Kumamoto J. Math
Vol. 2, 1~8 March(1989)

A REMARK ON NONNEGATIVE SOLUTIONS WITH COMPACT SUPPORT FOR DEGENERATE SEMILINEAR ELLIPTIC EQUATIONS

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(Received November, 25, 1988)

1. Introduction

In this note we study the existence of nonnegative solutions with compact support for the equation

(1) $$\Delta u + f(u) = 0 \qquad \text{in } \mathbf{R}^N,$$

where $N > 1$ and f satisfies the following conditions:

(A1) f is locally Lipschitz continuous on $(0, p_1]$.

(A2) $f(p_1) = 0$.

(A3) There exists $\alpha \in (0, p_1)$ such that $f(u) > 0$ on (α, p_1).

(A4) The integral $F(u) = \int_0^u f(s)\,ds$ exists for all $u \in (0, p_1]$

and satisfies

$$F(u) < F(p_1) \qquad \text{for } 0 \leq u < p_1$$

As for the equation (1), in the previous paper [3] we studied the Dirichlet problem for

(2) $$\begin{cases} \Delta u + f(u) = 0 & \text{in } B_R, \\ u = 0 & \text{on } \partial B_R, \end{cases}$$

where $B_R = \{x \in \mathbf{R}^N ; |x| < R\}$, and proved the following

Theorem (Fukagai & Yoshida [3]) *Under the conditions* (A1) – (A4) *there exists* R_0 *such that the equation* (2) *admits a positive radial solution* $u(x) = u(r)$, $r = |x|$, *for any* $R \geqq R_0$, *which satisfies* $u'(R) < 0$,

At that time the problem remained whether the equation (2) admits a nonnegative radial solution $u(r)$ which satisfies $u'(R) = 0$ and so we prove here the following theorem with the additional condition:

(A5) $$-\infty < \liminf_{s \to +0} \frac{f(s)}{s^\sigma} \leq \limsup_{s \to +0} \frac{f(s)}{s^\sigma} = -m < 0$$

with some σ such that $0 < \sigma < 1$

Theorem. *Under the conditions* (A1) – (A5) *there exist nontrivial nonnegative radial solutions* $u(x)$ *of* (1) *with compact support.*

Apply this theorem to the equation

$$\Delta(v^p) + v(1-v)(v-a) = 0 \quad \text{in } \mathbf{R}^N \tag{3}$$

with $p > 1$. Then if we put

$$u = v^p \text{ and } f(u) = u^{1/p}(1-u^{1/p})(u^{1/p}-a),$$

we have the following

Corollary. *If* $0 < a < (p+1)/(p+3)$, *then the equation* (3) *admits nontrivial nonnegative radial solutions* v *such that* $v^p \in C_0^2(\mathbf{R}^N)$.

In [4] N. Fukagai proves the theorem of this type by studying the asymptotic behavior of solutions for the associated ordinary differential equation:

$$y'' + \frac{N-1}{r}y' + f(y) = 0 \qquad \text{for } 0 < r < \infty,$$
$$y(0) = p, \, y'(0) = 0$$

with help of Strum's comparison theorem. In this note we remark that the above theorem holds in the framework of variational methods considered in [1, 2].

2. Preliminaries

First we recall the mountain pass theorem (cf. Berestyki and Lions [2]) and compactness and radial lemmas due to Strauss [6]. Let H be a real Hilbert space whose norm and scalar product will be denoted respectively by $\|\cdot\|_H$ and $(\cdot, \cdot)$. Let E be a real Banach space with norm $\|\cdot\|_E$ and $E \subset H \subset E'$ wich continuous injections. Then H is identified with its dual space. Let M be the manifold

$$M = \{x \in E ; \|x\|_H = 1\},$$

which is endowed with the topology inherited from E, and J denotes a functional $J : E \to \mathbf{R}$ which is of class C^1 on E. We denote by J_M the trace of J on M. Then J_M is a C^1 functional on M, and for any $x \in M$,

$$\langle J'_M(x), w \rangle = \langle J'(x), w \rangle \qquad \text{for } w \in T_xM,$$

where T_xM is the tangent space at $x \in M$, that is,

$$T_xM = \{y \in E; (x,y) = 0\},$$

and $\langle \cdot , \cdot \rangle$ is the duarity pairing of either E' and E. Thus $J'_M(x) \in (T_xM)'$ and the notation $\|J'_M(x)\|$ is the norm in the cotangent space $T'_xM = (T_xM)'$. Now we recall the weaker Palais-Smale condition (in short $(P-S^+)$):

$(P-S^+)$ *For any* $C_1, C_2 > 0$ *and any sequence* $\{x_n\}_{n\in N} \subset M$ *such that* $C_1 \leq J(x_n) \leq C_2$ *and* $\|J'_M(x_n)\| \to 0$, *there exists a subsequence* $\{x_{n_k}\}$ *which converges in M.*

Let $\Sigma(M)$ be the set of compact and symmetric (with respect to the origin) subsets of M. The genus of a set $A \in \Sigma(M)$, $\gamma(A)$ is defined by the least integer $n \geq 1$ such that there exists an odd continuous mapping $\varphi: A \to S^{n-1} = \{x \in \mathbf{R}^n; |x| = 1\}$. We set $\gamma(A) = \infty$ if such an integer does not exist. For $k \geq 1$ let $\Gamma_k = \{A \in \Sigma; \gamma(A) \geq k\}$.

Theorem A. 1 (Berestyki and Lions). *Let* $J: E \to \mathbf{R}$ *be an even functional of class* C^1. *We assume that J is bounded from above on M and that* J_M *satisfies the condition* $(P-S^+)$. *Let*

$$b_k = \sup_{A\in\Gamma_k} \inf_{x\in A} J(x).$$

Then b_k *is a critical value of J provided* $b_k > 0$.

Theorem A. 2 (Strauss [6]). *Let P and* $Q: \mathbf{R} \to \mathbf{R}$ *be continuous functions satisfying*

$$\frac{P(s)}{Q(s)} \to 0 \qquad \text{as } |s| \to \infty.$$

Let $\{u_n\}$ *be a sequence of measurable functions* : $\mathbf{R}^N \to \mathbf{R}$ *such that*

$$\sup_n \int_{\mathbf{R}^N} |Q(u_n(x))|\, dx < \infty$$

and

$$P(u_n(x)) \to v(x) \quad \text{a. e. in } \mathbf{R}^N, \text{ as } n \to \infty.$$

Then for any bounded Borel set B one has

$$\int_B |P(u_n(x)) - v(x)|\, dx \to 0 \qquad \text{as } n \to \infty$$

If one further assumes that

$$\frac{P(s)}{Q(s)} \to 0 \qquad as\ s \to 0$$

and

$u_n(x) \to 0$ *as* $|x| \to \infty$, *uniformly with respect to n, then* $P(u_n)$ *converges to v in* $L^1(\mathbf{R}^N)$ *as* $n \to \infty$.

As usual let $H^1(\mathbf{R}^N)$ be the Sobolev space and $\mathrm{D}^{1,2}(\mathbf{R}^N)$ the closure of $C_0^\infty(\mathbf{R}^N)$ for the norm

$$\|\varphi\|_{\mathrm{D}^{1,2}} = \{\int_{\mathbf{R}^N} |\nabla\varphi|^2 dx\}^{1/2}$$

Then we see, by Sobolev's inequality, the injection

$$\mathrm{D}^{1,2}(\mathbf{R}^N) \subset L^{2^*}(\mathbf{R}^N),$$

is continuous, where

$$2^* = \begin{cases} \text{any } p \text{ such that } 2 < P < \infty & \text{if } N = 2, \\ \dfrac{2N}{N-2} & \text{if } N > 2. \end{cases}$$

Theorem A. 3 (Strauss [6]). *Let* $N \geqq 2$. *Every radial function* $u \in H^1(\mathbf{R}^N)$ *is almost everywhere equal to a function U (x), continuous for* $x \neq 0$ *and such that*

$$|U(x)| \leq C_N |x|^{(1-N)/2} \|u\|_{H^1(\mathbf{R}^N)} \qquad for\ |x| \geq a_N,$$

where C_N *and* a_N *depend only on the dimension N.*

3. Proof of Theorem

Let us define $\tilde{f} : \mathbf{R} \to \mathbf{R}$ an follows:

$$\tilde{f}(s) = \begin{cases} f(s) & \text{on } [0, p_1] \\ 0 & \text{for } s \geq p_1. \end{cases}$$

For $s \leq 0, \tilde{f}$ is defined by $\tilde{f}(s) = -\tilde{f}(-s)$. Observe that by maximum principle, nonnegative solutions which have compact support for the equation (1) with $\tilde{f}$ are also nonnegative solutions which have compact support for (1) with f. Hence there is no loss of generality in replacing f by $\tilde{f}$, and so we keep also the same notation f for the modified function $\tilde{f}$. Consider the functional

(4) $\phi(u) = \frac{1}{2}\int_{R^N} |\nabla u|^2 dx - \int_{R^N} F(u)\,dx$

and put

(5) $T(u) = \int_{R^N} |\nabla u|^2 dx, \qquad V(u) = \int_{R^N} F(u)\,dx.$

Since $F(s) = F(-s)$, if u_c is a critical point of $\phi(u)$, then $|u_c|$ is also the critical point of ϕ. Hence we can take always a nonnegative critical point of ϕ, if exists. Let $H = D^{1,2}(\mathbf{R}^N)$. Then H is the Hilbert space with scalar product

$$(\varphi, \psi) = \int_{R^N} \nabla\varphi \cdot \nabla\psi \; dx.$$

and H is identified with its dual. Let's denote by $D_r^{1,2}(\mathbf{R}^N)$ the subspace of $D^{1,2}(\mathbf{R}^N)$ formed by the radial functions. Similarly we use the notations $H_r^1(\mathbf{R}^N)$. Put

$$E = D_r^{1,2}(\mathbf{R}^N) \cap L_r^{1+\sigma}(\mathbf{R}^N),$$

which is endowed with the graph norm, where σ is the same constant as in (A. 5). If $u \in E$, then $u \in L^{1+\sigma}(\mathbf{R}^N) \cap L^{2^*}(\mathbf{R}^N)$ and so $u \in L^2(\mathbf{R}^N)$ by Hölder's inequality, which implies $E \subset H_r^1(\mathbf{R}^N)$. Thus we have, as an immediate consequence of Theorems A. 2 and A. 3, for any p such that $1 + \sigma < p < 2^*$, the injection $E \subset L^p(\mathbf{R}^N)$ is compact. Put

$$M = \{u \in E\,; T(u) = 1\}.$$

Then our theorem is derived from the following

Proposition. *For all $k \geq 1$ there exists a critical value β_k of V_M given by*

$$\beta_K = \sup_{A \in \Gamma_k} \inf_{x \in A} V_M(x).$$

Moreover $\beta_k > 0$ and there exist a critical point $v_k \in M$ corresponding to B_k and $\theta_k > 0$ such that

$$-\Delta v_k = \theta_k f(v_k) \qquad \textit{in } \mathbf{R}^N$$

Proof. Apply Theorem A. 1. Then we can prove this proposition by checking the following:

(i) V_M is bounded from above.

(ii) V_M satisfies $(P - S^+)$

(iii) $\beta_k > 0$.

As for (i), from the conditions (A. 4) and (A. 5)

$$\alpha = \sup\{c > 0; F(s) \leq 0 \text{ for } |s| \leq c\}$$

is positive. On the other hand, since $F(s)$ is bounded, there exists a positive constant C such that

$$F(s) \leq C\,|s|^{2^*}$$

Thus we have, by Sobolev's inequality,

$$V_M(u) = \int_{R^N} F(u)\,dx \leq C \int_{R^N} |u(x)|^{2^*} \leq C',$$

since $u \in M$.

We proceed to (ii). Let C_1 and C_2 be any positive constants such that $C_1 < C_2$ and $\{u_n\}_{n\in N} \subset M$ a sequence such that $C_1 \leq V(u_n) \leq C_2$ and $\| V'_M(x_n)\| \to 0$ as $n \to \infty$. Then we prove there exists $C > 0$ such that $\|u_n\|_E \leq C$. If this fact is shown, then the remainder of the proof is done along the same line as in [2]. In what follows we use a notation C which implies a variable positive constant but dose not depend on $\{u_n\}$. Put

$$f_1(s) = (f(s) + ms^{\sigma})^+ \quad \text{and} \quad f_2(s) = f_1(s) - f(s) \quad \text{for } s \geq 0$$

and

$$f_i(s) = -f_i(-s) \quad \text{for } s < 0.$$

Clearly $f_i \geq 0$. If we put

$$F_i(t) = \int_0^t f_i(s)\,ds,$$

then for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

(6) $\quad F_1(s) \leq C_\varepsilon\,|s|^{2^*} + \varepsilon F_2(s),$

and further

(7) $\quad F_2(s) \geq \dfrac{m}{1+\sigma}|s|^{1+\sigma} \qquad s \in \mathbf{R},$

since $f_2(s) \geq ms^{\sigma}$ for $s \geq 0$ and $f_2(s) = -f_2(-s)$ for $s < 0$. Form (6) with $\varepsilon = \frac{1}{2}$ we have

(8) $\quad V(u_n) \leq C \displaystyle\int_{R^N} |u_n(x)|^{2^*}\,dx - \frac{1}{2}\int_{R^N} F_2(u_n(x))\,dx.$

Since $\|u_n\| \leq 1$, it follows from Sobolev's inequality that there exists a constast $C > 0$ such that

$$\int_{R^N} |u_n(x)|^{2^*}\,dx \leq C.$$

Since $V(u_n) \geqq C_1 > 0$, we have, from (8),

$$0 \leq \int_{\mathbf{R}^N} F_2(u_n(x))\,dx \leq C.$$

This together with (7) leads to

$$\int_{\mathbf{R}^N} |u_n(x)|^{1+\sigma}\,dx \leq C,$$

which means $\| u_n \|_E \leq C$, since

$$\int_{\mathbf{R}^N} |\nabla u_n(x)|^2\,dx = 1.$$

Thus (ii) holds.

Since the assertion (iii) follows from Theorem 10 in [2], the above Proposition holds. The proof is complete.

Finally we prove our Theorem. Apply the above Proposition. If we put, for any $k \geq 1$,

$$u_k(x) = v_k(x/\sqrt{\theta_k}),$$

then $u_k(x)$ is a weak solution of $\Delta u_k + f(u_k) = 0$. In what follows we omit the subcsript k in u_k. Since f is locally Lipschitz continuous on $\mathbf{R}^N - \{0\}$ and Hölder continuous at 0, we have $u \in C^2(\mathbf{R}^N)$. Furthermore since $u \in E \subset H^1_r(\mathbf{R}^N)$, it follows from Theorem A. 3 that

(9) $\qquad |u(x)| \leq C_N\, |x|^{(1-N)/2}\, \|u\|_{H^1(\mathbf{R}^N)} \qquad$ for $|x| \geq a_N$.

Now, from [5, Theorem5] we see the condition

(10) $\qquad \int_0 |F(s)|^{-1/2}\,ds < \infty$

is necessary and sufficient for nonnegative radial solutions u of (1) with a property that $u(x) \to 0$ as $|x| \to \infty$ to have compact support. But the condition (10) holds from (A. 5). Thus the proof is complete.

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