree $(n-1)$, since the right hand side is of degree n.

Here, suppose that $e_{n-1,n-k-1}, e_{n-2,n-k-1}, \ldots, e_{n-l,n-k-1}$ are determined as polynomials of degree $(n-2), (n-3), \ldots, (n-l)$, respectively. Then, for $1 \leq l \leq k-2$ the relation

$$\begin{aligned}
&(t-\lambda_{n-l-1})\left(e_{n-l-1,n-k-1} + e'_{n-l-1,n-k}\right) \\
&= \sum_{h=0}^{k-l-3} d_{n-l-1,n-l-1-h}e_{n-l-1-h,n-k} + d_{n-l-1,n-k+1}\varphi_{n-k} + e_{n-l,n-k}
\end{aligned}$$

determines $e_{n-l-1,n-k-1}$ as a polynomial of degree $(n-l-2)$, since the right hand side is of degree $n-l-1$, because of $n-k+1 \leq n-l-1$. We have thus proved that each $e_{j,i}$ $(2 \le j \le n,\ 0 \le i \le j-2)$ can be determined as a polynomial of degree $(j-1)$.

In the above procedure, for $j = n, n-1, \ldots, 2$ and $l = -1, 0, \ldots, j-3$ we have put

$$(6) \qquad e_{j+1,j-l-2} + \sum_{h=0}^{l+1} d_{j,j-h} e_{j-h,j-l-2} \Big|_{t=\lambda_j} = 0$$

and consequently obtained

$$(7) \qquad (t-\lambda_j)\, e'_{j,0} - e_{j+1,0} - \sum_{h=0}^{j-2} d_{j,j-h} e_{j-h,0} = d_{j,1}.$$

Since the left hand side of (7) is of degree j and however, the right hand side is a polynomial of degree 1, we have to assign zero to the coefficients of t^l $(l = j, j-1, \ldots, 2)$ in the left hand side.

By these relations (6) and (7), we can determine all $d_{j,i}$ as polynomials of degree 1. In fact, for each j we have $(j-1)$ relations of (6) and $(j-1)$ relations from (7), whence one can determine $2(j-1)$ coefficients of the $d_{j,i}$ $(i = j, j-1, \ldots, 2)$, and lastly obtain $d_{j,1}$ just by (7).

Consequently, we can determine

$$\sum_{j=1}^{n} 2j = n(n+1)$$

coefficients of the $d_{j,i}$ $(j = 1, 2, \ldots, n;\ i = 1, 2, \ldots, j)$ by the same number of coefficients of the polynomials P_j $(j = 0, 1, \ldots, n-1)$ of degree n.

Example. Consider

$$(t-\lambda_1)(t-\lambda_2)\, y'' = P_1(t) y' + P_0(t) y,$$

where

$$P_1(t) = a_1 t^2 + b_1 t + c_1, \quad P_0(t) = a_0 t^2 + b_0 t + c_0.$$

Then, putting $e_{2,1} = (t-\lambda_1)$, we have the following formulas of reduction:

① $(t-\lambda_2)\left(e_{2,0} + e'_{2,1}\right) = d_{2,2}(t-\lambda_1) - P_1,$

② $(t-\lambda_2)\, e'_{2,0} = d_{2,2} e_{2,0} + d_{2,1} - P_0,$

③ $d_{1,1} + e_{2,0} = 0.$

We seek

$$d_{2,2} = \beta_2 t + \alpha_2, \quad d_{2,1} = \beta_{2,1} t + \alpha_{2,1}, \quad d_{1,1} = \beta_1 t + \alpha_1.$$

① leads to

$$(\beta_2\lambda_2+\alpha_2)(\lambda_2-\lambda_1)=a_1\lambda_2^2+b_1\lambda_2+c_1$$

and

$$e_{2,0}=(\beta_2-a_1)t-\{a_1\lambda_2+b_1-\alpha_2-\beta_2(\lambda_2-\lambda_1)+1\}.$$

Then, ② becomes

$$\begin{aligned}&(t-\lambda_2)(\beta_2-a_1)\\&=\underset{\sim\sim\sim\sim\sim\sim\sim\sim}{\{\beta_2(\beta_2-a_1)-a_0\}}t^2+\{\beta_{2,1}+\alpha_2(\beta_2-a_1)+\gamma\beta_2-b_0\}t+(\alpha_{2,1}+\gamma\alpha_2-c_0),\end{aligned}$$

where

$$\gamma=-\{a_1\lambda_2+b_1-\alpha_2-\beta_2(\lambda_2-\lambda_1)+1\}.$$

Hence, we have

$$\begin{cases}\beta_2(\beta_2-a_1)-a_0=0,\\ \beta_{2,1}=-(\alpha_2-1)(\beta_2-a_1)-\gamma\beta_2+b_0,\\ \alpha_{2,1}=-\lambda_2(\beta_2-a_1)-\gamma\alpha_2+c_0\end{cases}$$

and ③ immediately leads to

$$\begin{cases}\beta_1=-(\beta_2-a_1),\\ \alpha_1=-\gamma.\end{cases}$$

From the above relations, we easily see that β_1,β_2 are roots of the equation

$$\nu^2-a_1\nu-a_0=0$$

and then

$$\begin{aligned}\alpha_2&=\frac{(\beta_1\lambda_2+\beta_2\lambda_1)\lambda_2+b_1\lambda_2+c_1}{\lambda_2-\lambda_1},\\ \gamma&=\frac{(\beta_1\lambda_2+\beta_2\lambda_1)\lambda_1+b_1\lambda_1+c_1}{\lambda_2-\lambda_1}-1.\end{aligned}$$

Substituting these results into other relations, we can consequently determine all $d_{j,i}$. We here pay an attention to the following values

$$\begin{aligned}\beta_2\lambda_2+\alpha_2&=\frac{a_1\lambda_2^2+b_1\lambda_2+c_1}{\lambda_2-\lambda_1},\\ \beta_1\lambda_1+\alpha_1&=\frac{a_1\lambda_1^2+b_1\lambda_1+c_1}{\lambda_1-\lambda_2}+1,\end{aligned}$$

the meaning of which will be explained in the next section.

4. Characteristic Exponents and Constants

The single linear differential equation

$$\varphi_n(t)y^{(n)} = P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y$$

has regular singularities at $t = \lambda_j$ ($j = 1, 2, \ldots, n$), which are roots of $\varphi_n(t) = 0$. The characteristic equation at each regular singular point $t = \lambda_j$ is

$$\rho(\rho-1)\cdots(\rho-n+1) = \frac{P_{n-1}(\lambda_j)}{\varphi_n'(\lambda_j)}\rho(\rho-1)\cdots(\rho-n+2).$$

So, the characteristic exponents are

$$\rho = 0, 1, \ldots, n-2, n-1+\frac{P_{n-1}(\lambda_j)}{\varphi_n'(\lambda_j)}, \tag{8}$$

which implies that near each regular singular point there exist $(n-1)$ holomorphic solutions and one nonholomorphic solution.

On the other hand, near the irregular singular point at infinity one can find formal solutions of the form

$$y(t) = e^{\nu t}t^{\rho}\sum_{s=0}^{\infty} h(s)t^{-s}.$$

The characteristic constant ν is one of roots of the characteristic equation

$$\nu^n = P_{n-1}^0\nu^{n-1} + P_{n-2}^0\nu^{n-2} + \cdots + P_1^0\nu + P_0^0, \tag{9}$$

where P_j^0 are the coefficients of t^n of $P_j(t)$, i.e.,

$$P_j(t) = P_j^0 t^n + \cdots \quad (j = 0, 1, \ldots, n-1).$$

For the system of linear differential equations

$$(t-B)\frac{dX}{dt} = (A+Ct)X,$$

there also exist $(n-1)$ holomorphic solutions and one nonholomorphic solution near each regular singular point $t = \lambda_j$. Then, the characteristic exponent ρ_j of the nonholomorphic solution is given by

$$\rho_j = a_{j,j} + \lambda_j c_{j,j}, \tag{10}$$

where $a_{j,j}$ and $c_{j,j}$ are the j-th diagonal elements of A and C, respectively.

As for the characteristic constants of formal solutions, it is also not difficult to see that they are equal to eigenvalues of C.

Now we shall show that the transformation of reduction described above preserves characteristic properties. To see this, we have only to consider the diagonal elements $d_{j,j}$.

From the relations

$$(11)\qquad (t-\lambda_j)\left(e_{j,j-2}+\varphi'_{j-1}\right)=e_{j+1,j-1}+d_{j,j}\varphi_{j-1}\quad (j=1,2,\ldots,n),$$

we have given formulas of determining $d_{j,j}$ as follows:

$$e_{j+1,j-1}(t)+d_{j,j}(t)\varphi_{j-1}(t)\ |_{t=\lambda_j}=0.$$

Each value of $d_{j,j}(\lambda_j)$ exactly corresponds to (10).

In order to calculate $d_{j,j}(\lambda_j)$, we multiply both sides of (11) by

$$\prod_{k=j+1}^{n}(t-\lambda_k)$$

and obtain

$$\left[\prod_{k=j}^{n}(t-\lambda_k)\right]e_{j,j-2}-\left[\prod_{k=j+1}^{n}(t-\lambda_k)\right]e_{j+1,j-1}$$
$$=d_{j,j}\left[\prod_{\substack{k=1\\k\neq j}}^{n}(t-\lambda_k)\right]-\left[\prod_{k=j}^{n}(t-\lambda_k)\right]\varphi'_{j-1}$$

Then, summing up these formulas from $j=1$ to $j=n$ and taking account of the notation that $e_{n+1,n-1}=-P_{n-1}$ and $e_{1,-1}=0$, we consequently obtain the formula

$$\sum_{j=1}^{n}d_{j,j}\left[\prod_{\substack{k=1\\k\neq j}}^{n}(t-\lambda_k)\right]=P_{n-1}+\sum_{j=2}^{n}\left[\prod_{k=j}^{n}(t-\lambda_k)\right]\varphi'_{j-1}.$$

If we put $t=\lambda_l$ in the above formula, then we have

$$d_{l,l}(\lambda_l)\left[\prod_{\substack{k=1\\k\neq l}}^{n}(\lambda_l-\lambda_k)\right]=P_{n-1}(\lambda_l)+\sum_{j=l+1}^{n}\left[\prod_{k=j}^{n}(\lambda_l-\lambda_k)\right]\varphi'_{j-1}$$
$$=P_{n-1}(\lambda_l)+(n-l)\left[\prod_{\substack{k=1\\k\neq l}}^{n}(\lambda_l-\lambda_k)\right],$$

whence

$$\rho_l\equiv d_{l,l}(\lambda_l)=\frac{P_{n-1}(\lambda_l)}{\varphi'_n(\lambda_l)}+(n-l).$$

This value differs only by an integer from the characteristic exponent of the single linear differential equation (2). Therefore, the behavior of solutions for the reduced system (1) is same as that for (2) near regular singularities.

Next we shall consider the characteristic constants at infinity. Since the matrix C for the reduced system of linear differential equations is triangular, the diagonal elements of C are eigenvalues. So, putting

$$d_{j,j} = \beta_j t + \alpha_j \quad (j = 1, 2, \dots, n),$$

we have only to investigate the values of β_j.

We take up the coefficients of the highest degree in the formula (5):

$$(t - \lambda_j)\left(e_{j,j-l-3} + e'_{j,j-l-2}\right) = e_{j+1,j-l-2} + \sum_{h=0}^{l+1} d_{j,j-h} e_{j-h,j-l-2}.$$

The coefficients of t^j are derived from $e_{j,j-l-3}$ in the left hand side and $e_{j+1,j-l-2}$, $d_{j,j}e_{j,j-l-2}$ in the right hand side. Hence, we have

$$(12) \qquad e^0_{j,j-l-3} - \beta_j e^0_{j,j-l-2} = e^0_{j+1,j-l-2} \quad (l = -1, 0, \dots, j-2),$$

where we have put

$$e_{j,k}(t) = e^0_{j,k} t^{j-1} + \cdots.$$

Multiplying (12) for $j = n$ by β_n^{n-l-2} and summing them up from $l = -1$ to $n-2$, we immediately obtain

$$\beta_n^n = P^0_{n-1}\beta_n^{n-1} + P^0_{n-2}\beta_n^{n-2} + \cdots + P^0_1\beta_n + P^0_0,$$

which implies that β_n is a root of the characteristic equation (9).

Also, multiplying (12) by β_j^{j-l-2} and summing them up from $l = -1$ to $j-2$, we obtain

$$\beta_j^j + e^0_{j+1,j-1}\beta_j^{j-1} + e^0_{j+1,j-2}\beta_j^{j-2} + \cdots + e^0_{j+1,1}\beta_j + e^0_{j+1,0} = 0.$$

These imply that β_j $(j = n, n-1, \dots, 1)$ are also roots of the characteristic equation (9). In fact, the relations (12) are the so-called Euclidean Algorithm:

$$\begin{array}{cccccc|l} 1 & e^0_{j+1,j-1} & e^0_{j+1,j-2} & \cdots & e^0_{j+1,1} & e^0_{j+1,0} & \beta_j \\ & \beta_j & \beta_j e^0_{j,j-2} & \cdots & \beta_j e^0_{j,1} & \beta_j e^0_{j,0} & \\ \hline 1 & e^0_{j,j-2} & e^0_{j,j-3} & \cdots & e^0_{j,0} & 0 & \end{array}$$

We have thus verified that the properties of solutions are not changed by our transformation of reduction.

5. Application

We here take up the fourth order single linear differential equation described in the introduction

$$t^2\left(t^2 - 1\right) y^{(4)} = P_3(t) y^{(3)} + P_2(t) y'' + P_1(t) y' + P_0(t) y.$$

Putting $\varphi_1 = (t-1)$, $\varphi_2 = \left(t^2-1\right)$ and $\varphi_3 = t\left(t^2-1\right)$, we have the following relations of determination of the $d_{j,i}(t)$:

① $t\left(e_{4,2}+\varphi_3'\right) = -P_3 + d_{4,4}\varphi_3$,

② $t\left(e_{4,1}+e_{4,2}'\right) = -P_2 + d_{4,4}e_{4,2} + d_{4,3}\varphi_2$,

③ $t\left(e_{4,0}+e_{4,1}'\right) = -P_1 + d_{4,4}e_{4,1} + d_{4,3}e_{3,1} + d_{4,2}\varphi_1$,

④ $te_{4,0}' = -P_0 + d_{4,4}e_{4,0} + d_{4,3}e_{3,0} + d_{4,2}e_{2,0} + d_{4,1}$,

⑤ $t\left(e_{3,1}+\varphi_2'\right) = e_{4,2} + d_{3,3}\varphi_2$,

⑥ $t\left(e_{3,0}+e_{3,1}'\right) = e_{4,1} + d_{3,3}e_{3,1} + d_{3,2}\varphi_1$,

⑦ $te_{3,0}' = e_{4,0} + d_{3,3}e_{3,0} + d_{3,2}e_{2,0} + d_{3,1}$,

⑧ $(t+1)\left(e_{2,0}+\varphi_1'\right) = e_{3,1} + d_{2,2}\varphi_1$,

⑨ $(t+1)\,e_{2,0}' = e_{3,0} + d_{2,2}e_{2,0} + d_{2,1}$,

⑩ $-e_{2,0} = d_{1,1}$.

The order of calculation is as follows:

$$\begin{array}{ccccccc}
\textcircled{1} & & & & & & \\
\Downarrow & \searrow & & & & & \\
\textcircled{2} & \Longrightarrow & \textcircled{5} & & & & \\
\Downarrow & \swarrow & \downarrow & \searrow & & & \\
\textcircled{3} & \Longrightarrow & \textcircled{6} & \Longrightarrow & \textcircled{8} & & \\
\Downarrow & \swarrow & \downarrow & \searrow & \downarrow & \searrow & \\
\textcircled{4} & \Longrightarrow & \textcircled{7} & \Longrightarrow & \textcircled{9} & \Longrightarrow & \textcircled{10}
\end{array}$$

Now, putting

$$d_{j,j} = c_j t + a_j, \quad d_{j,i} = c_{j,i} t + a_{j,i} \quad (j \neq i),$$

we first carry out the calculation:

$$\textcircled{1}\,(e_{4,2}, d_{4,4}) \quad \rightarrow \quad \textcircled{2}\,(e_{4,1}, d_{4,3}) \quad \rightarrow \quad \textcircled{5}\,(e_{3,1}, d_{3,3}).$$

According to our method, we can immediately obtain

$$e_{4,2} = c_4t^3 + (a_4+8)\,t^2 - c_4t - (a_4+4)\,,$$

$$\begin{cases} a_{4,3} = \left(\beta^2-4\right) - a_4\,(a_4+4)\,, \\ e_{4,1} = \left(c_4^2+\alpha^2\right)t^3 + \left\{c_4\,(2a_4+5) + c_{4,3}\right\}t^2 \\ \qquad + \left\{a_4\,(a_4+6) - c_4^2 + a_{4,3} - 3\beta^2 - \gamma + 14\right\}t - \left\{c_4\,(2a_4+3) + c_{4,3}\right\} \end{cases}$$

and

$$\begin{cases} a_3 = -a_4 - 4, \\ e_{3,1} = (c_3 + c_4)\, t^2 + (a_3 + a_4 + 6)\, t - (c_3 + c_4)\,. \end{cases}$$

We here make a remark on the relation ①. Since the right hand side include a factor t, we have not any relation between a_4 and c_4. So, we can assign any value to a_4. From now on, we put

$$a_4 = 0.$$

Next we calculate:

$$③\ (e_{4,0}, d_{4,2}) \quad \to \quad ⑥\ (e_{3,0}, d_{3,2}) \quad \to \quad ⑧\ (e_{2,0}, d_{2,2}).$$

We have

$$\begin{cases} a_{4,2} = -\left(\beta^2 - 4\right)(c_3 + c_4)\,, \\ e_{4,0} = c_4\left(c_4^2 + \alpha^2\right) t^3 + \left\{2c_4^2 + c_{4,3}\,(c_3 + 2c_4) + 5\alpha^2\right\} t^2 \\ \qquad + \left\{-c_4\left(c_4^2 + \beta^2 + \gamma + 4\right) + c_3\left(\beta^2 - 4\right) + c_{4,2}\right\} t \\ \qquad - \left\{2c_4\,(c_4 + c_{4,3}) + c_3 c_{4,3} + c_{4,2} - a_{4,2} + 8\beta^2 + 3\gamma\right\}, \end{cases}$$

$$\begin{cases} a_{3,2} = c_4 + 4c_3 - c_{4,3}, \\ e_{3,0} = \left(c_4^2 + c_3 c_4 + c_3^2 + \alpha^2\right) t^2 + (c_{4,3} + c_{3,2} - c_4 - 4c_3)\, t \\ \qquad - \left(c_4^2 + c_3 c_4 + c_3^2 + c_{3,2} - a_{3,2} + 2\beta^2 + \gamma\right) \end{cases}$$

and

$$\begin{cases} a_2 = c_2 - 1, \\ e_{2,0} = (c_2 + c_3 + c_4)\,(t - 1)\,. \end{cases}$$

The last formula gives $d_{1,1}$ by ⑩.

We proceed to the calculation:

$$④\ (d_{4,1}) \quad \to \quad ⑦\ (d_{3,1}) \quad \to \quad ⑨\ (d_{2,1}).$$

Following our method again, we assign zero to coefficients of t^k $(k \geq 2)$ in the relations. From coefficients of fourth, third and second degree in ④, ⑦ and ⑨, respectively, we obtain

$$\begin{cases} c_4^2\left(c_4^2 + \alpha^2\right) = 0, \\ c_3\left(c_3^2 + c_3 c_4 + c_4^2 + \alpha^2\right) + c_4\left(c_4^2 + \alpha^2\right) = 0, \\ \left(c_3^2 + c_3 c_4 + c_4^2 + \alpha^2\right) + c_2\,(c_2 + c_3 + c_4) = 0. \end{cases}$$

We here take

$$c_4 = -\alpha i, \quad c_3 = \alpha i, \quad c_2 = 0.$$

Then, the coefficient of t^3 in ④ yields

$$2c_4^2 + c_{4,3}\,(c_3 + 2c_4) + 5\alpha^2 = 0,$$

whence

$$c_{4,3} = -3\alpha i.$$

Furthermore, the coefficients of t^2 in ⑦ and ④ yields

$$c_{3,2} = 6\alpha i, \quad c_{4,2} = \alpha i \left(\alpha^2 - 2\beta^2 - \gamma - 12\right),$$

successively.

From those values of the $c_{j,i}$, we can determine the remaining values of the $a_{j,i}$, and hence all $d_{j,i}$ and $e_{j,i}$ as follows:

$$\begin{cases} e_{2,0}(t) = 0, \\ e_{3,0}(t) = \theta + 12, \\ e_{3,1}(t) = 2t, \\ e_{4,0}(t) = -12\alpha i t + \left\{5\alpha^2 - 8\beta^2 - 3\gamma - \theta\alpha i\right\}, \\ e_{4,1}(t) = -8\alpha i t^2 + (\theta + 24)\, t + 6\alpha i, \\ e_{4,2}(t) = -\alpha i t^3 + 8t^2 + \alpha i t - 4, \end{cases}$$

where we have put

$$\theta = \alpha^2 - 2\beta^2 - \gamma - 12.$$

Then, by the transformation (3) with the above $e_{j,i}$, we have

$$\begin{cases} d_{1,1}(t) = 0, \\ d_{2,1}(t) = -\theta - 12, \\ d_{2,2}(t) = -1, \\ d_{3,1}(t) = -\theta\alpha i t + \theta\alpha i - \alpha^2 - \gamma, \\ d_{3,2}(t) = 6\alpha i t + 6\alpha i, \\ d_{3,3}(t) = \alpha i t - 4, \\ d_{4,1}(t) = \xi t + \eta, \\ d_{4,2}(t) = \theta\alpha i t, \\ d_{4,3}(t) = -3\alpha i t + \left(\beta^2 - 4\right), \\ d_{4,4}(t) = -\alpha i t, \end{cases}$$

where the constants ξ, η are given by

$$\xi = \alpha^2\theta + 2\left(4\alpha^2 - 7\beta^2 - 3\gamma - 6\right)\alpha i,$$
$$\eta = -\beta^2(\theta + 14) + 4\alpha^2 - 2\gamma.$$

Thus we have reduced the fourth order single differential equation to a system of linear differential equations of another form (1) with

$$B = \operatorname{diag}(1, -1, 0, 0),$$
$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\theta - 12 & -1 & 1 & 0 \\ \theta\alpha i - \alpha^2 - \gamma & 6\alpha i & -4 & 1 \\ \eta & 0 & \beta^2 - 4 & 0 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\theta\alpha i & 6\alpha i & \alpha i & 0 \\ \xi & \theta\alpha i & -3\alpha i & -\alpha i \end{pmatrix}.$$

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