Kumamoto J. Math.
Vol. **19** (2006), 115–116

A remark on the functional calculus in C*-algebras and positive definite sequences

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(Received January 20, 2006)

Abstract. Let A be a C*-algebra with the identity 1 and a be a self-adjoint element in A with the norm $||a|| \leq 1$. If a continuous function f on the interval $[-1,1]$ satisfies the invariant property with respect to positive definite sequences, then the element $f(a)$ is of the form $f(a) = \sum_{n=0}^{\infty} c_n a^n$, with $c_n \geq 0 (n = 0,1,2,\ldots)$

1. Main theorem

In this paper we give a remark on the functional calculus in C*-algebras and positive definite sequences. Let A be a C*-algebra with the identity 1 and a be a self-adjoint element in A. From the functional calculus it follows that an element x in A belongs to the C*-subalgebra $C^*(a)$ generated by a and 1 if and only if there is a continuous function f on the spectrum sp(a) of a such that $x = f(a)$. If the element x is given by $x = \sum_{n=0}^{\infty} c_n a^n$, with $c_n \geq 0 (n = 0,1,2,\ldots)$, then the function f defined by $f(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n$ satisfies an invariant property with respect to positive definite sequences that for a positive definite sequence $\{\lambda_i\}$, the sequence $\{f(\lambda_i)\}$ is positive definite. The converse is also true([2]). Thus we have the following

Theorem 1. *Let A be a C^*-algebra with the identity* 1 *and a be a self-adjoint element in A with the norm $||a|| \leq 1$. If a continuous function f on the interval* $[-1,1]$ *satisfies the invariant property with respect to positive definite sequences, then the element $f(a)$ is of the form $f(a) = \sum_{n=0}^{\infty} c_n a^n$, with $c_n \geq 0 (n = 0,1,2,\ldots)$*

2. Proof of Theorem

Let f be a continuous function on the interval $[-1,1]$. If the function f satisfies the invariant property with respect to positive definite sequences, then from Theorem I([2]) it follows that f is of the form $f(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n (-1 < \lambda < 1)$, with $c_n \geq 0 (n = 0,1,2,\ldots)$. The uniqueness of the functional calculus completes the proof.

Mathematics Subject Classification (2000): 46L05, 42A82
Keywords: C*-algebra, functional calculus, positive definite sequence

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